Oscillons Near Hopf Bifurcations of Planar Reaction Diffusion Equations

by

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Oscillons are spatially localized, time-periodic structures that have been observed in many natural processes, often under temporally periodic forcing. Near Hopf bifurcations, such systems can be formally reduced to a forced complex Ginzburg–Landau (CGL) equation, with oscillons then corresponding to stationary localized patterns. In this thesis, stationary localized structures of the planar 2:1 forced CGL are investigated analytically and numerically.

Four distinct localized solutions to the steady-state planar forced CGL are studied: localized offsets from both the trivial background state and a nontrivial background state, in each case with both monotone tails and oscillatory tails. The existence of both monotone solutions is proved analytically in regions where two spatial eigenvalues associated with the appropriate linearization of the one-dimensional CGL collide at zero.

The numerical study complements the analytical results away from onset. The numerical study also investigates the existence of localized solutions with oscillatory tails. One particular outcome is that planar oscillons with oscillatory tails can exist in parameter regions where one dimensional oscillons cannot.
This dissertation by Kelly McQuighan is accepted in its present form by The Division of Applied Mathematics as satisfying the dissertation requirement for the degree of Doctor of Philosophy.

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To my mother, a fierce advocate for women’s rights, who raised me in the belief that women and men are equally capable in mathematics.
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Chapter One

Introduction
Broadly speaking, this thesis is concerned with the mathematical study of pattern formation. The term “pattern” can refer to a wide variety of physical phenomena, such as stripes (as in zebra stripes) or spots (as in leopard spots). Patterns can be domain filling, as for animal coats, or localized as in desert vegetation spots and rings (see Figure 1.1). Patterns can be stationary in time, as in all of the previous examples, or time periodic, as with the rotating spirals observed in the Belousov-Zhabotinsky chemical reaction (see Figure 1.2). They can be found naturally and created experimentally. The mathematical field of pattern formation seeks to understand the similarities and differences between these various phenomena.

1.1 Oscillons in experiments

This thesis is concerned in particular with the mathematical study of oscillons: spatially localized, temporally oscillating structures that emerge from a uniform background state in parametrically forced nonlinear systems.

Oscillons were first observed in an experiment performed by Umbanhowar et al.
using granular media [55]. In that work, the researchers filled a 127 mm diameter circular disk with an approximately 3 mm thick layer of bronze spheres. The bronze spheres were approximately 0.15-0.18 mm in diameter, so that the layer was approximately 17 spheres thick. The circular disk was then placed on a vibrating table and subjected to vertical vibrations. Oscillating heaps of beads were seen to emerge for certain values of the forcing frequency and amplitude. These heaps were seen to oscillate between a heap and a crater at half the forcing frequency; see Figure 1.3. The oscillons were observed in a variety of arrangements; see Figure 1.4.

Oscillons were subsequently observed in a variety of experimental settings including Newtonian fluids [2, 47, 64, 65], chemical reactions [42, 58, 60], and colloidal suspensions [35]. In chemical reactions, oscillons have also been observed in autonomous systems [56] and in systems subjected to global feedback [59]. The frequency of the pattern oscillation is often close to the forcing frequency or half the forcing frequency,
and we refer to these patterns as 1:1 and 2:1 resonance oscillons, respectively. See Figure 1.5 for some examples.

1.2 Mathematical models of oscillons

A variety of models have been proposed to explain pattern formation in temporally forced systems. The experiments in granular media pose particular challenges for theorists: unlike fluid flow and chemical reactions, there do not exist any equations based on first principles that regulate the evolution of granular particles. While molecular-dynamics type simulations have been successful at reproducing experimental observations in granular media [6, 67] (see Figure 1.6), such simulations provide limited insight into the mechanism behind oscillon formation.

As a result, several continuum equations have been proposed as models of granular media [9, 17, 43]. In these models, it is often assumed that the granular layer can be modeled by a thin sheet at height \( h = h(x, y, t) \) and that the bottom of the sheet layer can be modeled by the motion of a single inelastic bouncing ball. For example, in [43], the vertical velocity of the sheet \( u = u(x, y, t) \), with \( h_t = u \), evolves
Figure 1.4: Top view of various arrangements of oscillons in a granular system. The system was illuminated from above so that the bright spots correspond with heaps and the dark spots correspond with craters. The time between the panels in Figures a-c is one forcing period. Figure taken from [55, Figure 3].

according to

\[ u_t = \nu \Delta u - g + B(h, u, s, \dot{s}, \alpha), \]

where \( s \) is the position of the vibrating table and the term \( B \) specifies how the sheet bounces when it hits the vibrating table and is defined implicitly by

\[ u \to \dot{s} + \alpha(\dot{s} - u) \quad \text{when} \quad h = s. \]

In the above, \( \nu \) is the viscosity, \( g \) is gravitational acceleration, and \( 0 < \alpha < 1 \) is the coefficient of restitution. Lastly, \( \alpha = \alpha(\rho) \) is a monotonically decreasing function of the density \( \rho \), where conservation of mass considerations imply that \( \rho \) evolves according to

\[ \rho_t = D[\Delta \rho - \nabla \cdot (\rho \nabla h/h_0)], \]

where \( D \) is a diffusion coefficient and \( h_0 \) is a characteristic height. Continuum equations have also been proposed to model fluids [7], chemical reactions [56, 57], and
Newtonian fluid. The left panel is the time sequence for a single oscillon. The right two panels represent two different configurations of oscillons at a single snapshot in time. Figure taken from [2, Figures 1 and 2].

Belousov–Zhabotinsky chemical reaction. Time between panels equals the forcing period. Figure taken from [58, Figure 2].

Colloidal suspension. The time sequence for three different configurations are shown: a single oscillon, a pair of oscillons, and an oscillon triad. Two full driving periods are shown for each sequence. Figure taken from [35, Figure 1].

Figure 1.5: Oscillons observed in other materials.

A third approach to modeling such systems has been to utilize a temporally discrete, but spatially continuous, evolution operator to account for the time periodicity of oscillons [30, 32, 62, 63]. Such approaches evolve a continuous field variable at time $n$, $\xi_n(x)$, forward to time $n + 1$ via some map $\mathcal{F}$:

$$\xi_{n+1}(x) = \mathcal{F}[\xi_n(x)].$$
Figure 1.6: Top view of molecular dynamics simulations versus experiment for oscillons in granular media. Patterns (a)-(e) oscillate at half the forcing frequency; patterns (f)-(h) oscillate at one quarter of the forcing frequency. Figure taken from [6, Figure 1].

For example, in [63], $\xi_n(x)$ represents the height field of the granular layer at time $n$, and the function $\mathcal{F}$ is given by

$$\mathcal{F} [\xi_n(x)] := \mathcal{L} [M [\xi_n(x), r]]$$

where

$$M(\xi, r) = re^{-(\xi-1)^2/2}$$

is a period doubling map depending on the parameter $r$, and

$$\mathcal{L} [\zeta_n] = f(x) * \zeta_n(x)$$
couples the dynamics of granular particles at nearby $x$ locations. The function $f(x)$ rapidly decays for $x \gg \lambda$, where $\lambda$ is the effective range over which the grains interact.

Another approach has been to use phenomenological amplitude equations that are designed to capture the essential features of pattern-forming systems. Such equations include variations of Ginzburg–Landau [13, 26, 54], Swift–Hohenberg [14, 25], and nonlinear Schrödinger equations [3, 27, 41]. For example, in [54] the proposed model

$$\begin{align*}
\partial_t A &= \gamma A - (1 - i \omega) A + (1 + i b) \Delta A - |A|^2 A - \rho A \\
\partial_t \rho &= \alpha \nabla \cdot (\rho \nabla |A|^2) + \beta \Delta \rho,
\end{align*} \tag{2.1}$$

 couples a Ginzburg–Landau-like equation for $A \in \mathbb{C}$, the local complex amplitude for particles oscillating at half the driving frequency, to an equation for the density $\rho$, derived from conservation of mass considerations. The equation for $A$, without the coupling term $\rho A$, is commonly used to model an oscillating liquid layer where: the linear terms can be derived from the dispersion relation for parametrically driven granular waves; $\gamma A$ provides the forcing; and $|A|^2 A$ accounts for the nonlinear saturation of oscillations due to restitution.

As another example of a phenomenological equation, in [14], a Swift–Hohenberg-like equation was proposed to show that oscillons are expected generally in systems which undergo a hysteretic transition to square patterns. The amplitude $A \in \mathbb{C}$ was modeled by

$$\partial_t A = RA - (A_x^2 + 1)^2 A + b A^3 - c A^5 + e \nabla \cdot [(\nabla A)^2] - \beta_1 A(\nabla A)^2 - \beta_2 A^2 \Delta A$$

where the nonlinear terms were chosen according to the following considerations: $A \in \mathbb{C}$ must satisfy the reflection symmetry $A \rightarrow -A$ so that only odd orders in $A$
are allowed; the usual Swift–Hohenberg equation must be extended to fifth order to obtain a subcritical bifurcation; and the term proportional to the parameter $e$ was chosen because it is known to favor square patterns.

In all the above model approaches, numerical computations revealed that these equations support the wealth of patterns observed in experiments.

Analytical results on oscillons are scarce. It was shown that a parametrically driven damped nonlinear Schrödinger equation in two and three spatial dimensions

$$iA_t + \Delta A + 2|A|^2 A - A - hA - i\gamma A,$$

proposed in [41] to model Newtonian fluids has two exact ring shaped oscillon-like solutions that are also stable [4]. In the above model, $\gamma$ and $h$ are parameters controlling the damping and driving forces respectively, and $A \in \mathbb{C}$ is again the amplitude for particle oscillations at half the driving frequency. We are aware of two analytical results in the dissipative regime. The coupled Ginzburg–Landau model equation (2.1), proposed in [54], was shown to support localized solutions in one spatial dimension [15]. Oscillons were also found, along with other experimentally observed 2:1 resonant patterns, as solutions to the complex Ginzburg–Landau (CGL) equation

$$A_t = (1+i\alpha)\Delta A + (-\mu + i\omega)A - (1+i\beta)|A|^2 A + \gamma\overline{A}, \quad A \in \mathbb{C} \quad (2.2)$$

in one spatial dimension [8]. Both works used a weakly nonlinear analysis.

The success of equation (2.2) to describe oscillons in one-spatial dimension provides the motivation for this thesis. In this thesis, we prove the existence of oscillon solutions to the planar 2:1 forced complex Ginzburg–Landau equation near onset.
Our analysis is complemented by numerical continuation of solutions away from the bifurcation point.

1.3 Outline

We conclude this chapter with an outline of the remainder of this thesis.

In Chapter 2 we review the following background information: we formally derive the 2:1 forced complex Ginzburg–Landau equation (2.2) as an amplitude equation for parametrically forced reaction–diffusion equations near a Hopf bifurcation; we summarize the results on the one-dimensional forced CGL from [8]; and we review the blow-up coordinate technique, which will be instrumental in our analysis.

As we explain in Chapter 2, it was shown in [8] that there exist four distinct oscillon solutions to the one-dimensional CGL. These different solutions are shown in Figure 2.4. In Chapters 3 and 4 we prove the existence of two of these oscillon types (namely, the ones displayed in Figures 2.4a and 2.4c) as radially symmetric solutions to the planar forced CGL. We take a dynamical systems approach and prove the existence of small amplitude solutions near the bifurcation point. The idea for the proof is the same for both oscillon types, so we explain the approach in more detail in Chapter 3 than in Chapter 4. We model our work after a similar analysis of the planar Swift–Hohenberg equation from [38].

In Chapter 5 we present some numerical results. We use the numerical continuation package AUTO07p to study the existence and behavior of large amplitude oscillon solutions, away from onset. We have numerical results on three of the four oscillon types.
Chapter Two

Background
We begin with some background. In Section 2.1 we justify our use of the 2:1 forced complex Ginzburg–Landau (CGL) equation as a model equation for oscillons. In particular, we use a multiple scales expansion to show, formally, that all small amplitude oscillatory solutions to a 2:1 forced reaction–diffusion system near a Hopf bifurcation can be captured by the 2:1 forced CGL. In Section 2.2 we review known results on the forced CGL in one spatial dimension.

One tool that we employ in our analysis of the planar 2:1 forced CGL is “blow-up coordinates”. In Section 2.3 we develop intuition about blow-up coordinates by analyzing a toy problem.

### 2.1 Derivation of the 2:1 forced complex Ginzburg–Landau equation

We illustrate the derivation of the forced CGL for general forced systems by studying this issue in the context of forced reaction–diffusion systems. Although reaction–diffusion systems will likely not be good models for granular media or non-Newtonian fluids, we expect a similar derivation to hold in those cases. We consider a periodically forced parameter dependent reaction–diffusion system

\[ u_t = \Delta u + f(u; \nu) + \Gamma e^{2i\Omega t}, \quad u \in \mathbb{R}^m, \tag{1.1} \]

with \( \Gamma, \Omega, \nu \in \mathbb{R} \). We assume \( f(0; \nu) = 0 \) for all \( \nu \) so that equation (1.1) supports the homogeneous rest solution \( u(x, t) \equiv 0 \). It has been argued, see for instance [13, 18],
that the 2:1 forced complex Ginzburg–Landau (CGL) equation

\[ A_T = (1 + \text{i}\alpha)\Delta A + (-\mu + \text{i}\omega)A - (1 + \text{i}\beta)|A|^2 A + \gamma A, \quad A \in \mathbb{C} \quad \text{(1.2)} \]

is a modulation equation for (1.1) near a supercritical Hopf bifurcation of the rest state \( u = 0 \). In other words, \( A(X,T) \) captures the “slow” dynamics of the envelope for a carrier wave evolving on the fast time scale \( t \); see Figure 2.1. If the slow and fast scales could be completely separated, then one could write

\[ u(t;X,T) = \epsilon A(X,T) e^{\text{i}\Omega t}, \quad \text{(1.3)} \]

with \( X := \epsilon x \) and \( T := \epsilon^2 t \). The 2:1-ratio between the oscillation frequency of the forcing in (1.1) to that of the ansatz (1.3) is consistent with experimental findings.

