Math 566A1, Take Home Final Exam, April 30, 2008
Prof. Takashi Kimura

This exam is due at my office (MCS 234) at 11am on Monday May 12. The exam is open book and you may work with others although you must write up the solutions yourself. Good luck!

1. (40 points) Let G be a matrix Lie group. Let V be a $\mathbf{R}$vector space and Aut$V$ be the set of invertible linear transformations from V to itself. A *representation of G* is a smooth map $\rho : G \to \text{Aut}V$ such that $\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2)$ for all g_1, g_2 in G.

   An *isomorphism between two representations of G*, $\rho : G \to \text{Aut}(V)$ and $\rho' : G \to \text{Aut}(V')$, is an isomorphism of vector spaces $\phi : V \to V'$ such that $\rho'(g) \circ \phi = \phi \circ \rho(g)$ for all g in G.

   (a) Let $\mathbf{g}$ be the Lie algebra of G and let $\text{Ad} : G \to \text{Aut}(\mathbf{g})$ denote the map where $\text{Ad}(g)A := gAg^{-1}$ for all A in $\mathbf{g}$, and g in G. Show that Ad is a representation of G. It is called the *adjoint representation of G*.

   (b) Let $\mathbf{g}^*$ denote the dual vector space to $\mathbf{g}$. Let $\text{coAd} : G \to \text{Aut}(\mathbf{g}^*)$ be defined by
   $$(\text{coAd}(g)\alpha)(A) := \alpha(\text{Ad}(g^{-1})A)$$
   for all g in G, A in $\mathbf{g}$, and α in $\mathbf{g}^*$. Show that coAd is a representation of G. It is called the *coadjoint representation of G*.

   (c) Define the map $\eta : \mathbf{g} \times \mathbf{g} \to \mathbf{R}$ via $\eta(A, B) := -\frac{1}{2}\text{tr}(AB)$ where tr denotes the trace. Show that η is a symmetric, bilinear form satisfying
   $$\eta(\text{Ad}(g)A, \text{Ad}(g)B) = \eta(A, B)$$
   for all A, B in $\mathbf{g}$ and g in G. This bilinear form η on $\mathbf{g}$ is said to be *G-invariant.*

   (d) If η is any G-invariant, symmetric, bilinear, nondegenerate form on $\mathbf{g}$ then it is said to be a *G-invariant metric on* $\mathbf{g}$. Show that when $\mathbf{g} = \mathbf{so}(3)$ then η is a G-invariant metric and show that $\{L_1, L_2, L_3\}$ forms an orthonormal basis for $\mathbf{so}(3)$.

   (e) If $\mathbf{g}$ has a G-invariant metric then show that the map $\phi : \mathbf{g} \to \mathbf{g}^*$ defined by $\phi(A) := \eta(A, \cdot)$ is an isomorphism between the adjoint and coadjoint representations of G. Therefore, as representations of SO(3), $\mathbf{so}(3)$ and $\mathbf{so}(3)^*$ are isomorphic.

   (f) Let K be the Lie group consisting of all matrices
   $$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$
   where a is any positive real number and b is any real number. Is the adjoint representation of K is isomorphic to the coadjoint representation of K?

(g) Consider the standard representation of $SO(3)$, $\rho : SO(3) \to \mathrm{Aut}(\mathbf{R}^3)$, defined by $\rho(g)v = g \cdot v$ where $g \cdot v$ denotes matrix multiplication where v is a column vector. Consider the map $\psi : \mathbf{so}(3) \to \mathbf{R}^3$ given by $\psi(\sum_{i=1}^3 \xi^i L_i) := \sum_{i=1}^3 \xi^i e_i$ where e_i is the i-th standard unit vector in $\mathbf{R}^3$. Argue that ψ is an isomorphism of representations of $SO(3)$.

2. (30 points) Let $M = \mathbf{R}^4$ with standard coordinates (r^1, r^2, p_1, p_2) and the symplectic form $\omega = dp_1 \wedge dr^1 - dp_2 \wedge dr^2$. Let G denote the subgroup of $SO(1,1)$ consisting of matrices of the form

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

for all t in $\mathbf{R}$.

(a) Show that the action of G on M given by $S_g(r,p) := (gr, pg^{\mathrm{T}})$ for all (r,p) in M and g in G preserves the symplectic form ω, *i.e.* $S_g^*\omega = \omega$ for all g in G.

(b) Show that the Lie algebra $\mathbf{g}$ is 1-dimensional with a basis

$$\lambda := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(c) Find the vector field, $\widehat{\lambda}$, on M associated to λ.

(d) Find a function Φ^λ (namely, the moment map) whose Hamiltonian vector field is equal to $\widehat{\lambda}$.

(e) Consider a Hamiltonian system (M, ω, H) where the Hamiltonian function of the system is of the form

$$H := \frac{1}{2}\left((p_1)^2 - (p_2)^2\right) + V(r^1, r^2)$$

for some function V which depends only upon the variables (r^1, r^2). What conditions should the potential function V satisfy which would insure that Φ^λ would be a conserved quantity of this Hamiltonian system?

3. (30 points) Let $M := \mathbf{R}^4$ with standard coordinates (r^1, r^2, p_1, p_2) with symplectic form and the symplectic form $\omega = \sum_{i=1}^2 dp_i \wedge dr^i$. Let an element $g \in \mathrm{SO}(2)$

$$g = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

act upon M via the map $S_g : M \to M$ defined by

$$S_g(r^1, p_1, r^2, p_2) := (r^1 \cos t - p_1 \sin t, r^1 \sin t + p_1 \cos t, r^2 \cos t - p_2 \sin t, r^2 \sin t + p_2 \cos t).$$

(a) Show that $S_g^*\omega = \omega$ for all g in $SO(2)$.

(b) Show that the Lie algebra $\mathbf{so}(2)$ is one dimensional with a basis element

$$L := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(c) Find the vector field $\widehat{L}$ associated to L.

(d) Find the function Φ^L whose Hamiltonian vector field is $\widehat{L}$. Denote Φ^L by ϕ for short.

(e) For all constants c in $\mathbf{R}$, what is the submanifold $\phi^{-1}(c)$ of M?

(f) The reduced phase space $M//_c\mathrm{SO}(2) := \phi^{-1}(c)/\mathrm{SO}(2)$ is a familiar manifold. What is it? Describe the symplectic form $\overline{\omega}$ on $M//_c\mathrm{SO}(2)$.

4. (Bonus question: 10 points) Generalize the construction in the previous problem to $M = \mathbf{R}^{2n}$ with a similar action of $\mathrm{SO}(2)$? What are $\phi^{-1}(c)$ and $M//_c\mathrm{SO}(2)$ in these cases?