Math 564A1, Midterm Exam
Introduction to Topology

(1) (10 points) Consider the diagonal map \( \Delta : X \to X \times X \) taking \( \Delta(x) := (x, x) \).
   (a) Prove that \( \Delta \) a continuous function.
   (b) Prove that \( X \) is Hausdorff if and only if \( \Delta(X) \) is a closed subset of \( X \times X \).

(2) Prove that any two disjoint compact subsets of a Hausdorff space possess disjoint open neighborhoods.

(3) (10 points) Prove that \([0, 1) \times [0, 1)\) is homeomorphic to \([0, 1) \times [0, 1)\).

(4) (10 points) Let \( X \times Y \) have the product topology. Show that \( A \times B = A \times B \).

(5) (10 points) Prove that \( \mathbb{R}^2 \) is homeomorphic to \( S^2 - \{ p \} \) where \( p \) is any point of the unit sphere \( S^2 \).

(6) (10 points) Crossley 5.8

(7) (10 points) Crossley 5.10

(8) (10 points) Crossley 5.11

(9) (10 points) Let \( X \) be a set. A metric on \( X \) is a function \( d : X \times X \to \mathbb{R} \) such that for all \( x, y, \) and \( z \) in \( X \), the following conditions are satisfied:
   - **Positivity:** \( d(x, y) \geq 0 \) where equality holds if and only if \( x = y \),
   - **Symmetry:** \( d(x, y) = d(y, x) \),
   - **Triangle Inequality:** \( d(x, y) + d(y, z) \geq d(x, z) \).
   Choose a metric \( d \) on \( X \) then for all \( x \) in \( X \) and \( r > 0 \), the open ball of radius \( r \) centered about \( x \) is
     \[ B_r(x) := \{ y \in X \mid d(x, y) < r \} . \]
   The metric topology on \( X \) induced by the metric \( d \) is the topology on \( X \) which has a basis given by the collection of all open balls \( B_r(x) \).
   (a) Define a metric on \( \mathbb{R} \) by the formula \( d(x, y) := |x - y| \). Show that the metric topology on \( \mathbb{R} \) induced by \( d \) agrees with the standard topology on \( \mathbb{R} \).
   (b) For all \( n \geq 1 \) and positive integers \( p \), let
     \[ d_p(x, y) := \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}} \]
   for all \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) in \( \mathbb{R}^n \). Assume that \( d_p \) defines a metric on \( \mathbb{R}^n \). Prove that all of these metrics induce the same topology on \( \mathbb{R}^n \).
   (c) Let \( X \) and \( Y \) be topological spaces with metric topologies induced from metrics \( d_X \) on \( X \) and \( d_Y \) on \( Y \), respectively. Let \( f : X \to Y \) be a function. Prove that \( f \) is a continuous function if and only if given any \( x \) in \( X \), and \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( d_Y(f(x), f(y)) < \epsilon \) whenever \( d_X(x, y) < \delta \).