In equation (1.2), \( \alpha, \beta, \mu, \omega, \gamma \in \mathbb{R} \) are real parameters which can be obtained from the original equation (1.1). In this section we use a multiple scales analysis to formally derive (1.2) as the amplitude equation for (1.1). We remark that since the ansatz (1.3) explicitly factors out the time oscillations, oscillons correspond with localized steady state solutions to (1.2); i.e., spatially localized solutions to

\[ 0 = (1 + \text{i}\alpha)\Delta A + (-\mu + \text{i}\omega)A - (1 + \text{i}\beta)|A|^2 A + \gamma A, \quad A \in \mathbb{C}. \]
2.1.1 Hypotheses

We assume that the unforced reaction–diffusion system

$$u_t(x, t) = D \Delta u(x, t) + f(u(x, t); \nu), \quad u \in \mathbb{R}^m, \ x \in \mathbb{R}^n \quad (1.4)$$

supports a homogeneous rest solution $u(x, t) = u_0$ so that $f(u_0, \nu) = 0$ for all $\nu$; without loss of generality, we assume $u_0 = 0$. We assume that $f$ is smooth in both $u$ and $\nu$ so that we can Taylor expand

$$f(u, \nu) = f_u(0; 0)u + N_2[u, u] + N_3[u, u, u] + f_\nu(0; 0)\nu + f_{\nu u}(0; 0)\nu \nu + \ldots \quad (1.5)$$

where $N_2[\ldots]$ and $N_3[\ldots]$ are the appropriate bilinear and trilinear forms. In what follows, we consider only isotropic solutions to (1.4) so that we can take $n = 1$ without loss of generality.

We furthermore assume that $u = 0$ is a stable solution of (1.4) for $\nu > 0$ and that it undergoes a supercritical Hopf bifurcation as $\nu$ is decreased through zero. This last assumption is necessary so that equation (1.4) supports temporally oscillating solutions. It can be formalized as follows: Let $\mathcal{L}(\nu) := D \Delta + f_u(0; \nu)$, the linearization of the unforced reaction–diffusion equation (1.4) about $u = 0$, be a densely defined operator $L^2(\mathbb{R}; \mathbb{R}^m) \rightarrow L^2(\mathbb{R}; \mathbb{R}^m)$ for $\nu$ fixed, with $\text{dom}(\mathcal{L}(\nu)) = H^2(\mathbb{R}; \mathbb{R}^m)$. It is well known that the essential spectrum of $\mathcal{L}$ can be computed

$$\text{spec}(\mathcal{L}(\nu)) = \{ \lambda \in \mathbb{C}; d(\lambda, ik; \nu) = 0 \text{ for some } k \in \mathbb{R} \}$$
Figure 2.2: Essential spectrum for $\mathcal{L}$, the unforced reaction–diffusion equation, linearized about $u = 0$. We assume that the essential spectrum of $\mathcal{L}$ crosses the imaginary axis with nonzero speed as $\nu$ is decreased through zero.

where

$$d(\lambda, \Theta; \nu) := \det[\Theta^2 D + f_u(0; \nu) - \lambda], \quad \Theta \in \mathbb{C} \quad (1.6)$$

is the dispersion relation associated with $\partial_t u = \mathcal{L}u$. We define $\lambda = \lambda_{\pm}(k; \nu)$ to be the curves which satisfy $d(\lambda_{\pm}(k; \nu), ik; \nu) = 0$. Then (1.4) undergoes a Hopf bifurcation precisely when the curves $\lambda_{\pm}(0; \nu)$ cross the imaginary axis with nonzero speed at $\lambda = \pm i\omega_0$, $\omega_0 \neq 0$, as $\nu$ is decreased through zero. See Figure 2.2. More precisely, we assume Hypothesis 2.1.1.

**Hypothesis 2.1.1.** The dispersion relation (1.6) satisfies

$$d(i\omega_0, 0; 0) = 0, \quad \Re \left( \frac{d\Theta}{d\lambda}(i\omega_0, 0; 0) \right) < 0$$

$$\Re \left( \frac{d\nu}{d\lambda}(i\omega_0, 0; 0) \right) > 0$$

and $d(i\omega, ik; 0) \neq 0$ for $(\omega, k) \neq (\pm \omega_0, 0)$.

In other words, we hypothesize that the Taylor expansion of the dispersion rela-
d(\lambda, \Theta; \nu) = d(i\omega_0, 0; 0) + d_\lambda(i\omega_0, 0; 0)(\lambda - i\omega_0) + d_{\Theta\Theta}(i\omega_0, 0; 0)\Theta^2
+ d_\nu(i\omega_0, 0; 0)\nu + O((\lambda - i\omega_0)^2 + \Theta^4 + \nu^2)

is of the form

\[ d(\lambda, \Theta; \nu) = c_1 \left( (\lambda - i\omega_0) + \tilde{c}_2 \Theta^2 + \tilde{c}_3 \nu + O((\lambda - i\omega_0)^2 + \Theta^4 + \nu^2) \right) \]

with \( \text{Re}(\tilde{c}_2) := c_2/c_1 < 0 \) and \( \text{Re}(\tilde{c}_3) := c_3/c_1 > 0 \). Thus

\[ d(\lambda, ik; \nu) = c_1 \left( (\lambda - i\omega_0) - \tilde{c}_2 k^2 + \tilde{c}_3 \nu + O((\lambda - i\omega_0)^2 + \Theta^4 + \nu^2) \right). \tag{1.7} \]

This expansion shows that \( d(i\omega_0, 0; 0) = 0 \) and \( (\partial_\lambda d)(i\omega_0, 0; 0) = c_1 \neq 0 \) so that the Implicit Function Theorem (see Appendix A.4) applies. Thus, there exists a function \( \lambda_+(k; \nu) \) of the form

\[ \lambda_+(k; \nu) = i\omega_0 + \tilde{c}_2 k^2 - \tilde{c}_3 \nu + O(k^4 + \nu^2) \tag{1.8} \]

such that \( d(\lambda_+(k; \nu), ik; \nu) = 0 \) for all \( k, \nu \) small enough. It is now straightforward to check that

\[ \lambda_+(0; 0) = i\omega_0, \]
\[ \text{Re} \lambda_+(k; 0) < 0 \text{ for } k > 0, \]
\[ \text{and } \text{sgn}(\text{Re} \lambda_+(0; \nu)) = -\text{sgn}(\nu), \]

as desired and as is shown in Figure 2.2. Then \( \lambda_-(k; \nu) = \overline{\lambda_+(k; \nu)} \), as is also shown in Figure 2.2, since the coefficients of operator \( \mathcal{L} \) are real.
As a consequence of Hypothesis 2.1.1, the kinetic equation

\[ u_t = f(u; \mu) \]  \hspace{1cm} (1.9)

undergoes a Hopf bifurcation at \((u; \mu) = (0, 0)\) and the dynamics of (1.9) near \(u \equiv 0\) can be reduced to a two-dimensional center manifold. We assume that the Hopf bifurcation of (1.9) is supercritical. More precisely, we assume Hypothesis 2.1.2.

**Hypothesis 2.1.2.** Under an appropriate scaling of \(|A|\), \(A \in \mathbb{C}\), the normal form of the kinetic equation \(u_t = f(u; \nu)\) on the center manifold at \(\nu = 0\) is given, up to cubic order, by

\[ A_t = i\omega_0 A - (1 + i\beta)|A|^2 A, \quad \beta \in \mathbb{R}. \]

We review the rationale behind Hypothesis 2.1.2. It is well known that the normal form near a Hopf bifurcation with characteristic frequency \(\omega_0\) is, for all sufficiently small \(\nu\),

\[ \dot{z} = \lambda z + (c_1 + ic_2)|z|^2z + O(|z|^4) \quad z \in \mathbb{C} \]  \hspace{1cm} (1.10)

where

\[ \lambda = \lambda(\nu) = \alpha(\nu) + i\omega(\nu) \]

with \(\alpha(0) = 0, \omega(0) = \omega_0, \text{sgn}(\alpha(\nu)) = -\text{sgn}(\nu)\), and \(\alpha, \omega, c_1, c_2 \in \mathbb{R}\) [33, Lemma 3.3, 3.6]. By writing \(z(t) = r(t)e^{i\theta(t)}\) in polar form we see that (1.10) is equivalent
(a) Supercritical Hopf bifurcation, $c_1 < 0$.  
(b) Subcritical Hopf bifurcation, $c_1 > 0$.

**Figure 2.3:** Supercritical versus subcritical Hopf bifurcation. Plotted is the radius of the cycle $r$ versus the parameter $\nu$. The zero solution exists for all $\nu$. A limit cycle with $r > 0$ bifurcates at $\nu = 0$; it exists for either positive or negative $\nu$, depending on the sign on $c_1$. Solid lines represent stable solutions whereas dotted lines represent unstable solutions.

to

\[
\dot{r} = \alpha(\nu) r + c_1 |r|^2 r + O(|r|^4)
\]
\[
\dot{\phi} = \omega(\nu) + c_2 |r|^2 + O(|r|^3).
\]

It is now easy to check that a limit cycle with $\dot{r} = 0$ exists provided

\[
|r|^2 \approx -\frac{\alpha(\nu)}{c_1} > 0.
\]

Thus, with $c_1 < 0$ the cycle exists for $\nu < 0$, whereas with $c_1 > 0$ the cycle exists for $\nu > 0$; see Figure 2.3. Since the trivial state destabilizes for $\nu < 0$, this means that the cycle is stable with $c_1 < 0$, in which case we say that the trivial state underwent a **supercritical bifurcation**, and that the limit cycle is unstable for $c_1 > 0$, in which case we say that the trivial state underwent a **subcritical bifurcation**. Thus, by Hypothesis 2.1.2, there exists a stable limit cycle to the kinetic equation $u_t = f(u; \nu)$ for $\nu < 0$.

Lastly, we assume $\lambda = \pm i\omega_0$ are simple eigenvalues for $f_u(0; 0)v_0 = \lambda v_0$:

**Hypothesis 2.1.3.** There is a unique, up to scalar multiples, nonzero eigenvector $v_0 \in \mathbb{C}^m$ satisfying $f_u(0; 0)v_0 = \lambda v_0$ with $\lambda = i\omega_0$. Thus, $\overline{v}_0$ is the unique nonzero eigenvector satisfying $f_u(0; 0)\overline{v}_0 = -\overline{\lambda} \overline{v}_0$ with $\overline{\lambda} = -i\omega_0$. 

Remark 2.1.4. Hypothesis 2.1.3 implies that there exists a unique, up to scalar multiples, nonzero eigenvector $w_0 \in \mathbb{C}^m$ satisfying $f u(0; 0)^T w_0 = \lambda w_0$ with $\lambda = i \omega$ (and hence $w_0$ is the unique nonzero eigenvector satisfying $f u(0; 0) w_0 = \bar{\lambda} w_0$ with $\bar{\lambda} = -i \omega$). Furthermore, $w_0$ can be normalized so that $v_0 \cdot w_0 = 1$. This is a direct application of Proposition 2.1.5 below since $\lambda = i \omega$ is a simple eigenvalue.

Proposition 2.1.5. Let $A \in \mathbb{R}^{n \times n}$ be a real square matrix. Then

(i) $\lambda$ is an eigenvalue for $A$ if, and only if, it is also an eigenvalue for $A^T$.

Let $\{\lambda_j\}_{j=1}^k$, $k \leq n$, be the collection of all eigenvalues for $A$, counted with geometric multiplicity. Also let $\{v^r_j\}_{j=1}^k$ and $\{w^r_j\}_{j=1}^k$ be the associated right eigenvectors for matrices $A$ and $A^T$, respectively; similarly, let $\{v^l_j\}_{j=1}^k$ and $\{w^l_j\}_{j=1}^k$ be the associated left eigenvectors. We assume that all vectors are normalized. Finally, let $\lambda_\ell$ be a simple eigenvalue for $\ell \in \{1, \ldots, k\}$, so that the algebraic multiplicity is 1. Then,

(ii) $v^r_\ell$ is the unique (up to scalar multiples) right eigenvector for $A$ if, and only if, $(v^r_\ell)^T$ is the unique (up to scalar multiples) left eigenvector for $A^T$, both associated with eigenvalue $\lambda_\ell$; and

(iii) $v^r_i \cdot w^r_\ell = \delta_{i\ell}$, including for eigenvectors $v_i$ associated with non-simple eigenvalues $\lambda_i$.

Furthermore, if $v^r_{i,m}$ is a generalized eigenvector associated with a non-simple eigenvalue $\lambda_i$, $m \in \{1, \ldots, p_i\}$ (where we define $v_{i,0} := v_i$), then

(iv) $v^r_{i,k} \cdot w^r_\ell = 0$. 

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Proof. (i) $\lambda \in \mathbb{C}$ is an eigenvalue for $A$ if, and only if,

$$0 = \det(A - \lambda I).$$

But since $\det(A) = \det(A^T)$, this implies

$$0 = \det((A - \lambda I)^T) = \det(A^T - \lambda I).$$

(ii) For any right eigenvector $v^r_j$ of $A$,

$$(v^r_j)^T A^T = \lambda j (v^r_j)^T$$

by taking the transpose of both sides. Thus $(v^r_j)^T$ is the left eigenvector of $A$. The converse follows from taking the transpose of $v^l_j A^T = \lambda j v^l_j$. The statement follows for $j = \ell$ since $v^r_\ell$ is unique.

(iii) Let $V_R \in \mathbb{C}^{n \times n}$ be the matrix whose columns consist of the eigenvectors and generalized eigenvectors for $A$

$$\{v^r_{j,m}\}_{j=1,...,k, \, m=0,...,p_j}$$

and let $J \in \mathbb{C}^{n \times n}$ be the Jordan normal form of $A$, ordered so that

$$AW_R = W_R J \quad \text{or, equivalently,} \quad W_R^{-1} A = JW_R^{-1}. \quad (1.11)$$

In general $J$ is not diagonal. However, if $v_i$ is an eigenvector associated with eigenvalue $\lambda_i$, then there exists a $k$ so that $J_{ik} = \lambda_i \delta_{ik}$ and $J_{kj} = \lambda_k \delta_{kj}$, where $J_{ij}$ denotes the entry in the $i$th row and $j$th column of matrix $J$. Let $v_\ell$ be the unique eigenvector
associated with the simple eigenvalue $\lambda_\ell$. It is then straightforward to observe from (1.11) that $v^*_\ell$ is the $k$th column of $W_R$ and $v^t_\ell$ is the $k$th row of $W_R^{-1}$, for some $k$. Thus, $v^*_\ell v^t_\ell = (v^*_\ell)^T (v^t_\ell)^T = \delta_{\ell t}$, even for non-simple eigenvalues $\lambda_i$. But by Property (ii) we have $(v^l_i)^T = w^*_i$ so that $v^*_i \cdot w^*_i = \delta_{i\ell}$.

