Math 566A1, Take Home Midterm Exam, March 17, 2008
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This exam is due in class on Monday March 31. The exam is open book and you may work with others although you must write up the solutions yourself. Good luck!

1. (20 points) Let M be an n-dimensional manifold. A *metric*, g, on M is a non-degenerate, symmetric bilinear form which takes two vector fields X and Y to a smooth function $g(X, Y)$ on M. In other words, g is a bilinear form such that

   Symmetry: $g(X, Y) = g(Y, X)$ for all vector fields X and Y, and

   Nondegeneracy: if $g(X, Y) = 0$ for all vector fields Y then $X = 0$.

   Furthermore, a metric is said to be *Riemannian* iff g is positive definite, *i.e.*

   Positive definiteness: $g(X, X) \geq 0$ for all vector fields X where the equality holds if and only if $X = 0$.

   (a) Show that if g is a symmetric bilinear form which is positive definite then g is nondegenerate.

   (b) Let $M = \mathbf{R}^n$ with Cartesian coordinates $(x^1, x^2, \ldots, x^n)$ and define g via the formula
   $$g = \sum_{i=1}^{n} dx^i \otimes dx^i. \tag{1}$$

   i. Show that g is a Riemannian metric and is, indeed, the ordinary scalar product on $\mathbf{R}^n$.

   ii. When $n = 2$, express g in polar coordinates (r, θ).

   (c) Let $M = \mathbf{R}^n$ with Cartesian coordinates $(x^1, x^2, \ldots, x^n)$ and define h via the formula
   $$h = -dx^1 \otimes dx^1 + \sum_{i=2}^{n} dx^i \otimes dx^i.$$

   i. Show that h is a metric but not a Riemannian metric. It is called the Lorentzian (or Minkowski) metric on $\mathbf{R}^n$.

   ii. When $n = 2$, express h in terms of the coordinates (z, w) where $z := -x^1 + x^2$ and $w := x^1 + x^2$. (z, w) are called light-cone coordinates.

   (d) Let $m > 0$ be a constant. Let $M = \{\,(t, r) \,|\, r > 2m, t \in \mathbf{R}\}$ be the 2-dimensional manifold and define g via the formula
   $$q = -\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr$$

   Show that q is a metric but not a Riemannian metric. It is called the (1+1)-dimensional Schwarzschild metric and it corresponds, in general relativity, to a (static, chargeless) black hole of mass m (in units where $G = c = 1$) where r is a radial coordinate and t is time.

2. (20 points) Let $M = \mathbf{R}^3$ regarded as a smooth 3 dimensional manifold with Cartesian coordinates (x^1, x^2, x^3). Let g be usual metric (the scalar product) given by Equation (1). Define the *standard volume form* ν in $\Omega^3(M)$ via

$$\nu := dx^1 \wedge dx^2 \wedge dx^3.$$

Given a vector field X, define α_X in $\Omega^1(M)$ by

$$\alpha_X := g(X, \cdot)$$

and define β_X in $\Omega^2(M)$ by

$$\beta_X := \nu(X, \cdot, \cdot).$$

(a) Show that both α_X and β_X are unique for a given vector field X.

(b) For all functions f in $\Omega^0(M) = C^\infty(M)$, define the vector field ∇f on M via the equality of 1-forms

$$g(\nabla f, \cdot) := df.$$

Show that ∇f agrees with the usual definition of the gradient (vector field) of f from vector calculus.

(c) For all vector fields X, define the vector field $\nabla \times X$ via the equality of 2-forms

$$\nu(\nabla \times X, \cdot, \cdot) := d\alpha_X.$$

Show that $\nabla \times X$ agrees with the usual definition of the curl of X from vector calculus.

(d) For all vector fields X, define the (scalar) function $\nabla \cdot X$ via the equality of 3-forms

$$(\nabla \cdot X)\nu := d\beta_X.$$

Show that $\nabla \cdot X$ agrees with the usual definition of the divergence of X from vector calculus.

(e) Show that the identities $\nabla \times \nabla f = 0$ and $\nabla \cdot \nabla \times X = 0$ are equivalent to the identities $d(df) = 0$ and $d(d\alpha_X) = 0$, respectively. This shows that the correct generalization of "div, grad, and curl" on a general (possibly non-3dimensional) manifold, once we have chosen a metric g and a volume form ν, is in terms of differential forms, the exterior derivative d, and the wedge product.

3. (40 points) Let S^2 denote the unit sphere in $\mathbf{R}^3$, *i.e.*

$$S^2 := \{ (x^1, x^2, x^3) \in \mathbf{R}^3 \,|\, (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \}.$$

Consider the following maps:

(a) $\gamma' : U' := (-1, 1) \times (0, 2\pi) \to S^2$ where

$$\gamma'(h, \theta) := (\sqrt{1 - h^2}\cos(\theta), \sqrt{1 - h^2}\sin(\theta), h),$$

(b) $\gamma'' : U'' := (-1,1) \times (-\pi,\pi) \to S^2$ where

$$\gamma''(h,\theta) := (\sqrt{1-h^2}\cos(\theta), \sqrt{1-h^2}\sin(\theta), h),$$

(c) $\gamma : U := \{(u,v) \,|\, u^2+v^2 < 1\} \to S^2$ where

$$\gamma(u,v) := (u, v, \sqrt{1-u^2-v^2})$$

(d) $\widetilde{\gamma} : \widetilde{U} := \{(\widetilde{u},\widetilde{v}) \,|\, \widetilde{u}^2+\widetilde{v}^2 < 1\} \to S^2$ where

$$\widetilde{\gamma}(\widetilde{u},\widetilde{v}) := (\widetilde{u}, \widetilde{v}, -\sqrt{1-\widetilde{u}^2-\widetilde{v}^2}).$$

(a) Show that $\{\gamma', \gamma'', \gamma, \widetilde{\gamma}\}$ is an atlas for S^2.

(b) Consider a 2 form ω on S^2 which, in local coordinates γ' (as well as in local coordinates γ'') is given by the expression

$$\omega := dh \wedge d\theta.$$

Express ω in local coordinates γ and in local coordinates $\widetilde{\gamma}$.

(c) Show that ω is a symplectic form on S^2.

(d) Define the "height function" $H : S^2 \to \mathbf{R}$ via $H(x^1,x^2,x^3) := x^3$ for all (x^1,x^2,x^2) in S^2. Show that H is a smooth function on S^2.

(e) Calculate its associated Hamiltonian vector field X_H in the local coordinates (h,θ). What happens to X_H at the north and south poles?

(f) Write down and solve Hamiltonian's Equations in local coordinates (h,θ).

(g) Give an explicit formula for the Hamiltonian flow $\Gamma_t : S^2 \to S^2$. Interpret this flow as an action of the Lie group S^1 on S^2.

(h) Describe the space of orbits S^2/S^1.

4. (20 points) What manifold is the Lie group $SO(1,1)$ diffeomorphic to? What about the Lie group $O(1,1)$?