

**Math 124, Solutions to Practice Questions for Exam #2, April 23, 2008**

1. Which of the following series converges? Explain your answer.

(a)

$$\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \cdots$$

This is an alternating series which can be written as

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}.$$

Since  $\frac{1}{\ln(n+1)} > \frac{1}{\ln(n+2)}$  for all  $n$  and

$$\lim_{n \rightarrow \infty} \frac{1}{\ln(n+1)} = 0,$$

the series converges by the Alternating Series Test.

(b)

$$\sum_{n=1}^{\infty} \frac{n}{(\sin n)^2}$$

Since  $(\sin n)^2 \leq 1$  for all  $n$ ,  $\frac{n}{(\sin n)^2} \geq n$  for all  $n$ . Taking limits of both sides

$$\lim_{n \rightarrow \infty} \frac{n}{(\sin n)^2} \geq \lim_{n \rightarrow \infty} n = \infty$$

which is nonzero. Therefore, the series diverges by the Divergence Test.

(c)

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Consider the function  $f(x) = \frac{1}{x(\ln x)^2}$ . It is certainly positive when  $x \geq 2$ . It is decreasing since its numerator is constant and its denominator is increasing. Thus, the function  $f(x)$  is positive and decreasing on the interval  $x \geq 2$ . Furthermore,

$$\lim_{x \rightarrow \infty} \frac{1}{x(\ln x)^2} = 0$$

hence, we can use the Integral Test which states that the series converges if and only if  $\int_2^{\infty} f(x)dx$  converges. The latter is

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Therefore, the series converges.

(d)

$$\sum_{n=1}^{\infty} \frac{n^2}{8n^7 + 6n^2 + 5}$$

Use the Limit Comparison Test. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^5}.$$

Both series have positive terms and, in addition,

$$\lim_{n \rightarrow \infty} \frac{\frac{n^2}{8n^7 + 6n^2 + 5}}{\frac{1}{n^5}} = \frac{1}{8}.$$

By the Limit Comparison Test, both series either converge or both series diverge. However,  $\sum_{n=1}^{\infty} \frac{1}{n^5}$  converges as it is a p-series with  $p = 5 > 1$ .

(e)

$$\sum_{n=1}^{\infty} 2^{2n+1} 5^{-n}$$

Notice that

$$\sum_{n=1}^{\infty} 2^{2n+1} 5^{-n} = \sum_{n=1}^{\infty} (2) \left(\frac{4}{5}\right)^n = \sum_{n=1}^{\infty} \left(\frac{8}{5}\right) \left(\frac{4}{5}\right)^{n-1}$$

which is a geometric series which converges since  $|\frac{4}{5}| < 1$ . In fact, we even know that the series is equal to

$$\frac{\frac{8}{5}}{1 - \frac{4}{5}} = 8.$$

(f)

$$\sum_{n=1}^{\infty} a_n$$

where  $a_1 = 7$  and  $a_{n+1} = \frac{2n+3}{5n+9} a_n$  for all  $n \geq 1$ .

Let's use the Ratio Test.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2n+3}{5n+9} \right| = \frac{2}{5}$$

but  $\frac{2}{5} < 1$ , therefore, the series converges.

- Write the repeating decimal  $0.\overline{38}$  as a ratio of integers.

This is a geometric series since

$$0.\overline{38} = .38 + .0038 + .000038 + \dots = \sum_{n=0}^{\infty} (.38) \frac{1}{100^n} = \frac{.38}{1 - \frac{1}{100}} = \frac{38}{99}.$$

- Consider the following series

$$s = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

How many terms in the series must one sum up in order to obtain  $s$  correct to within an accuracy of 0.00001?

Recall that the series  $s$  converges by the Integral Test. By the Remainder Estimate for the Integral Test, the remainder  $R_n = s - s_n$  where  $s_n = \sum_{k=1}^n \frac{1}{k^3}$  satisfies

$$R_n \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}.$$

We want

$$\frac{1}{2n^2} < 0.00001$$

which is equivalent to  $n > \sqrt{50000} \cong 223.607$ . Therefore,  $s_{224}$  (which is the sum of the first 224 terms) is equal to  $s$  up to an accuracy of 0.00001.

4. Suppose that we know that a power series  $\sum_{n=0}^{\infty} c_n(x-1)^n$  converges when  $x = 4$  and diverges at  $x = -6$ .

We know that for any power series  $\sum_{n=0}^{\infty} c_n(x-a)^n$ , there exists a number  $R$ , called the radius of convergence, such that if  $|x-a| < R$  then the series converges, if  $|x-a| > R$  then the series diverges, and if  $|x-a| = R$  then the series could either converge or diverge.

In our situation  $a = 1$ , clearly. Since the series converges for  $x = 4$ , it must be true that  $|4-1| = 3 \leq R$ . On the other hand, the series diverges for  $x = -6$  so  $|-6-1| = 7 \geq R$ . So the only thing we know is that

$$3 \leq R \leq 7.$$

- (a) What can one say about the convergence of the series at  $x = -1$ ?

Since  $|-1-1| = 2 < R$ , the series converges at  $x = -1$ .

- (b) What can one say about the convergence of the series at  $x = -7$ ?

Since  $|-7-1| = 8 > R$ , the series diverges at  $x = -7$ .

- (c) What can one say about the convergence of the series at  $x = -2$ ?

Since all we know is that  $|-2-1| = 3 \leq R$ , the series could either diverge or converge since  $R$  could be equal to 3. So we can't say anything without any more information.