(iv) Since $\lambda_\ell$ is simple, $\ell \neq i$ since otherwise there would not be any generalized eigenvectors by (i). Then $v^r_{i,m} \cdot w^r_\ell = 0$ follows from the same reasoning as in (iii) since the generalized eigenvectors $v^r_{i,m}$ are included in the matrix $W_R$.  

Under Hypotheses 2.1.1, 2.1.2, and 2.1.3 we add near 2:1 resonant forcing to (1.4) for $\nu =: \epsilon^2 \hat{\mu}$ small

$$u_t(x, t) = D\Delta u(x, t) + f(u(x, t), \epsilon^2 \hat{\mu}) + \epsilon^2 \hat{\gamma} v e^{2i(\omega_0 - \epsilon^2 \hat{\omega}) t}. \quad (1.12)$$

The parameters $\hat{\mu}$, $\hat{\gamma}$ and $\hat{\omega}$ are real and bounded. The vector $v \in \mathbb{R}^m$ allows the amplitude of forcing to be different for each system variable. The scalings $\nu = \epsilon^2 \hat{\mu}$, $\epsilon^2 \hat{\gamma}$, and $\epsilon^2 \hat{\omega}$ are necessary for consistency in the scaling. One does not need to assume these scalings a priori: they arise naturally in the analysis below. However, we skip this part of the argument so as to emphasize the most relevant aspects of the analysis below.

2.1.2 Multiple scales expansion

We use a multiple scales expansion to model the evolution of small amplitude solutions to (1.12). This idea is explained further in, for example, [28, Chapter 7]. We define $X := \epsilon x$ and $T := \epsilon^2 t$. As shown in Figure 2.1, $X$ and $T$ capture the solution envelope as it evolves over the “long” space and “slow” time scales. We assume that
the evolution of $u$ on each time or space scale is independent of the other scales, i.e., $u = (x, t; X, T) = a(x, t)A(X, T)$. Since $X$ and $T$ depend implicitly on $x$ and $t$, respectively, the derivatives in (1.12) become

$$
\partial_t u(x, t; X, T) \mapsto \frac{\partial u}{\partial t} + \epsilon^2 \frac{\partial u}{\partial T} =: u_t + \epsilon^2 u_T \tag{1.13a}
$$

and

$$
\Delta u(x, t; X, T) \mapsto \frac{\partial^2 u}{\partial x^2} + \epsilon^2 \frac{\partial^2 u}{\partial X^2} =: \Delta_x u + \epsilon^2 \Delta_X u. \tag{1.13b}
$$

By Hypothesis 2.1.1, on the short space scale $\Delta_x u = 0$; we therefore write $u = u(t; X, T)$ and the spatial Laplacian operator $\Delta$ becomes

$$
\Delta u(t; X, T) \mapsto \epsilon^2 \Delta_X u. \tag{1.13b}
$$

One can see from the derivatives (1.13) that the relative scaling of $X$ and $T$ ensures that both slow variables affect $u$ over the same scale $O(1/\epsilon^2)$.

**Remark 2.1.6.** The analysis below is only valid when the long and short scales can be completely separated. See Remark 2.1.9. This assumption generally does not hold.

We plug the Taylor expansion (1.5) and derivatives (1.13) into equation (1.12) to get

$$
u_t - f_u(0; 0)u = -\epsilon^2 u_T + \epsilon^2 D\Delta_X u + N_2[u, u] + N_3[u, u, u] + \epsilon^2 f_{uv}(0; 0)u \hat{\mu} + \ldots + \epsilon^2 \gamma e^{2i\omega t}e^{-2i\omega T}. \tag{1.14}$$
Next, we expand solutions to (1.14) near \( u = 0 \)

\[
u(t; X, T) = \epsilon u_1(t; X, T) + \epsilon^2 u_2(t; X, T) + \epsilon^3 u_3(t; X, T) + c.c + O(\epsilon^4). \tag{1.15}\]

We match terms in (1.14) at successively higher orders in \( \epsilon \).

\[
\begin{align*}
O(\epsilon) : & \quad \mathcal{L}_0 u_1 = 0 \\
O(\epsilon^2) : & \quad \mathcal{L}_0 u_2 = N_2 [u_1, u_1] + \tilde{\gamma} v e^{2i\omega_0 t} e^{-2i\omega T} \\
O(\epsilon^3) : & \quad \mathcal{L}_0 u_3 = -\partial_T u_1 + D\Delta_X u_1 + 2N_2 [u_1, u_2] + N_3 [u_1, u_1, u_1] + \tilde{\mu} f_{u\nu}^0 u_1 
\end{align*} \tag{1.16a-1.16c}
\]

where \( \mathcal{L}_0 := \partial_t - f_u(0; 0) \) is defined as a map

\[
\mathcal{L}_0 : L^2([0, 2\pi/\omega_0]/\sim) \to L^2([0, 2\pi/\omega_0]/\sim),
\]

with \( \text{dom}(\mathcal{L}_0) = H^1([0, \frac{2\pi}{\omega_0}]/\sim) \). We remark that by \( L^2([0, \frac{2\pi}{\omega_0}]/\sim) \)
(resp. \( H^1([0, \frac{2\pi}{\omega_0}]/\sim) \)) we mean the set of square integrable (resp. \( H^1 \)) functions \( u(t; X, T) \) which are \( 2\pi/\omega_0 \) periodic in \( t \) and where the variables \( X \) and \( T \) are defined
on all of \( \mathbb{R} \); this definition of the spaces is consistent with our assumption that the
short scale \( t \) can be completely separated from the long scales \( X \) and \( T \). In equations
(1.16), the expressions \( f_{\nu}^0 \) and \( f_{u\nu}^0 \) are shorthand for the Taylor expansion vector
\( f_{\nu}(0; 0) \) and matrix \( f_{u\nu}(0; 0) \), respectively.

**Matching at \( O(\epsilon) \) and \( O(\epsilon^2) \)**

We first solve (1.16a). By Hypothesis 2.1.3 the only nontrivial solutions to \( \mathcal{L}_0 v(t) = 0 \)
are linear combinations of \( e^{i\omega_0 t} v_0 \) and \( e^{-i\omega_0 t} \overline{v_0} \). Hence, \( u_1 = A_1(X, T) e^{i\omega_0 t} v_0 + c.c. \)

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with $A_1(X, T) \in \mathbb{C}$.

Next consider the $O(\epsilon^2)$ terms. We substitute the expression for $u_1$ into (1.16b). Since $N_2[u_1, u_1]$ on the fast time-scale consists of constants and $e^{\pm 2i\omega_0 t}$ terms, $u_2$ must be of the form $u_2 = u_{2,0} + u_{2,1} e^{i\omega_0 t} + u_{2,2} e^{2i\omega_0 t} + c.c.$, where $u_{2,j} := u_{2,j}(X, T)$. Thus, solving (1.16b) reduces to solving

$$f_u^0 u_{2,0} = -2|A_1|^2 N_2[v_0, \bar{v}_0]$$

$$(i\omega_0 I - f_u^0) u_{2,1} = 0$$

$$(2i\omega_0 I - f_u^0) u_{2,2} = A_1^2 N_2[v_0, v_0] + \hat{\gamma} v e^{-2i\omega_0 T}. \quad (1.17)$$

As with the $O(\epsilon)$ terms we can solve the second equation $u_{2,1} = A_{2,1}(X, T)v_0$. We further assume $\det(f_u^0) \neq 0$ and $\det(2i\omega_0 I - f_u^0) \neq 0$ so that we can solve for $u_{2,0}$ and $u_{2,2}$:

$$u_{2,0} = (f_u^0)^{-1} (-2|A_1|^2 N_2[v_0, \bar{v}_0])$$

$$u_{2,2} = (2i\omega_0 I - f_u^0)^{-1} (A_1^2 N_2[v_0, v_0] + \hat{\gamma} v e^{-2i\omega_0 T}). \quad (1.18)$$
The Fredholm Alternative and matching at $O(\epsilon^3)$

Finally, we solve (1.16c), using the solutions $u_1$ and $u_2$ found above.

$$L_0u_3 = -\partial T A_1 e^{i\omega_0 t} v_0 + \Delta_X A_1 e^{i\omega_0 t} D v_0 + \tilde{\mu} A_1 e^{i\omega_0 t} f^0_{u_0}$$

$$+ 2 \left[ A_1 N_2 [v_0, u_{2.1}] + e^{i\omega_0 t} (A_1 N_2 [v_0, u_{2.0}] + \overline{A_1 N_2 [v_0, u_{2.2}])} \right]$$

$$+ e^{2i\omega_0 t} A_1 N_2 [v_0, u_{2.1}] + e^{3i\omega_0 t} A_1 N_2 [v_0, u_{2.2}] \right] + 3|A_1|^2 A_1 e^{i\omega_0 t} N_3 [v_0, v_0]$$

$$+ A_4^3 e^{3i\omega_0 t} N_3 [v_0, v_0] + c.c.$$

$$=: f_3(u_1, u_2) \quad (1.19)$$

Since $L_0$ is not invertible, equation (1.19) may not be solvable for $u_3$, depending on $A_1(X, T)$. We use the following form of the Fredholm Alternative.

**Definition 2.1.7** (Fredholm Operator). [68, §5.4] Let $X$ and $Y$ be Hilbert spaces over $\mathbb{K}$, where $\mathbb{K}$ is some field. We say that a continuous linear operator $L$ is a **Fredholm operator** if ran $L$ is closed and if both dim ker $L < \infty$ and codim ran $L < \infty$. We then define the **Fredholm index**

$$\text{ind}(L) := \text{dim ker } L - \text{codim ran } L < \infty.$$

If an operator $L : X \to Y$ is Fredholm, then so is its adjoint $L^* : Y \to X$, where $L^*$ is defined so that $\langle Lu, v \rangle_Y = \langle u, L^* v \rangle_X$ for all $u \in X$ and $v \in Y$. Furthermore, $\text{ind } L^* = - \text{ind } L$.

**Theorem 2.1** (Fredholm Alternative). [68, Theorem 5.G]. Let $X$ and $Y$ be Hilbert spaces over some field $\mathbb{K}$. Suppose that the continuous linear operator $L : X \to Y$ is Fredholm. Then, for each $b \in Y$, the equation $Lu = b$ has a solution for some $u \in X$. 

25
if, and only if, $\langle b, v \rangle_Y = 0$ for all solutions $v$ of the adjoint equation $L^* v = 0$.

The Fredholm Alternative can be thought of as the infinite-dimensional version of the Fundamental Theorem of Linear Algebra, which we will use in the proof of Proposition 2.1.8.

**Theorem 2.2** (Fundamental Theorem of Linear Algebra). [50, Theorem 4.B] Let $A \in \mathbb{C}^{n \times n}$ be a matrix. Then

$$\text{ran}(A) \perp \ker(A^*)$$

where $A^*$ is the conjugate transpose of $A$. In other words, there exists a vector $x$ satisfying $Ax = b$ if, and only if, $\langle b, y \rangle = 0$ for all $y$ such that $A^* y = 0$.

**Proposition 2.1.8.** $L_0$ is Fredholm with index $\text{ind} L_0 = 0$ when considered as an operator

$$L_0 : L^2 ([0, 2\pi/\omega_0] / \sim) \to L^2 ([0, 2\pi/\omega_0] / \sim)$$

for functions of the form $u(t; X, T) = a(t) A(X, T)$, with dom $L_0 = H^1 ([0, 2\pi/\omega_0] / \sim)$.

**Proof.** Let

$$u(t; X, T) = a(t) A(X, T) v \in H^1 ([0, 2\pi/\omega_0] / \sim),$$

with $v \in \mathbb{R}^m$. Since $\{ e^{i\omega_0 n t} \}_{n \in \mathbb{Z}}$ forms a complete basis for periodic functions, we can expand

$$a(t) v = \sum_{n \in \mathbb{Z}} a_n e^{i\omega_0 n t} \quad (1.20)$$
with $a_n \in \mathbb{C}^m$. Because
\[
||\partial_t a(t)||^2_{L^2([0,2\pi/\omega_0]/\sim)} = \int_0^{2\pi/\omega_0} \left| \sum_{n \in \mathbb{Z}} i\omega_0 n a_n e^{i\omega_0 nt} \right|^2 dt = \omega_0^2 \sum_{n \in \mathbb{Z}} n^2 |a_n|^2,
\]
we must have
\[
\sum_{n \in \mathbb{Z}} n^2 |a_n|^2 < \infty
\]
so that $u(t; X, T) \in H^1([0,2\pi/\omega_0]/\sim)$. It is now straightforward to compute the action of $\mathcal{L}_0$ on $u(t; X, T)$:
\[
\mathcal{L}_0 u(t; X, T) = A(X, T) (\partial_t - f_0^0) \sum_{n \in \mathbb{Z}} a_n e^{i\omega_0 nt} \\
= A(X, T) \sum_{n \in \mathbb{Z}} (i\omega_0 n I - f_0^0) a_n e^{i\omega_0 nt}.
\]
There are three properties to verify:

(i) $\operatorname{dim} \ker \mathcal{L}_0 < \infty$ (in fact, $\operatorname{dim} \ker \mathcal{L}_0 = 2$);

(ii) $\operatorname{codim} \operatorname{ran} \mathcal{L}_0 < \infty$ (in fact, $\operatorname{codim} \operatorname{ran} \mathcal{L}_0 = 2$); and

(iii) $\operatorname{ran} \mathcal{L}_0$ is closed.

