Math 722A1, Final Exam
Differential Topology II

(1) (10 points) Let g be the usual scalar product on $\mathbb{R}^n$. The n-dimensional orthogonal group $O(n)$, for $n \geq 1$, is defined by

$$O(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid g(Av, Aw) = g(v, w) \,\forall v, w \in \mathbb{R}^n\}.$$

Find the Lie algebra of $O(n)$, denoted by $o(n)$.

(2) (10 points) Lee Problem 20-6

(3) (10 points) Lee Problem 20-7

(4) (10 points)

(a) Let M be an n-dimensional connected smooth manifold and let p be a point of M. Use the Mayer-Vietoris sequence to relate the deRham cohomology of $M - \{\, p \,\}$ to the deRham cohomology of M.

(b) Calculate the deRham cohomology of $\mathbb{R}^n - \{p, q\}$ where p and q are distinct points in $\mathbb{R}^n$.

(5) (10 points) Let $\mathbb{C}^\times$ denote the nonzero complex numbers. It is a 2 dimensional Lie group under multiplication. Consider the $(2n+2)$-dimensional smooth manifold

$$P := \{\, (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} | (z_0, z_1, \ldots, z_n) \neq (0, 0, \ldots, 0)\}$$

It has a (right) action $P \times \mathbb{C}^\times \to P$ defined by

$$(z_0, z_1, \ldots, z_n) \cdot \lambda := (z_0\lambda, z_1\lambda, \ldots, z_n\lambda).$$

Define two points $(z_0, z_1, \ldots, z_n)$ and $(z'_0, \ldots, z'_n)$ in P to be equivalent if and only if

$$(z'_0, z'_1, \ldots, z'_n) = (z_0\lambda, z_1\lambda, \ldots, z_n\lambda)$$

for some λ in $\mathbb{C}^\times$, i.e. each equivalence class $[z_0, z_1, \ldots, z_n]$ is an orbit of $\mathbb{C}^\times$. The complex projective space CP^n is the set of such equivalence classes.

(a) Show that CP^n is a smooth $2n$-dimensional manifold.

(b) Let $\pi : P \to CP^n$ be the projection map $\pi(z_0, z_1, \ldots, z_n) := [z_0, z_1, \ldots, z_n]$. Show that $\pi : P \to CP^n$ is a principal $\mathbb{C}^\times$-bundle.