5. Consider the function  $f(x) = \frac{3x^4}{5x-7}$ .

- (a) Write  $f(x)$  as a power series.

$$\frac{3x^4}{5x-7} = -\frac{3x^4}{7} \frac{1}{1-\frac{5x}{7}} = -\frac{3x^4}{7} \sum_{n=0}^{\infty} \left(\frac{5x}{7}\right)^n$$

using  $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$  and plugging in  $u = \frac{5x}{7}$ . Multiplying through, one obtains

$$f(x) = \sum_{n=0}^{\infty} -\frac{(3)5^n}{7^{n+1}} x^{n+4}.$$

- (b) Find its radius of convergence.

Its radius of convergence,  $R$ , is equal to the radius of convergence of  $\sum_{n=0}^{\infty} \left(\frac{5x}{7}\right)^n$ . But the latter converges if and only if  $|\frac{5x}{7}| < 1$  or, equivalently, when  $|x| < \frac{7}{5}$ . Therefore,  $R = \frac{7}{5}$ .

- (c) Find its interval of convergence.

The series converges on the interval  $(-\frac{7}{5}, \frac{7}{5})$ .

6. Consider the function  $f(x) = \tan^{-1}(x^3)$ .

- (a) Write  $f(x)$  as a power series.

We have that

$$\tan^{-1} u = \int \frac{1}{1+u^2} du = \int \sum_{n=0}^{\infty} (-u^2)^n du = C + \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{2n+1}$$

where  $C$  is an integration constant. However, since  $\tan^{-1}(0) = 0$ ,  $C = 0$ . Therefore,

$$\tan^{-1} u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{2n+1}.$$

Now we just set  $u = x^3$  to obtain

$$\tan^{-1} x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}.$$

- (b) Find its radius of convergence.

The series converges if  $|u| = |x^3| < 1$  which is equivalent to  $|x| < 1$ . Therefore, the radius of convergence is 1.

7. Consider the series  $\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n n}$ .

- (a) Find its radius of convergence.

Since

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(x+3)^{(n+1)}}{2^{(n+1)}(n+1)}}{\frac{(x+3)^n}{2^n n}} \right| = \lim_{n \rightarrow \infty} \frac{|x+3|}{2} \frac{n}{n+1} = \frac{|x+3|}{2},$$

the series converges if  $|x+3| < 2$  and diverges if  $|x+3| > 2$  by the Ratio Test. Therefore, the radius of convergence is 2.

- (b) Find its interval of convergence.

We need only check  $x = -3 \pm 2$ . If  $x = -3 + 2 = -1$  then the series is

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is the Harmonic series and, hence, diverges. If  $x = -3 - 2 = -5$  then the series is

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges by the Alternating Series Test. Therefore, the interval of convergence is  $[-5, -1)$ .

8. Find the Taylor series centered at 1 of the function  $f(x) = x^{\frac{2}{3}}$ .

Notice that  $f^{(0)}(x) = x^{\frac{2}{3}}$  and for all  $n \geq 1$ ,

$$f^{(n)}(x) = \left(\frac{2}{3}\right)\left(\frac{2}{3} - 1\right) \cdots \left(\frac{2}{3} - n + 1\right) x^{\frac{2}{3} - n}.$$

The we have

$$\sum_{n=0}^{\infty} f^{(n)}(1) \frac{(x-1)^n}{n!} = \sum_{n=0}^{\infty} c_n \frac{(x-1)^n}{n!}$$

where  $c_n = \left(\frac{2}{3}\right)\left(\frac{2}{3} - 1\right) \cdots \left(\frac{2}{3} - n + 1\right)$  if  $n \geq 1$  and  $c_0 = 1$ .

9. (a) Find the MacLauren series of the function  $f(x) = \ln(3+x)$ .

Notice that  $f^{(0)}(x) = \ln(3+x)$  and for all  $n \geq 1$ ,

$$f^{(n)}(x) = (-1)^{n-1} (n-1)! (3+x)^{-n}.$$

Plugging in 0, we obtain

$$\sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!} = \ln 3 + \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! 3^{-n} \frac{x^n}{n!} = \ln 3 + \sum_{n=1}^{\infty} (-1)^{n-1} 3^{-n} \frac{x^n}{n}.$$

- (b) Find its radius of convergence.

Since

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n 3^{-(n+1)} \frac{x^{n+1}}{n+1}}{(-1)^{n-1} 3^{-n} \frac{x^n}{n}} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{3} \frac{n}{n+1} = \frac{|x|}{3},$$

the Ratio Test implies that the radius of convergence is 3.

10. (a) Find a power series expression for the following integral:

$$\int e^{-x^4} dx$$

We use the formula

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

then setting  $u = -x^4$ ,

$$\int e^{-x^4} dx = \int \left( \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} \right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \int x^{4n} dx \right) = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{4n+1}}{4n+1}$$

where  $C$  is an integration constant.

- (b) Find a series representation for the following:

$$\int_0^2 e^{-x^4} dx$$

Just take the answer in the previous answer, plug in  $x = 2$  and  $x = 0$  and take the difference to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{2^{4n+1}}{4n+1}$$

11. Calculate

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)! 2^{2n}}$$

Recall the power series expansion for  $\sin x$  which is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Plugging in  $x = \frac{\pi}{2}$ , we obtain

$$\sin \frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)! 2^{2n+1}}.$$

Multiplying through by 2, we obtain

$$2 \sin \frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} 2}{(2n+1)! 2^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)! 2^{2n}}$$

which is our desired series. The answer is therefore  $2 \sin \frac{\pi}{2} = 2$ .