(i) The last condition of Hypothesis 2.1.1 states
\[
\det(-Dk^2 + f_0^0 - i\omega I) \neq 0 \quad \text{for} \quad (\omega, k) \neq (\pm \omega_0, 0),
\]
so that $\mathcal{L}_0(a_n e^{i\omega_0 nt}) \neq 0$ for all $a_n$ provided $n \neq \pm 1$. Furthermore, by Hypothesis 2.1.3, for $n = \pm 1$, $\mathcal{L}_0(a_1 e^{i\omega_0 t}) = 0$ if, and only if, $a_1 = \alpha v_0$ for some constant scalar $\alpha$; similarly, $\mathcal{L}_0(a_{-1} e^{-i\omega_0 t}) = 0$ if, and only if, $a_{-1} = \alpha \overline{v}_0$. Thus,
\[
\ker \mathcal{L}_0 = \operatorname{span}\{v_0 e^{i\omega_0 t}, \overline{v}_0 e^{-i\omega_0 t}\}.
\]
(ii) By the same hypotheses, \((i\omega_0 nI - f_0^0)\) is invertible if, and only if, \(n \neq \pm 1\). Thus, for any function of the form

\[ h(t) = h_n e^{i\omega_0 nt}, \]

the equation \(L_0 g(t) = h(t)\) is solvable for \(g(t)\) with

\[ g(t) = (i\omega_0 nI - f_0^0)^{-1} h_n e^{i\omega_0 nt} \]

provided \(n \neq \pm 1\). For \(n = 1\), by the Fundamental Theorem of Linear Algebra, \(h_1 \in \text{ran}(i\omega_0 I - f_0^0)\) if, and only if, \(h_1 \cdot \overline{y} = 0\) for all \(y \in \ker(-i\omega_0 I - (f_0^0)^T)\), where, by Remark 2.1.4,

\[ \ker(-i\omega_0 I - (f_0^0)^T) = \text{span} \{\overline{w}_0\}. \]

Next we expand an arbitrary vector \(v\) in terms of the generalized eigenvectors \(\{v_j\}_{j=0}^{m-1}\)

\(\text{for } (i\omega_0 I - f_0^0)\)

\[ v = \sum_{j=0}^{m-1} c_j v_j \]

(possible since \(\{v_j\}_{j=0}^{m-1}\) spans \(\mathbb{C}^m\)), where the eigenvectors are ordered so that \(v_0\) is the unique eigenvector associated with the simple eigenvalue \(\lambda_0 = 0\). By Proposition 2.1.5 we have that \(v_j \cdot w_0 = \delta_{0j}\), which shows that \(\text{ran}(i\omega_0 I - f_0^0) = v_0^\perp\). A completely analogous argument shows that \(\text{ran}(-i\omega_0 I - f_0^0) = \overline{v}_0^\perp\). Therefore,

\[ \text{ran } L_0 = L^2([0, 2\pi/\omega_0]/\sim) \setminus \text{span } \{v_0 e^{i\omega_0 t}, \overline{v}_0 e^{-i\omega_0 t}\}. \]

(iii) It is well known that \(L^2([0, 2\pi/\omega_0]/\sim)\) is a Banach space (see, for example, [19, Theorem 6.6]), and hence closed. Let \(\{h^m\}\) be a sequence of functions with
\( h^m \in \text{ran} \mathcal{L}_0 \) for every \( m \). Thus,

\[
    h^m(t) = \sum_{n \in \mathbb{Z}} h_n^m e^{i\omega_0 nt} \quad \text{with} \quad h_1^m \in v_0^\perp \text{ and } h_{-1}^m \in \overline{v_0}^\perp.
\]

Consider an arbitrary function \( g(t) = c_1 v_0 e^{i\omega_0 t} + c_2 \overline{v_0} e^{-i\omega_0 t} \). Then, for every \( m \),

\[
    ||g(t) - h^m(t)||_{L^2([0, \frac{2\pi}{\omega_0})/\sim)} = \int_0^{2\pi/\omega_0} \left| c_1 v_0 e^{i\omega_0 t} + c_2 \overline{v_0} e^{-i\omega_0 t} - \sum_{n \in \mathbb{Z}} h_n^m e^{i\omega_0 nt} \right|^2 dt
    = \int_0^{2\pi/\omega_0} \left( |c_1 v_0 - h_1^m|^2 + |c_2 \overline{v_0} - h_{-1}^m|^2 \right) dt
    = \int_0^{2\pi/\omega_0} \left( (c_1^2 + c_2^2)|v_0|^2 + |h_1^m|^2 + |h_{-1}^m|^2 \right) dt.
\]

where the last line follows by the Pythagorean Theorem (see, for example, [19, Theorem 5.23]) since \( v_0 \perp h_1^m \) and \( \overline{v_0} \perp h_{-1}^m \). Thus, for all \( m \),

\[
    ||g(t) - h^m(t)||_{L^2([0, \frac{2\pi}{\omega_0})/\sim)} \geq 2\pi(c_1^2 + c_2^2)|v_0|^2 / \omega_0
\]

so that there exists a sequence of function \( \{h^m(t)\} \) with

\[
    h^m(t) \xrightarrow{m \to \infty} (c_1 v_0 e^{i\omega_0 t} + c_2 \overline{v_0} e^{-i\omega_0 t})
\]

if, and only if, \( c_1 = c_2 = 0 \).

**Remark 2.1.9.** If the time scales can not be completely separated, then the operator \( \mathcal{L}_0 \) is no longer Fredholm. This is related to the fact that \( \text{ran} \partial_t \) is not closed when \( \partial_t \) is considered as an operator \( \partial_t : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \), with \( \text{dom} \partial_t = H^1(\mathbb{R}) \). This is shown through the following counter-example, taken from [44]: Let \( h(t) \in L^2(\mathbb{R}) \) be any odd, continuous, bounded, square integrable function with

\[
    h(t) = \frac{1}{t^{3/2}} \quad |t| \geq 1.
\]
The function \( h(t) \) is not in the domain of \( \partial_t \) since \( \partial_t g(t) = h(t) \) is solved by

\[
g(t) = a + 2 - \frac{2}{\sqrt{t}} \quad t \geq 1
\]

for any \( a \in \mathbb{R} \) and \( g(t) \notin H^1(\mathbb{R}) \). However, \( h(t) \) can be approximated arbitrarily closely by a sequence of functions \( h_n(t) \) that are in the domain of \( \partial_t \). In fact, defining

\[
h_n(t) := \begin{cases} 
\frac{1}{t^{2/n}} & t \geq 1 \\
h(x) & |t| \leq 1
\end{cases}
\]

with \( h(t) = -h(-t) \) and

\[
g_n(t) = a + \frac{2}{1 + \frac{2}{n}} \left(1 - t^{-\left(\frac{1}{2} + \frac{1}{n}\right)}\right) \quad t \geq 1,
\]

one can easily check that \( g_n(t) \in H^1(\mathbb{R}) \), \( h_n(t) \in L^2(\mathbb{R}) \), \( g_n \to g \), and \( \partial_t g_n = h_n \to h \).

We proceed applying the Fredholm Alternative solvability condition to \( L_0 u = f_3(u_1, u_2) \). The space \( L^2([0, \frac{2\pi}{\omega_0}] / \sim) \) is equipped with the periodic inner product

\[
\langle u, v \rangle = \int_0^{2\pi/\omega_0} u \overline{v} dt.
\]
Thus, we seek \( f_3(u_1, u_2) \) so that

\[
\int_0^{2\pi/\omega_0} f_3(u_1, u_2) w(t) dt = 0, \tag{1.21}
\]

where \( w(t) \) solves \( L_0^* w(t) = 0 \), where \( L_0^* = -\partial_t - f_u(0; 0) \). By Remark 2.1.4, \( w(t) = \)
\( e^{i\omega_0 t} w_0 + c.c. \) We recall from (1.18) and (1.19) that

\[
\begin{align*}
f_3(u_1, u_2) := & -\partial_T A_1 e^{i\omega_0 t} v_0 + \Delta_X A_1 e^{i\omega_0 t} D v_0 + \tilde{\mu} A_1 e^{i\omega_0 t} f^0_{u^\omega} v_0 \\
& + 2 \left[ A_1 N_2[v_0, \bar{u}_{2,1}] + e^{i\omega_0 t} \left( A_1 N_2[v_0, u_{2,0}] + \bar{A}_1 N_2[v_0, u_{2,2}] \right) \right] \\
& + e^{2i\omega_0 t} A_1 N_2[v_0, u_{2,1}] + e^{3i\omega_0 t} A_1 N_2[v_0, u_{2,2}] \right] + 3 |A_1|^2 A_1 e^{i\omega_0 t} N_3[v_0, v_0, \bar{v}_0] \\
& + A_1^3 e^{3i\omega_0 t} N_3[v_0, v_0, v_0] + c.c.
\end{align*}
\]

and

\[
\begin{align*}
u_{2,0} &= (f^0_{u})^{-1} \left( -2 |A_1|^2 N_2[v_0, \bar{v}_0] \right) \\
u_{2,2} &= (2i\omega_0 I - f^0_{u})^{-1} \left( A_1^2 N_2[v_0, v_0] + \bar{\gamma} v e^{-2i\omega T} \right).
\end{align*}
\]

Since the limits of the \( L^2 \) integral in (1.21) go from 0 to \( 2\pi/\omega_0 \), any terms in the integrand

\[
f_3(u_1, u_2) \left( e^{-i\omega_0 t} w_0 + c.c. \right)
\]

of the form \( e^{im\omega_0 t}, m \neq 0 \), will automatically integrate to zero. The remaining \( m = 0 \) terms enforce the condition

\[
\partial_T A_1 v_0 \cdot w_0
\]

\[
= \left\{ \Delta_X A_1 D v_0 + \tilde{\mu} A_1 f^0_{u^\omega} v_0 \\
+ 2A_1 N_2[v_0, u_{2,0}] + 2\bar{A}_1 N_2[v_0, u_{2,2}] + 3 |A_1|^2 A_1 N_3[v_0, v_0, \bar{v}_0] \right\} \cdot w_0. \tag{1.22}
\]

By substituting the expressions for \( u_{2,0} \) and \( u_{2,2} \) into equation (1.22), we get the
condition

\[ \partial_T A_1 v_0 \cdot w_0 = \left\{ \Delta_X A_1 D v_0 + \tilde{\mu} A_1 f^0_{uu} v_0 ight. \]
\[ \quad + |A_1|^2 A_1 \left( -4 N_2 \left[ v_0, (f^0_u)^{-1} N_2 [v_0, \overline{v_0}] \right] ight) \]
\[ \quad + 2 N_2 \left[ \overline{v}_0, (2i \omega_0 I - f^0_u)^{-1} N_2 [v_0, v_0] \right] + 3 N_3 [v_0, v_0, \overline{v}_0] \right) \]
\[ \quad + 2 \tilde{\gamma} A_1 e^{-2i \omega T} N_2 [\overline{v}_0, (f^0_u - 2i \omega_0 I)^{-1} v] \} \cdot w_0. \] (1.23)

Parameter transformations to simplify the equation

We show that (1.23) is equivalent to the 2:1 forced complex Ginzburg–Landau equation

\[ A_T = (1 + i \alpha) \Delta_X A + (-\mu + i \omega) A - (1 + i \beta) |A|^2 A + \gamma \overline{A} \] (1.24)
under Hypothesis 2.1.2. We first normalize \( v_0 \cdot w_0 = 1; \) this is possible by Remark 2.1.4. Next, we remove the nonautonomous term \( \overline{A}_1 e^{-2i \omega T} \) in (1.23) through the transformation \( A_1 =: B_1 e^{-i \omega T} \) to get

\[ \partial_T B_1 - i \tilde{\omega} B_1 = \left\{ \Delta_X B_1 D v_0 + \tilde{\mu} B_1 f^0_{uu} v_0 \right. \]
\[ \quad + |B_1|^2 B_1 \left( -4 N_2 \left[ v_0, (f^0_u)^{-1} N_2 [v_0, \overline{v}_0] \right] \right) \]
\[ \quad + 2 N_2 \left[ \overline{v}_0, (2i \omega_0 I - f^0_u)^{-1} N_2 [v_0, v_0] \right] + 3 N_3 [v_0, v_0, \overline{v}_0] \right) \]
\[ \quad + 2 \tilde{\gamma} B_1 N_2 [\overline{v}_0, (2i \omega_0 I - f^0_u)^{-1} v] \} \cdot w_0 \]
\[ = : (a + ib) \Delta_X B_1 - (c + id) \tilde{\mu} B_1 \]
\[ - (e + if) |B_1|^2 B_1 + (g + ih) \tilde{\gamma} B_1 \] (1.25)
with \( a, c, e \in \mathbb{R}^+ \) and \( b, d, f, g, h \in \mathbb{R} \). The signs of \( a, c, \) and \( e \) follow from Hypotheses 2.1.1-2.1.3 as follows:

(i) \((a \text{ and } c \text{ follow from Hypothesis 2.1.1 and 2.1.3})\) Hypothesis 2.1.1 states that the dispersion relation (1.6)

\[
d(\tilde{\lambda}, ik; \nu) = \det \left( -i\omega_0 I + f_u^0 - Dk^2 - \tilde{\lambda} I + f_{ub}^0 \nu + O(\nu^2) \right) \quad k \in \mathbb{R}
\]

can be expanded near \((\tilde{\lambda}, k, \nu) = (i\omega_0, 0, 0)\) as (equation (1.7))

\[
d(\tilde{\lambda}, ik; \nu) = c_1 \left[ \tilde{\lambda} - \tilde{c}_2 k^2 + \tilde{c}_3 \nu + O(k^4 + \nu^2) \right]
\]

where \( \tilde{\lambda} := \lambda - i\omega_0 \). We have already shown (equation (1.8)) that this in turn implies, by the Implicit Function Theorem, that there exists a curve \( \tilde{\lambda}(k; \nu) \) near \((\tilde{\lambda}, k, \nu) = (0, 0, 0)\) given by

\[
\tilde{\lambda}(k; \nu) = \tilde{c}_2 k^2 - \tilde{c}_3 \nu + O(k^4 + \nu^2),
\]

with \( \text{Re} \tilde{c}_2 < 0 \) and \( \text{Re} \tilde{c}_3 > 0 \), such that \( d(\tilde{\lambda}(k; \nu), ik; \nu) = 0 \). Thus, by definition of the dispersion relation, there exists an eigenvector \( \tilde{v} \) which solves

\[
(-i\omega_0 I + f_u^0 - Dk^2 + f_{ub}^0 \nu + O(\nu^2)) \tilde{v} = \tilde{\lambda}(k; \nu) \tilde{v} \quad (1.26)
\]

for all \( k, \nu \) small enough. Furthermore, by Hypothesis 2.1.3, \( \tilde{v} \) is a small perturbation of \( v_0 \) (see, for example [31, Chapter II, §2.1]) where \( v_0 \) solves

\[
(-i\omega_0 I + f_u^0) v_0 = 0.
\]
In particular,

\[
\tilde{v} = v_0 + O(k^2 + \nu)
\]

\[
= v_0 + (a_1 k^2 + a_2 \nu) v_0 + (a_3 k^2 + a_4 \nu) v_0^\perp + O(k^4 + \nu^2), \quad (1.27)
\]

where \(v_0^\perp\) is the orthogonal complement of \(v_0\). We substitute (1.27) into (1.26)

\[
(-i\omega_0 I + f_0^u - Dk^2 + f_{u\nu}^0 \nu + O(\nu^2))(v_0 + O(k^2 + \nu))
\]

\[
= (\tilde{c}_2 k^2 - \tilde{c}_3 \nu + O(k^4 + \nu^2))(v_0 + O(k^2 + \nu))
\]

and match terms at successive orders of \(k^2\) and \(\nu\), noting that the condition at \(O(1)\) is automatically satisfied by definition of the vector \(v_0\):

- **O\((k^2)\):** \[a_3(-i\omega_0 I + f_0^u) v_0^\perp - Dv_0 = \tilde{c}_2 v_0.\] Thus, \(Dv_0 \cdot w_0 = -\tilde{c}_2;\)

- **O\((\nu)\):** \[a_4(-i\omega_0 I + f_0^u) v_0^\perp + f_{u\nu}^0 v_0 = -\tilde{c}_3 v_0.\] Thus, \(f_{u\nu}^0 v_0 \cdot w_0 = -\tilde{c}_3.\)

In the above computations we have used that \((-i\omega_0 I + f_0^u) v_0 = 0\) and, by Proposition 2.1.5(iii), \(v_0^\perp \cdot w_0 = 0.\)

(ii) (e follows from Hypothesis 2.1.2) Hypothesis 2.1.2 states that the normal form of the kinetic equation \(u_t = f(u; \nu)\) on the center manifold at \(\nu = 0\) is given, up to cubic order, by

\[
A_t = i\omega_0 A - (1 + i\beta)|A|^2 A, \quad \beta \in \mathbb{R}. \quad (1.28)
\]

Since \(v_0 \cdot w_0 = 1\), \(v_0^\perp \cdot w_0 = 0\), and \(v_0\) is the unique eigenvector satisfying

\[
(-i\omega_0 I + f_0^u) v_0 = 0,
\]

taking the dot product with \(w_0\) in equation (1.25) effectively restricts to the
center manifold. Thus, the cubic terms in equation (1.25) are equivalent to the cubic terms in equation (1.28). In Hypothesis 2.1.2, the coefficient $e$ was already rescaled to 1. We leave $e \in \mathbb{R}^+$ for now and explicitly explain the rescaling below.

