Math 722A1, Homework 1
Differential Topology I

(1) (10 points) Let S^n be the unit n-sphere with the standard metric g. Let RP^n denote n-dimensional projective space and let $\pi : S^n \to RP^n$ be the canonical projection map $\pi(p) := [p]$ where $[p]$ is the equivalence class of p.
   (a) Show that there is unique metric h on RP^n such that $\pi^* h = g$.
   (b) Show that every geodesic $\gamma : R \to RP^n$ with respect to the metric h is closed (that is, there is a number a such that $\gamma(t+a) = \gamma(t)$ for all t), and that every two geodesics intersect exactly once.

(2) (10 points) Let M be a smooth, n-dimensional manifold. Let ∇ be a connection on M. Let p be a point in M and let $(U, (x^i))$ be a chart containing p then let Γ^k_{ij} for all $i, j, k = 1, \ldots, n$ be functions defined by the equation

$$\sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x^k} := \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}.$$

Similarly, let $(\widetilde{U}, (\widetilde{x}^i))$ denote another chart about p and define $\widetilde{\Gamma}^k_{ij}$ for all $i, j, k = 1, \ldots, n$ be functions defined by the equation

$$\sum_{k=1}^{n} \widetilde{\Gamma}^k_{ij} \frac{\partial}{\partial \widetilde{x}^k} := \nabla_{\frac{\partial}{\partial \widetilde{x}^i}} \frac{\partial}{\partial \widetilde{x}^j}.$$

How are $\{ \Gamma^k_{ij} \}$ and $\{ \widetilde{\Gamma}^k_{ij} \}$ related?

(3) (10 points) Let M be a smooth manifold.
   (a) Let ∇ and ∇' be two connections on M. For all vector fields X and Y on M, define
   $$A(X, Y) := \nabla'_X Y - \nabla_X Y.$$
   Show that A is a covariant tensor of type 2.
   (b) Let ∇ be a connection on M and let B be any covariant tensor of type 2 then define
   $$\nabla'_X Y := \nabla_X Y + B(X, Y).$$
   Show that ∇' is a connection on M.
   (c) What can one conclude about the set of all connections on M?

(4) (10 points) Problem 17-4 of Lee.
(5) (10 points) Problem 17-5 of Lee.
(6) (10 points) Prove part (e) of Proposition 18.9 (see page 474) of Lee.
(7) (10 points) Problem 18-2 of Lee.
(8) (10 points) Problem 18-4 of Lee.
(9) (10 points) Problem 18-6 of Lee.