Next, we simplify (1.25) through some rescalings. We first rescale the space variable $X \mapsto \sqrt{a}X$ (i.e., writing $X := \sqrt{a} \tilde{X}$ and then, with abuse of notation, $X := \tilde{X}$). We also rescale the magnitude by defining $C_1 := \sqrt{e}B_1$. Together, these transformations turn (1.25) into

$$
\partial_T C_1 - i \hat{\omega} C_1 = (1 + i\alpha)\Delta X C_1 - (c + id)\hat{\mu} C_1 \\
- (1 + i\beta)|C_1|^2 C_1 + (g + ih)\gamma C_1
$$

(1.29)

with $\alpha, \beta \in \mathbb{R}$. We also remove the imaginary part of the complex conjugate term by defining $A(X, T) = e^{i\phi}C_1(X, T)$ and choosing $\phi$ so that $\hat{\gamma}e^{-2i\phi}(g + ih) =: \gamma$ for some $\gamma \in \mathbb{R}$. Finally, we redefine $\mu := c\hat{\mu}$ and $\omega := -d\hat{\mu} + \hat{\omega}$ to obtain the forced complex Ginzburg–Landau equation (1.24).

### 2.1.3 Rigorous derivation of the unforced complex Ginzburg–Landau equation

The preceding formal derivation provides intuition about the relationship between solutions to the forced complex Ginzburg–Landau equation (1.2) and small amplitude oscillatory solutions to a periodically forced reaction–diffusion equation (1.12). There are two major difficulties to making the argument rigorous. Firstly, in general the long and short scales will not be completely separable; thus the operator $\mathcal{L}_0$ is
not Fredholm and we can not use the Fredholm Alternative solvability condition.

Secondly, there is no guarantee that the higher order terms $O(\epsilon^4)$ in the ansatz

$$u(t; X, T) = \epsilon u_1(t; X, T) + \epsilon^2 u_2(t; X, T) + \epsilon^3 u_3(t; X, T) + c.c + O(\epsilon^4)$$

remain bounded for all time. Although, to our knowledge, no rigorous derivation is known for the forced CGL, we believe that methods used to justify the unforced CGL can be extended to our case. We therefore end this section by reviewing the results of two complementary approaches used to show that

$$A_T = (1 + i\alpha)\Delta X A - \mu A - (1 + i\beta)|A|^2A$$

(1.30)
captures the dynamics of small amplitude oscillatory solutions to

$$u_t(x, t) = D\Delta u(x, t) + f(u(x, t), \epsilon^2 \hat{\mu})$$

(1.31)

near a supercritical Hopf bifurcation.

(i) In [45] it is assumed that solutions to (1.31) are time periodic with frequency $\omega \approx \omega_0$. For the sake of clarity we review [45, Chapter 2 §4.1], in which the proof is for $x \in \mathbb{R}$. Radially symmetric solutions in higher dimensions were considered in [45, Chapter 4]. Letting $v := u$ and $w := u_x$ and rescaling time $\omega t \mapsto t$ we obtain the first-order differential equation

$$v_x = w$$

$$w_x = D^{-1}(\omega \partial_t u - f(u; \epsilon^2 \hat{\mu}))$$

(1.32)

where $v$ and $w$ are $2\pi$ periodic functions in time. Formally linearizing (1.32) around the $x$-dynamics equilibrium $(v, w)(x, \cdot) \equiv 0$ and decoupling into the Fourier coeffi-
cients \((v^\ell, w^\ell)\) results in the infinite dimensional ODE

\[
\begin{align*}
v_x^\ell &= w^\ell \\
w_x^\ell &= D^{-1}\left(\omega \ell u^\ell - f_u(u; \epsilon^2 \hat{\mu})u^\ell\right).
\end{align*}
\tag{1.33}
\]

It was then shown that (1.33) has a 4-dimensional center manifold, and that the dynamics on this center manifold can be captured by (1.30).

(ii) In [46] it was shown that, given a function \(A(X, T)\) which solves (1.30) on a bounded time interval \(t \in [0, T_0]\), one can find a solution \(u(x, t)\) to (1.31) of the form

\[
u(x, t) = \epsilon A(\epsilon x, \epsilon^2 t)e^{i\omega_0 t} + \epsilon^2 \mathcal{R}(x, t; \epsilon) + \text{c.c.}
\]

where the error term \(\mathcal{R}(x, t; \epsilon)\) has the estimate

\[
\sup_{t \in [0, T_0]} ||\mathcal{R}(x, t; \epsilon)||_{C^m(\mathbb{R}^n; \mathbb{C})} < C(T_0).
\]

The two approaches are complementary. In [45] it was assumed a priori that solutions are periodic with frequency close to \(\omega_0\); under this assumption, equation (1.30) is a normal form for (1.31) for all time. By contrast, in [46] no assumptions are made about the form of the remainder terms \(\mathcal{R}\); as a result, one can only show that these terms remain bounded for time scales on the order of \(1/\epsilon^2\).
2.2 Forced complex Ginzburg–Landau equation in 1-spatial dimension

As discussed in Section 2.1, the 2:1 forced complex Ginzburg–Landau (CGL) equation

\[ u_t = (1 + i\alpha)\Delta u + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma \bar{u}, \quad u \in \mathbb{C} \]  

(2.1)

captures the dynamics of small amplitude oscillatory solutions. Thus, oscillons correspond with steady state solutions to (2.1). In this section, we review known results for the steady state CGL in one space dimension

\[ 0 = (1 + i\alpha)u_{xx} + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma \bar{u}, \quad u \in \mathbb{C}, \quad x \in \mathbb{R} \]  

(2.2)

from [8]. We remark that in equations (2.1) and (2.2) we have used \( u \) as the amplitude variable, rather than \( A \) as in equation (1.2); this is for consistency with our notation in Chapters 3-5.

It was shown in [8] that (2.2) supports two types of localized solutions. The first localized solution, referred to as a standard oscillon, can be thought of as a homoclinic orbit connecting to the trivial background state \( u = 0 \) in the limits \( x \to \pm \infty \) as shown in Figures 2.4a-2.4b. The second localized solution, referred to as a reciprocal oscillon, can be thought of as a homoclinic orbit connecting to a nontrivial background state \( u_{\text{unif}}^+ \neq 0 \) in the limits \( x \to \pm \infty \) as shown in Figures 2.4c-2.4d. Reciprocal oscillon solutions to the CGL were originally reported in [66]. In both cases, the localized solution may have monotone or oscillatory tails.
Figure 2.4: Localized solutions to equation (2.2). The dotted line represents $u \equiv 0$.

Since the planar radially symmetric steady state CGL

$$0 = (1 + i \alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i \omega)u - (1 + i \beta)|u|^2u + \gamma \bar{u}$$

reduces to the one-dimensional case (2.2) in the far field $r = \infty$, we expect that the bifurcation curves in one space-dimension may hold for the planar CGL. We remark, however, that this expectation is not always true. For example, in the steady state quadratic-cubic Swift–Hohenberg equation

$$0 = -(1 + \Delta)^2u - \mu u + \nu u^2 - \kappa u^3 + O(u^4) \quad (2.3)$$

it has been shown there are two major differences between the one-dimensional and planar cases:

(i) (Localized solutions exist for a larger range of parameter values [36]) Define

$$c_3^0 := \frac{3\kappa}{4} - \frac{19\nu^2}{18}.$$ 

The one-dimensional analysis predicts that “spot A” solutions, similar to Figure 2.4b, will bifurcate into the region $\mu < 0$ provided that also $c_3^0 < 0$. This
is because the spot A solutions can be thought of as the “gluing” of a trivial solution to a periodic solution [29], and the periodic solution only exists for $c_3^0 < 0$; see Figure 2.5. However, it was shown in [36] that spot A solutions to the planar Swift–Hohenberg equation bifurcate into the region $\mu < 0$ for all values of $c_3^0$.

(ii) (Non-monotone localized solutions have different scaling [38]) In general, the magnitude of spot solutions is expected to scale with the bifurcation parameter like $||u|| = O(\sqrt{\mu})$. This scaling is true for spot A in all spatial dimensions. However, there exists a second “spot-like” solution to (2.3), first discovered numerically in [37] for the planar Swift–Hohenberg equation and denoted “spot B”. It was shown in [38] that the scaling of spot B depends on the spatial dimension $n$:

$$||u|| = O\left(\mu^{\frac{5-n}{8}}\right) = \begin{cases} O(\sqrt{\mu}) & : n = 1 \\ O(\mu^{3/8}) & : n = 2 \end{cases}$$

so that spot B is larger in dimension 2 than in dimension 1.

Therefore, we use the bifurcation curves for the one-dimensional CGL\(^1\) as a starting point for our planar analysis, but with the caution that the results may not be the same. In the remainder of this chapter we review the relevant results from [8].

\(^1\)We remark that [8] uses $\mu$ instead of $-\mu$; therefore, all bifurcation curves are flipped across the $\mu = 0$ plane relative to ours.
2.2.1 Homogeneous solutions

Standard oscillons (Figures 2.4a-2.4b) bifurcate with the destabilization of the trivial solution $u \equiv 0$; reciprocal oscillons (Figures 2.4c-2.4d) bifurcate with the destabilization of a nontrivial homogeneous solution $u = u_{\text{unif}}$. We first show that these homogeneous solutions exist. It is straightforward to check that $u \equiv 0$ is a solution to (2.2) for all parameter values. It was shown in [8] that equation (2.2) also admits non-trivial uniform solutions $u_{\text{unif}}^\pm := R^\pm e^{i\phi^\pm}$ where

$$
(R^\pm)^2 := \frac{\omega\beta - \mu \pm \sqrt{(1+\beta^2)\gamma^2 - (\omega + \beta\mu)^2}}{1 + \beta^2}
$$

(2.4)

and $\phi^\pm$ solves

$$
\cos 2\phi^\pm = \frac{(R^\pm)^2 + \mu}{\gamma}, \quad \sin 2\phi^\pm = \frac{\omega - \beta (R^\pm)^2}{\gamma}.
$$

(2.5)

The nontrivial uniform solutions $u_{\text{unif}}^\pm$ bifurcate in a saddle node along the curve

$$
\Gamma_b := \{(\mu, \gamma) : (1 + \beta^2)\gamma^2 = (\omega + \beta\mu)^2\}
$$

(2.6)

provided $\omega\beta > \mu$, where we have considered $\omega$ and $\beta$ to be fixed. See Figure 2.9. On $\Gamma_b$

$$
R_0^2 := (R^\pm)^2 = \frac{\omega\beta - \mu}{1 + \beta^2}.
$$

(2.7)

We write $u = v + iw$ with $v, w \in \mathbb{R}$ and define $u_{\text{unif}}^\pm =: v^\pm + iw^\pm$, where $v^\pm = R^\pm \cos \phi^\pm$ and $w^\pm = R^\pm \sin \phi^\pm$. We also define $\gamma_b := \gamma_b(\mu)$ to be the value of $\gamma$ so that $(\mu, \gamma_b) \in \Gamma_b$. Then, with $\gamma = \gamma_b + \epsilon^2$, it was shown in [8] that $u_{\text{unif}}^\pm$ can be
expanded
\[
\begin{pmatrix}
v^± \\
w^±
\end{pmatrix} = \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} + \epsilon \begin{pmatrix} v^±_1 \\ w^±_1 \end{pmatrix} + \epsilon^2 \begin{pmatrix} v^±_2 \\ w^±_2 \end{pmatrix} + \ldots
\] (2.8)

where
\[
\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} \eta_b \\ 1 \end{pmatrix} \Upsilon_0, \quad \text{with } \Upsilon_0 := \frac{R_0}{\sqrt{1 + \eta_b^2}},
\]
\[
\eta_b := \beta + \text{sgn}(\omega + \beta \mu)\sqrt{1 + \beta^2}
\] (2.9)

and
\[
\begin{pmatrix} v^+_1 \\ w^+_1 \end{pmatrix} = \begin{pmatrix} \xi_b \\ 1 \end{pmatrix} \Upsilon_1, \quad \text{with } \xi_b := \frac{\eta_b \omega + (1 - \beta \eta_b)R_0^2}{\omega - (\beta + \eta_b)R_0^2},
\]
\[
\Upsilon_1 := \text{sgn}[\xi_b \eta_b + 1] \sqrt{\frac{\eta_b}{(\xi_b \eta_b + 1)(\xi_b - \eta_b)}}.
\] (2.10)

2.2.2 Spatial eigenvalue analysis

The bifurcation curves for the one-dimensional CGL were computed in [8] using a spatial eigenvalue analysis. We seek nontrivial solutions to the linearization of (2.2) about some homogeneous solution \( u = u_{\text{hom}} \) of the form \( U = e^{ik_j x} U_0 \) where \( U := (\text{Re}(u), \text{Im}(u)) \) and \( U_j \) is the eigenvector associated with spatial eigenvalue \( k_j \). The linearization will be of the form
\[
0 = \Delta U + CU,
\] (2.11)
Figure 2.6: Possible spatial eigenvalue arrangements for the bifurcation of localized solutions. Spatial eigenvalues collide on the imaginary axis and then move away.

where the matrix $C := C(u_{\text{hom}}) \in \mathbb{R}^2$ depends on the solution $u_{\text{hom}}$. Localized solutions may bifurcate as small-amplitude solutions from a homogeneous state $u = u_{\text{hom}}$ only if the (2.11) has spatial eigenvalues $\{k_j\}$ with $\text{Re}(k_j) = 0$ for some $j$. Furthermore, localized solutions bifurcate into the parameter region where these spatial eigenvalues move away from the imaginary axis: this ensures that the stable and unstable manifolds are as high dimensional as possible. Equation (2.11) has four spatial eigenvalues since there are two space derivatives and $U \in \mathbb{R}^2$; these eigenvalues obey the symmetry $k \mapsto -k$ due to the Laplacian. Thus, there are two cases to consider, as shown in Figure 2.6.

(i) On the bifurcation curve $k_1 = k_2 = 0$ and $k_3 = -k_4 = m \in \mathbb{R}$. Localized solutions bifurcate into the region where $k_1$ and $k_2$ split along the real axis of the complex plane; see Figure 2.6a. This bifurcation gives rise to localized solutions with monotone tails (Figures 2.4a and 2.4c).

(ii) On the bifurcation curve $k_1 = k_2 = k_3 = k_4 = im$ with $m \in \mathbb{R}$. Localized solutions bifurcate into the region where $k_j$ split away from the imaginary axis so that $\{k_j\} \cap \mathbb{R} = \emptyset$; see Figure 2.6b. This bifurcation results in standard oscillons with oscillatory tails (Figures 2.4b and 2.4d).
The difference between the standard oscillons (Figures 2.4a and 2.4b) and reciprocal oscillons (Figures 2.4c and 2.4d) is that the linearization (2.11) is computed using \( u \equiv 0 \) and \( u = u_{\text{unif}} \), respectively.

### Standard oscillon bifurcation curves

The linearization of (2.2) about \( u = 0 \)

\[
0 = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} -\mu + \gamma & -\omega \\ \omega & -\mu - \gamma \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}, \tag{2.12}
\]

has four spatial eigenvalues which satisfy

\[(1 + \alpha^2)k^4 + 2(\alpha \omega - \mu)k^2 + \mu^2 + \omega^2 - \gamma^2 = 0. \tag{2.13}\]

In Figure 2.7 we illustrate the spatial eigenvalues \( k \), plotted in the complex plane, for various values of \( \omega \) and \( \gamma \). The two bifurcation cases are as follows:

(i) Define

\[\Gamma_0 := \{(\mu, \gamma) : \gamma = \sqrt{\mu^2 + \omega^2}\},\]

set \( \mu_m := \alpha \omega + \frac{m^2}{2} \), and pick \( \gamma_m \) so that \((\mu_m, \gamma_m) \in \Gamma_0\). Then, with

\[(\mu, \gamma) = (\mu_m, \gamma_m - \epsilon^2),\]

\(k_{1,2}\) split into \(O(\epsilon)\) eigenvalues along the real axis, whilst \(k_{3,4} \approx \pm m\), as shown in Figure 2.7. Thus, standard monotone oscillons may bifurcate for \( \mu > \alpha \omega \) into the region below the curve \( \Gamma_0 \).
Figure 2.7: Plotted is the \((\mu, \gamma)\)-plane with \(\alpha, \beta, \text{ and } \omega \) fixed. The inlays are the spatial eigenvalues associated with the linearization of (2.2) about \(u = 0\) plotted in the complex plane. There are two bifurcation cases to consider: (i) into the region below \(\Gamma_0\) (purple dashed line) for \(\mu > \alpha \omega\), and (ii) into the region below \(\Gamma_a\) (green dash-dotted line) for \(\mu < \alpha \omega\). Non-trivial uniform solutions \(u_{\text{unif}}^{\pm}\) emerge at a fold bifurcation at \(\Gamma_b\) (maroon dotted line); the fold curve \(\Gamma_b\) intersects \(\Gamma_0\) at \(\mu = \beta \omega\) where \(u_{\text{unif}}^{+} = u_{\text{unif}}^{-} = 0\).

(ii) Define

\[ \Gamma_a := \{ (\omega, \gamma) : (1 + \alpha^2)\gamma^2 = (\omega + \alpha \mu)^2 \}, \]

set \(\mu_m := \alpha \omega - \frac{m^2}{2}\), and pick \(\gamma_m\) so that \((\mu_m, \gamma_m) \in \Gamma_a\). Then, with

\[ (\mu, \gamma) = (\mu_m, \gamma_m - \epsilon^2), \]

\[ k_j = \pm O(\epsilon) \pm i(m + O(\epsilon)), \]

as shown in Figure 2.7. Thus, standard oscillatory oscillons may bifurcate for \(\mu < \alpha \omega\) into the region below the curve \(\Gamma_a\).

We remark that in [8] it was found that localized solution of either type only bifurcate provided that also \(\mu < \beta \omega\). The same condition remains true in the planar case; it is related to a subcriticality condition and will be discussed in Chapter 3.
Reciprocal oscillon bifurcation curves

The linearization of (2.2) about $u = u_{\text{unif}}$ 

$$0 = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} \mu - \gamma & \omega \\ -\omega & \mu + \gamma \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} x x + \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix} \left( (R^\pm)^2 I + 2 \begin{pmatrix} (v^\pm)^2 & v^\pm w^\pm \\ v^\pm w^\pm & (w^\pm)^2 \end{pmatrix} \right) \begin{pmatrix} v \\ w \end{pmatrix}$$ (2.14)

has four spatial eigenvalues which satisfy

$$0 = (1 + \alpha^2) k^4 + 2 \left( \alpha \omega - \mu - 2(1 + \alpha \beta) |u_{\text{unif}}^\pm|^2 \right) k^2 + 4 \left( (\beta \omega - \mu) |u_{\text{unif}}^\pm|^2 + \gamma^2 - \mu^2 - \omega^2 \right).$$ (2.15)

In Figure 2.8 we illustrate the spatial eigenvalues $k$, plotted in the complex plane, for various values of $\omega$ and $\gamma$. The two bifurcation cases are as follows:

(i) As discussed in Section 2.2.1, on the curve

$$\Gamma_b := \{ (\mu, \gamma) : (1 + \beta^2) \gamma^2 = (\omega + \beta \mu)^2 \}$$

we have

$$|u_{\text{unif}}^+|^2 = |u_{\text{unif}}^-|^2 = R_0^2 = \frac{\omega \beta - \mu}{1 + \beta^2}.$$

A straightforward computation shows that on $\Gamma_b$, $k_1 = k_2 = 0$ whereas $k_3$ and $k_4$ satisfy

$$(1 + \alpha^2) k_j^2 = 2(\mu - \alpha \omega + 2(1 + \alpha \beta) R_0^2) = \frac{2z(\omega - \omega_z)}{1 + \beta^2}$$ (2.16)
Figure 2.8: Plotted is the \((\mu, \gamma)\)-plane with \(\alpha, \beta, \) and \(\omega\) fixed. The inlays are the spatial eigenvalues associated with the linearization of (2.2) about \(u = u_{\text{unif}}^+\) plotted in the complex plane. There are two bifurcation cases to consider: (i) into the region above \(\Gamma_b\) (maroon dotted line) for \(z(\omega - \omega_z) > 0\), and (ii) into the region above \(\Gamma_d\) (orange dash-dot-dotted line) for \(z(\omega - \omega_z) < 0\). Non-trivial uniform solutions \(u_{\text{unif}}^\pm\) emerge at a fold bifurcation at the intersection of \(\Gamma_b\) with \(\Gamma_0\) (purple dashed line), which occurs at \(\mu = \beta \omega\) where \(u_{\text{unif}}^+ = u_{\text{unif}}^- = 0\).

where\(^2\)

\[
z := \alpha(1 - \beta^2) - 2\beta \quad \text{and} \quad \omega_z := \frac{\mu(1 - \beta^2 + 2\alpha\beta)}{z}.
\]

Hence, \(k_3 = -k_4 \in \mathbb{R}\) provided \(z(\omega - \omega_z) < 0\). In this case, near \(\Gamma_b\) the zero eigenvalues \(k_1, k_2\) associated with the upper branch solution \(u_{\text{unif}}^+\) split along the real axis; see Figure 2.8. In contrast, the eigenvalues associated with the lower branch solution \(u_{\text{unif}}^-\) split along the imaginary axis; see Figure 2.9. Thus, monotone reciprocal oscillons may bifurcate for \(z(\omega - \omega_z) < 0\) into the region above the curve \(\Gamma_b\); such solutions can be written as localized offsets from the upper branch solution

\[
u(x) = u_{\text{unif}}^+ + \tilde{u}(x),
\]

\(^2\)We remark that [8] defines \(-z\) instead of \(z\). Hence, all inequalities involving \(z\) are reversed in [8] from the inequalities discussed in this section.
Figure 2.9: Nontrivial uniform solutions to (2.2) bifurcate along the curve $\Gamma_b$. The insets show the spatial eigenvalues of the linearization of (2.2) about the upper and lower branches plotted in the complex plane for $z\omega < z\omega_z$; for $z\omega > z\omega_z$, the spatial spectrum is rotated by 90 degrees. At $\omega = \omega_3$, we have $u^\pm_{\text{unif}} = 0$; for $\omega < \omega_3$, the fold ceases to exist.

with $\tilde{u}(x) \to 0$ as $|x| \to \infty$.

(ii) The bifurcation curve for oscillatory solutions, denoted $\Gamma_d$ in [8], is difficult to compute explicitly for all parameter values. The curve $\Gamma_d$ is tangent to $\Gamma_b$ at the point $\omega = \omega_z$ since all four spatial eigenvalues $k_j$ are zero there. Thus, near $\omega = \omega_z$, $\Gamma_d$ is close enough to $\Gamma_b$ that one can use the expansion (2.8) to compute $\Gamma_d$:

$$
\Gamma_d := \left\{ (\mu, \gamma) \mid 4\mu^2(1 + \beta^2)(\alpha - \beta)^2(\gamma^2 - \gamma_b^2) = \left(z(\mu^2 + \omega_\zeta) - \text{sgn}(\alpha - \beta)\sqrt{1 + \alpha^2(1 + \beta^2)\gamma_0}\right)^2 \right\}
$$

where $\gamma_b = \{ \gamma : (\mu, \gamma) \in \Gamma_b \}$ and similarly for $\gamma_0$. On $\Gamma_d$ the spatial eigenvalues $k_j = \pm ik_d$ with

$$
k_d := \sqrt{-\mu + \alpha \omega - 2|u^+_{\text{unif}}(\gamma_d)|^2(1 + \alpha \beta)}.
$$

They will be purely imaginary on $\Gamma_d$ provided that $z(\omega - \omega_z) > 0$. In general, we can compute $\Gamma_d$ numerically using $\text{FSOLVE}$ in Matlab to solve

$$
(\alpha \omega - \mu - 2(1 + \alpha \beta)|u^\pm_{\text{unif}}|^2)^2 = 4(1 + \alpha^2) \left((\beta \omega - \mu)|u^+_{\text{unif}}|^2 + \gamma^2 - \mu^2 - \omega^2\right).
$$
since this is the curve along which the spatial eigenvalues, given by (2.15), occur in pairs.

This concludes our review of the bifurcation curves for localized solutions to (2.2) in one-space dimension from [8].

2.3 Blowup coordinates

In our analysis of the forced complex Ginzburg–Landau equation we will need to understand the dynamics near a non-hyperbolic fixed point. One tool for analyzing such situations is “blow-up” coordinates. In order to build intuition about the information captured by blow-up coordinates, we consider the following system of ordinary differential equations

\[ \begin{align*}
    x_t &= x^2 - 2xy \\
    y_t &= y^2 - 2xy,
\end{align*} \]

(3.1)

taken from [11, p. 73-75]. System (3.1) has one equilibrium, \((x, y) = (0, 0)\), and it can easily be shown that this equilibrium is not hyperbolic. Hence, the linearization of equation (3.1) about \((x, y) = (0, 0)\) provides no information about the dynamics nearby.

We consider three blow-ups of equation (3.1): a radial blow-up in Section 2.3.1 and directional blow-ups in the \(x\) and \(y\) directions in Section 2.3.2. In all cases we “zoom-in” on a non-hyperbolic fixed point by “blowing it up” to either a sphere (radial blow-up) or a line (directional blow-ups). These blow-ups are visualized in
Radial blow-up: a point is blown-up to a sphere.

Directional blow-up: a point is blown-up to a line.

Figures 2.10: Radial versus directional blow-up of a point.

Figures 2.10a and 2.10b, respectively. The “zooming-in” is generally accomplished by slowing down time. The hope is that we will then be able to distinguish a finite number of critical directions that organize the dynamics.

The radial blow-up is easiest to understand intuitively, but it is harder to implement in higher dimensions. Hence, directional blow-ups are more commonly employed and, in fact, are what we use in our analysis of the 2:1 forced CGL in this thesis. By considering the effect of directional blow-ups of the same equation (3.1) in Section 2.3.2, we build intuition for the relationship between these different approaches. In particular, we will see that directional blow-ups are equivalent to a projection onto the real projective $n$-space $\mathbb{RP}^n$. Moreover, since $\mathbb{RP}^n$ is the quotient space of the $n$-sphere $\mathbb{S}^n$, obtained by identifying opposite points, we expect the radial and directional blow-ups to contain similar information. See Figure 2.11. We remark that directional blow-ups only cover half of the $n$-sphere. Moreover, they miss points; for example, in Figure 2.11 the projection misses the points $(x_1, \ldots, x_n, 0)$. Therefore, multiple directional blow-ups are usually necessary in order to capture all information about a vector field.
2.3.1 Radial blowup

In this section we consider a radial blow-up of (3.1). This is the blow-up employed in [11, p. 73-75]. We first transform system (3.1) into polar coordinates

\[ x = r \cos \theta, \quad y = r \sin \theta \]

and

\[ r_t = r^2(\cos^3 \theta - 2 \cos^2 \theta \sin \theta - 2 \cos \theta \sin^2 \theta + \sin^3 \theta) \]

\[ \theta_t = 3r(\cos \theta \sin^2 \theta - \cos^2 \theta \sin \theta). \quad (3.2) \]

Equation (3.2) has a single fixed point at \( r = 0 \). This is the same point as in the original coordinates \((x, y) = (0, 0)\), and a simple computation shows that it is still not hyperbolic. Thus, the transformation into polar coordinates has not yet yielded any new information.

We observe, however, that there is an Euler multiplier, namely \( r \), in the equations for both \( r_t \) and \( \theta_t \). We can remove this Euler multiplier by rescaling time by \( \tau := rt \).

As mentioned in the introduction to this section, it is this slowing down of time that really allows us to “zoom-in” on the fixed point. We note that since \( r > 0 \) the reparametrization of time does not alter the direction of the vector field [11, Prop.
The system (3.2) becomes

\[ r_{\tau} = r(\cos^3 \theta - 2 \cos^2 \theta \sin \theta - 2 \cos \theta \sin^2 \theta + \sin^3 \theta) \]

\[ \theta_{\tau} = 3 \cos \theta \sin \theta (\sin \theta - \cos \theta). \]  

(3.3)

Now \( r = 0 \) is a necessary, but not sufficient, condition for equilibrium. In particular, equation (3.3) distinguishes directions along which \( \theta \) is invariant, i.e., \( \theta = 0, \pi/4, \pi/2, \pi, 5\pi/4, \) and \( 3\pi/2 \). These equilibria are all hyperbolic and their stability can be analyzed by considering the linearization of (3.3) about each point. Letting \((r, \theta) = (0, \theta^*)\), with \( \theta^* \in [0, \pi/4, \pi/2, \pi, 5\pi/4, 3\pi/2] \),

\[
c^*_s := (1 - 3 \cos \theta^* \sin \theta^*)(\sin \theta^* + \cos \theta^*)
\]

\[
= \begin{cases}
1 & : \theta^* = 0, \pi/2 \\
\frac{-1}{\sqrt{2}} & : \theta^* = \pi/4 \\
-1 & : \theta^* = \pi, 3\pi/2 \\
\frac{1}{\sqrt{2}} & : \theta^* = 5\pi/4 \\
\end{cases}
\]

the linearized equation is

\[
\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} c^*_s & 0 \\ 0 & -3c^*_s \end{pmatrix} \begin{pmatrix} r \\ \theta \end{pmatrix}.
\]

Thus, the fixed points \((r, \theta^*)\) are all hyperbolic and, in fact, are saddle points. We also observe from (3.3) that the ring \( r = 0 \) is invariant, as are each of the lines \( \theta = \theta^* \). The resulting flow field is shown in Figure 2.12a. To obtain the flow in the original coordinates we “shrink” the circle back to the origin. Since the direction of the flow will not change when we undo the reparametrization of time \( \tau = rt \), the flow field for equation (3.1) is the same, as depicted in Figure 2.12b.
Flow field near \( r = 0 \) in the blow-up coordinates.

(a) Flow field near \((x, y) = (0, 0)\) in the original coordinates.

(b) Flow field near \((x, y) = (0, 0)\) in the original coordinates.

**Figure 2.12:** Flow field in radial blow-up coordinates versus original coordinates. The trajectory in the original coordinates is colored the same as its corresponding trajectory in the blow-up coordinates. The green points in (a) represent fixed points \((r, \theta) = (0, \theta_*)\) in the blow-up coordinates; the red ring in (a) and red point in (b) represent the fixed point \((x, y) = (0, 0)\) in the original coordinates.

### 2.3.2 Directional blow-up

In our analysis of the 2:1 forced (CGL) we will employ a directional rather than a radial blow-up. In Sections 2.3.2 and 2.3.2 we redo the blow-up analysis of (3.1) using a \( y \) and \( x \) directional blow-up, respectively. We see that the resulting flow field in the original coordinates, depicted in Figures 2.14b and 2.17b respectively, is the same as the flow field obtained by the radial blow-up, depicted in Figure 2.12b, with the following differences:

(i) In the \( y \) (resp. \( x \)) directional blow-up coordinates, the line \( y = 0 \) (resp. \( x = 0 \)) is captured by \((\bar{x}, \bar{y}) = (\infty, x/\bar{x})\) (resp. \((\bar{x}, \bar{y}) = (y/\bar{y}, \infty))\), as is shown in Figure 2.13 (resp 2.16); this corresponds with the fact that the point \((x_1, \ldots, x_{n_1}, 0) \in \mathbb{S}^n\) corresponds with the point \((\infty, \ldots, \infty, 1) \in \mathbb{R}P^n\) in Figure 2.11. Hence, the directional blow-up cannot capture interesting dynamics along the line \( x = 0 \) (resp. \( y = 0 \)).
Figure 2.13: Transformation of a region in the original coordinates to a region in the
$y$-directional blow-up coordinates. The dotted lines are lines of constant slope in the
original $(x, y)$ coordinates. These correspond with the thin vertical lines in the blow-up
coordinates, with $x = \tilde{x}$ precisely when $y = 1$ as shown along the thin blue horizontal line.
The thick red line $\tilde{y} = 0$ represents the blown-up fixed point $(x, y) = (0, 0)$ in the original
coordinates.

(ii) The point $(x, y)$ is indistinguishable from the point $(-x, -y)$ in any directional
blow-up; this corresponds with the fact that $\mathbb{RP}^n$ can be obtained by identifying
antipodal points on the sphere $S^n$, as shown in Figure 2.11. Thus, the $y$ (resp. $x$) directional blow-up results in the flow field for $x > 0$ (resp. $y > 0$). The
flow field for $x < 0$ (resp. $y < 0$) is captured by a $-y$ (resp. $-x$) directional
blow-up.

For a general $n$-dimensional vector field, one may need to perform several blow-ups,
i.e., in the $x_i$, $-x_i$, and $x_j$ directions, to overcome these difficulties.

$y$-Directional blowup

Similar to the radial blow-up, we begin with a coordinate transformation. We de-
define $\tilde{x} := x/y$, $\tilde{y} := y$, so that $\tilde{x}$ is the inverse slope of a given line and $y$ is a
parametrization along that line, as is shown in Figure 2.13. Inverting this transfor-
mation, $(x, y) = (\tilde{x}\tilde{y}, \tilde{y})$, we see that the line $\tilde{y} = 0$ corresponds with $(x, y) = (0, 0)$
for every value of $\tilde{x}$; in other words, the fixed point has been ”blown-up” to the line.
\( \tilde{y} = 0 \). See again Figure 2.13.

Vector field (3.1) in these new coordinates is

\[
\begin{align*}
\tilde{x}_t &= \frac{x_t}{y} - \frac{x y_t}{y^2} = 3\tilde{y}(\tilde{x}^2 - \tilde{x}) \\
\tilde{y}_t &= y_t = \tilde{y}^2(1 - 2\tilde{x}),
\end{align*}
\]

and we see that \( \tilde{y} = 0 \) is an equilibrium point for all values of \( \tilde{x} \), as expected. However, these points are again not hyperbolic. We remove the Euler multiplier \( \tilde{y} \) from each equation by rescaling time \( \tau := \tilde{y} t \), restricted to \( \tilde{y} > 0 \), and obtain the vector field

\[
\begin{align*}
\tilde{x}_\tau &= 3\tilde{x}(\tilde{x} - 1) \\
\tilde{y}_\tau &= \tilde{y}(1 - 2\tilde{x}).
\end{align*}
\]  

System 3.5 has equilibria \((\tilde{x}, \tilde{y}) = (0, 0)\) and \((1, 0)\). The linearization of (3.5) about the fixed points \((\tilde{x}, \tilde{y}) = (0, 0)\) and \((1, 0)\) is

\[
\begin{pmatrix}
\tilde{x}_\tau \\
\tilde{y}_\tau
\end{pmatrix} = \begin{pmatrix}
-3 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\tilde{x}_\tau \\
\tilde{y}_\tau
\end{pmatrix} = \begin{pmatrix}
3 & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix},
\]

respectively. Thus, the fixed points \((0, 0)\) and \((0, 1)\) are both hyperbolic, and, in fact, are saddle points. We also observe from (3.5) that the lines \( \tilde{x} = 0 \) and \( \tilde{x} = 1 \) are invariant. The resulting flow field is shown in Figure 2.14a. To obtain the flow in the original coordinates we “shrink” the line \( \tilde{y} = 0 \) back to the origin. Additionally, by inverting \( x = \tilde{x} y \), the vertical lines \( \tilde{x} = 0 \) and \( \tilde{x} = 1 \) becomes the lines \( x = 0 \) and \( y = x \), respectively. Thus we can transform any flow trajectory on bounded intervals \( \tilde{x} \in [-C, C] \) into the original coordinates; this corresponds with the conical
Flow field near $\tilde{y} = 0$, for $\tilde{y} > 0$, in the blow-up coordinates.

Flow field near $(x, y) = (0, 0)$ in the original coordinates.

Figure 2.14: Flow field in $y$-directional blow-up coordinates versus original coordinates. The trajectory in the original coordinates is colored the same as its corresponding trajectory in the blow-up coordinates. The green points in (a) represent fixed points $(\tilde{x}, \tilde{y}) = (0, 0), (1, 0)$ in the blow-up coordinates; the red line in (a) and red point in (b) represent the fixed point $(x, y) = (0, 0)$ in the original coordinates. The dotted lines in (b) are the boundary of the conical region $|x/y| < C$.

Since the direction of the flow will not change when we undo the reparametrization of time $\tau = \tilde{y}t$, provided $\tilde{y} > 0$, the flow field for equation (3.1) is as depicted in Figure 2.14b.

To capture the dynamics for $y < 0$ we can argue by symmetry: if $(\tilde{x}(\tau), \tilde{y}(\tau))$ is a solution to vector field (3.5), then so is $(\tilde{x}(-\tau), -\tilde{y}(-\tau))$, resulting in the flow field depicted in Figure 2.15a. Then the flow field in Figure 2.15b can be obtained from Figure 2.15a by again shrinking the line $\tilde{y} = 0$ back to the origin and inverting $x = \tilde{x}y$. We could alternatively have argued directly from the symmetry in the original vector field: if $(x(t), y(t))$ is a solution to (3.1), then so is $(-x(-t), -y(-t))$. The flow field in Figure 2.15b is the same as in Figure 2.12b except that the dynamics along the axis $y = 0$ are unknown. To properly capture the dynamics on $y = 0$ we need to use a different blow-up; we will see below that the $x$-directional blow-up does indeed capture the dynamics along this line.
Flow field near $\tilde{y} = 0$, for $\tilde{x} > 0$, in the blow-up coordinates.

Flow field near $(x, y) = (0, 0)$ in the original coordinates.

**Figure 2.15:** Flow field in $y$-directional blow-up and $(-y)$-directional coordinates versus original coordinates. The flow field in the lower half plane in (a) is obtained from the flow field in the upper half plane by using the symmetry $(x, y, t) \mapsto (-x, -y, -t)$ in the vector field (3.5). The flow field in the lower half plane in (b) is obtained from the flow field in the upper half plane by using the symmetry $(x, y, t) \mapsto (-x, -y, -t)$ in the vector field (3.1). The dotted lines in (b) are the boundary of the conical region $|x/y| < C$.

If we had not been able to use symmetry, we could have instead used the $(-y)$-directional blow-up $\tilde{x} := x/y$, $\tilde{y} := -y > 0$, $\tilde{\tau} := \tilde{y}t$, to capture the dynamics in the lower half plane. We observe that $\tilde{x} = \tilde{x}$, $\tilde{y} = -\tilde{y}$, and $\tilde{\tau} = -\tau$; thus, this blow-up is equivalent to understanding trajectories $(\tilde{x}(-\tau), -\tilde{y}(-\tau))$, which we have already analyzed by using symmetry.
Figure 2.16: Transformation of a region in the original coordinates to a region in the $x$-directional blow-up coordinates. The dotted lines are lines of constant slope in the original $(x, y)$ coordinates. These correspond with the thin horizontal lines in the blow-up coordinates, with $y = \hat{y}$ precisely when $x = 1$ as shown along the thin blue vertical line. The thick red line $\hat{x} = 0$ represents the blown-up fixed point $(x, y) = (0, 0)$ in the original coordinates.

x-Directional blowup

Finally, we perform an $x$-directional blow-up for comparison with the $y$-directional blow-up. We will see that this transformation captures different trajectories than the $y$-directional blow-up. We define $\hat{x} := x$, $\hat{y} := y/x$. Now $\hat{y}$ captures lines of constant slope and $x$ is a parametrization along each line, as is shown in Figure 2.16. We furthermore slow down time by transforming $\tau := \hat{x}t$. The resulting vector field is

$$
\hat{x}_\tau = \hat{x}(1 - 2\hat{y})
$$

$$
\hat{y}_\tau = 3\hat{y}(\hat{y} - 1).
$$

System 3.6 has equilibria $(\hat{x}, \hat{y}) = (0, 0)$ and $(0, 1)$; moreover, the lines $\hat{y} = 0$ and $\hat{y} = 1$ are invariant. A standard stability analysis shows that the resulting flow field is as shown in Figure 2.17a. To obtain the flow in the original coordinates we “shrink” the line $\hat{x} = 0$ back to the origin. Additionally, by inverting $y = x\hat{y}$, the
Flow field near $\hat{x} = 0$, for $\hat{y} > 0$, in the blow-up coordinates.

(b) Flow field near $(x, y) = (0, 0)$ in the original coordinates.

Figure 2.17: Flow field in $x$-directional blow-up coordinates versus original coordinates. The trajectory in the original coordinates is colored the same as its corresponding trajectory in the blow-up coordinates. The green points in (a) represent fixed points $(\hat{x}, \hat{y}) = (0, 0), (0, 1)$ in the blow-up coordinates; the red line in (a) and red point in (b) represent the fixed point $(x, y) = (0, 0)$ in the original coordinates. The dotted lines in (b) are the boundary of the conical region $|y/x| < C$.

Horizontal lines $\hat{y} = 0$ and $\hat{y} = 1$ become the lines $y = 0$ and $y = x$, respectively. Thus we can transform any flow trajectory on bounded intervals $\hat{y} \in [-C, C]$ into the original coordinates; this corresponds with the conical region $|y/x| < C$. Since the direction of the flow will not change when we undo the reparametrization of time $\tau = \hat{x}t$, provided $\hat{x} > 0$, the flow field for equation (3.1) is as depicted in Figure 2.17b. We can determine the flow field in the left half plane through the same symmetry argument as with the $y$-directional blow-up; the result will be the same as in Figure 2.12b except that the dynamics along the line $x = 0$ are undefined.

Observe that the $x$-directional blow-up captures the dynamics along the $x$ axis but not the $y$ axis, and the reverse is true for the $y$-directional blow-up. Hence these two blow-ups provide complementary information about the system.
Chapter Three

Standard oscillons
3.1 Introduction

In this chapter, we prove the existence of small-amplitude standard oscillons with monotone tails (see Figure 3.1 for a representation in one spatial dimension) in the stationary planar radial forced complex Ginzburg–Landau (CGL) equation

$$0 = (1 + i\alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma \bar{u} \quad (1.1)$$

near onset. These results are taken from our work [40]. We use the bifurcation curve for the one-dimensional CGL

$$\Gamma_0 := \{(\mu, \gamma) : \gamma = \sqrt{\mu^2 + \omega^2}\} \quad (1.2)$$

from Section 2.2.2. As was argued in Section 2.2.2, summarized from [8], standard oscillons may bifurcate for the one-dimensional CGL into the region $\gamma < \gamma_0$ (where $\gamma_0$ is defined so that $(\mu, \gamma_0(\mu)) \in \Gamma_0$), provided that also $\mu > \alpha \omega$. The argument relies on a spatial eigenvalue analysis: standard oscillons with monotone tails are expected to bifurcate into the region where all four spatial eigenvalues associated with the linearization of (1.1) about $u \equiv 0$ are purely real; on $\Gamma_0$, two of these eigenvalues are located at the origin of the complex plane. See Figure 3.2 for an illustration.

In this chapter, we show that this result also holds for the planar CGL. We now state our main result.

**Theorem 3.1.** Fix $\alpha \omega < \mu < \beta \omega$ and let $\gamma = \sqrt{\mu^2 + \omega^2} - \epsilon^2$. Then there is an
\( \epsilon_0 > 0 \) so that (1.1) has a nontrivial stationary localized radial solution of amplitude \( O(\epsilon) \) for each \( \epsilon \in (0, \epsilon_0) \).

The requirement \( \mu > \alpha \omega \) is the spatial eigenvalue condition discussed above. In Section 3.3.5 we will show that \( \text{sign}(\mu - \omega \beta) \) equals the sign of the leading-order nonlinear term in an appropriate center-manifold reduction of (1.1) near \( \epsilon = 0 \). Therefore, the condition \( \mu < \omega \beta \) ensures that the bifurcation from the curve \( \gamma(\omega, \mu) = \sqrt{\mu^2 + \omega^2} \) is subcritical and leads to localized patterns. The subcriticality plays a key role in the analysis, as explained in Section 3.4.2.

### 3.1.1 Outline

We conclude this introduction to this chapter by giving an overview of the proof methodology. The idea is to construct the core manifold as the set of solutions of (1.1) that stay bounded as \( r \to 0 \) for \( r \in [0, r_0] \) for some \( r_0 < \infty \). We will separately construct the far-field manifold as the set of all solutions of (1.1) that decay to zero.
Figure 3.3: Schematic overview of our strategy: we construct two sets of solutions to (1.1), the core manifold (solutions that are bounded as \( r \to 0 \)) and the far-field stable manifold (solutions that decay as \( r \to \infty \)). Any solution lying in the intersection of these manifolds is a localized solution. For fixed \( r \), both manifolds have a two-dimensional parametrization in four-dimensional space. The far-field stable manifold is identified with the strong stable foliation associated with solutions on the center manifold that converge to zero as \( r \to \infty \) exponentially as \( r \to \infty \). A solution lying in both sets will, by definition, be a bounded localized solution. A schematic is shown in Figure 3.3.

In the proof of Theorem 3.1 we are confronted with two main difficulties, both of which are related to the far field. Firstly, in the parameter region we are interested in, the linearization of (1.1) about \( u = 0 \) in the far field has two eigenvalues close to the origin and, in addition, one strong stable and one strong unstable direction. Our far-field analysis should therefore be performed on an appropriate center manifold. On the other hand, the final matching between the core and far-field manifolds is performed in the original four-dimensional space; hence, in order to carry out the final matching analysis, we need to employ a stable foliation over solutions in the center manifold, denoted by \( \mathcal{F}^s \) in Figure 3.3, and derive appropriate expansions of all foliations and manifolds.

Secondly, we encounter difficulties when analyzing the flow on the center manifold. The small-amplitude solutions on the center manifold are, to leading order, of the form \( A(r) = \epsilon A_{\exp}(\epsilon r) \) for some bounded function \( A_{\exp}(s) \). Thus, we can expect uniform expansions of an exponentially decaying solution \( A_*(r) \) to be possible on the
interval \( s \in [s_0, \infty) \), or \( r \geq s_0/\epsilon \). In particular, we expect to lose control over bounds on \( A_*(r) \) at \( r = r_0 \) for \( \epsilon \) small enough. To resolve this issue and obtain meaningful bounds near the matching point at \( r = r_0 \), we employ the blow-up coordinates from [38] to mediate between exponential decay in the far field and algebraic behavior in the core. Blow-up coordinates in general were reviewed in Section 2.3.

This chapter is organized as follows. In Section 3.2, we construct the core manifold. In Section 3.3, we establish the existence and expansions of the far-field center-stable and center manifolds, as well as the stable foliation. We also derive the reduction of equation (1.1) to the center manifold. Finally, we carry out the matching analysis between the core and far-field manifolds. In Section 3.4, we analyze the flow on the center manifold using geometric blow-up techniques, in order to find the slowly decaying solution \( A_*(r) \).

### 3.2 Bounded solutions near the core

In this section, we construct the set of all bounded solutions of the planar, radial, stationary forced Ginzburg–Landau equation

\[
0 = (1 + i\alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma \bar{u} \quad (2.1)
\]

on \( r \in [0, r_0] \) with \( r_0 < \infty \) fixed.
3.2.1 The linearization about $u = 0$

We decompose $u$ into its real and imaginary parts via $u = u_1 + i u_2$ with $u_1, u_2 \in \mathbb{R}$, and define $\tilde{U} = (u_1, u_2)^T \in \mathbb{R}^2$. Motivated by the one-dimensional spatial dynamics discussed in Section 2.2.2, we set $\mu_m := \alpha \omega + \frac{m^2}{2}$ with $m^2 > 0$, and pick $\gamma_m$ so that $(\mu_m, \gamma_m) \in \Gamma_0$. Lastly, we rescale the radial variable $r$ via $r \mapsto r/\sqrt{\alpha^2 + 1}$. The resulting equation is

$$0 = \tilde{U}_{rr} + \frac{1}{r} \tilde{U}_r - C_1 \tilde{U} - \epsilon^2 C_2 \tilde{U} - |\tilde{U}|^2 C_3 \tilde{U}$$

(2.2)

where

$$C_1 := \begin{pmatrix} \frac{m^2}{2} - \gamma_m & \omega + \alpha \mu_m + \alpha \gamma_m \\ -\omega - \alpha \mu_m + \alpha \gamma_m & \frac{m^2}{2} + \gamma_m \end{pmatrix},$$

$$C_2 := \begin{pmatrix} 1 & -\alpha \\ -\alpha & -1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 + \alpha \beta & \alpha - \beta \\ \beta - \alpha & 1 + \alpha \beta \end{pmatrix}.$$  

Equation (2.2) can also be written as the first-order system

$$\partial_r U = A(1/r)U + F(U, \epsilon^2)$$

(2.3)
\[ \mathbf{U} = (\tilde{U}, \tilde{V})^T \in \mathbb{R}^4, \]

where
\[ \mathcal{A}(\kappa) := \begin{pmatrix} 0 & I \\ C_1 & -\kappa I \end{pmatrix} \]
and
\[ \mathcal{F} \left( (\tilde{U}, \tilde{V})^T ; \nu \right) = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} \]
\[ \mathcal{F} \left( (\tilde{U}, \tilde{V})^T ; \nu \right) := \begin{pmatrix} 0 \\ \nu C_2 \tilde{U} + |\tilde{U}|^2 C_3 \tilde{U} \end{pmatrix}. \]

We linearize (2.3) at \( \epsilon = 0 \) about \( U \equiv 0 \) and obtain
\[ V_r = \mathcal{A}(1/r)V. \quad (2.4) \]

Before giving a set of linearly independent solutions to equation (2.4), we first consider the eigenvalues and eigenvectors of \( \mathcal{A}(0) \) where
\[ \mathcal{A}(0) := \begin{pmatrix} 0 & I \\ C_1 & 0 \end{pmatrix} \]
is independent of \( r \), and where \( \mathcal{A}(1/r) \rightarrow \mathcal{A}(0) \) as \( r \rightarrow \infty \). The eigenvalues and eigenvectors of \( \mathcal{A}(1/r) \) are related to those of the matrix \( C_1 \). The matrix \( C_1 \) has eigenvalues \( \lambda_0 = 0 \) and \( \lambda_1 = m^2 \) with associated eigenvectors
\[ \tilde{U}_0 = \begin{pmatrix} \alpha \gamma_m + \omega + \alpha \mu_m \\ \gamma_m - \frac{m^2}{2} \end{pmatrix}, \quad \tilde{U}_1 = \begin{pmatrix} \alpha \gamma_m + \omega + \alpha \mu_m \\ \gamma_m + \frac{m^2}{2} \end{pmatrix}, \]
respectively. Then the eigenvalues of the matrix \( \mathcal{A}(0) \) are \( \nu^c = 0 \) (with multiplicity
We now return to equation (2.4): $V_r = A(1/r)V$. The four linearly independent solutions $\{V_j(r)\}_{j=1}^4$ to (2.4) are given by

$$V_1 = \begin{pmatrix} \tilde{U}_0 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \tilde{U}_0 \ln r \\ \tilde{U}_0 r \end{pmatrix},$$

$$V_3 = \begin{pmatrix} \tilde{U}_1 I_0(mr) \\ \tilde{U}_1 mI_1(mr) \end{pmatrix}, \quad V_4 = \begin{pmatrix} \tilde{U}_1 K_0(mr) \\ -\tilde{U}_1 mK_1(mr) \end{pmatrix},$$

(2.6)

where $I_0(z)$ and $K_0(z)$ are the zeroth-order modified Bessel functions [1, §9.6]. The asymptotic behaviors of $I_0(z)$ and $K_0(z)$ will be important in the analysis and are displayed in Table 3.1. Note that only $V_1(r)$ and $V_3(r)$ are bounded as $r \to 0$.

For $r$ large enough, we will see that the far-field stable and unstable manifolds will remain close, in an appropriate sense, to the stable and unstable subspaces $E_{r+}^s$. 

### Table 3.1: The asymptotic behavior of the zeroth-order modified Bessel functions for small and large arguments quoted from [1, (9.6.12)-(9.6.13), (9.7.1)-(9.7.2)], respectively, where $\gamma = \lim_{n \to \infty} \left( \sum_{j=1}^n \frac{1}{n} - \ln n \right)$ is the Euler-Mascheroni constant.

<table>
<thead>
<tr>
<th>$z \to 0$</th>
<th>$z \to \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_0(z)$</td>
<td>$1 + O(z^2)$</td>
</tr>
<tr>
<td>$K_0(z)$</td>
<td>$-\ln \left( \frac{z}{2} \right) I_0(z) - \gamma + O(z^2)$</td>
</tr>
</tbody>
</table>
and $E_{r+}^u$, respectively. We will also see that the space $E_{r+}^c$ gives appropriate center-manifold coordinates. Finally, we note that normalized solution vectors $V_3(r)/|V_3(r)|$ and $V_4(r)/|V_4(r)|$ converge to unit vectors in $E_{r+}^u$ and $E_{r+}^s$, respectively. Furthermore, the center subspace $E_{r+}^c$ is actually invariant under the linearization (2.4), as discussed in the following remark.

**Remark 3.2.1.** We have $\text{span}\{V_1(r), V_2(r)\} = E_{r+}^c$ for all $r$ since

$$c_1V_1(r) + c_2V_2(r) = (c_1 + c_2\ln r)V_1^0 + \frac{c_2}{r}V_2^0$$

for all $c_1, c_2 \in \mathbb{R}$, where $V_1^0 := V_1$ and $V_2^0 := (0, \tilde{U}_0)^T$.

### 3.2.2 Construction of the core manifold

We now construct the set of small-amplitude solutions to (2.3) that stay bounded on intervals of the form $[0, r_0]$ with $0 < r_0 < \infty$ fixed. We call this set the “core manifold” and denote it by $\tilde{W}^{cu}_0$. We define $P_{r_0}^{cu}(r_0)$ to be the projection onto $\text{span}\{V_1(r_0), V_3(r_0)\}$ along $\text{span}\{V_2(r_0), V_4(r_0)\}$.

**Definition 3.2.2.** Throughout this chapter, we say that $f(x) = O_{r_0}(x)$ if, for each fixed $r_0$, there are positive constants $C = C(r_0)$ and $\delta = \delta(r_0)$ such that $|f(x)| \leq C|x|$ for all $x$ with $|x| < \delta$. An analogous convention holds for all other Landau symbols used in this chapter.

**Lemma 3.2.3.** Fix $0 < r_0 < \infty$ and let $d = (d_1, d_3) \in \mathbb{R}^2$. Then there exist constants $\rho_1, \rho_2, \epsilon_0 > 0$ so that, for $\epsilon \leq \epsilon_0$,

$$\tilde{W}^{cu}_0(\epsilon) := \left\{ (U(r), r) \left| \begin{array}{l} U(r) \text{ satisfies (2.3) for } r \in [0, r_0] \text{ with } \\ \sup_{0 \leq r \leq r_0} |U(r)| < \rho_1, \ |P_{r_0}^{cu}(r_0)U(r_0)| < \rho_2 \end{array} \right. \right\}$$
is a smooth three-dimensional submanifold of \( \mathbb{R}^5 \). Moreover, there are smooth functions \((g_2, g_4)(d_1, d_3; \epsilon)\) with \((g_2, g_4)(d_1, d_3; \epsilon) = O_{r_0}(\epsilon^2|d| + |d|^3)\) so that \( U \in \widetilde{W}_{cu}^\epsilon(\epsilon) \) if, and only if,

\[
U(r_0) = d_1 V_1(r_0) + g_2(d_1, d_3; \epsilon) V_2(r_0) + d_3 V_3(r_0) + g_4(d_1, d_3; \epsilon) V_4(r_0)
\]

(2.7)

with \(|d| = |(d_1, d_3)| < \rho_2\).

**Proof.** The proof follows from a standard application of the variation-of-constants formula on a bounded interval as in, for instance, [36, proof of Lemma 1]. The details are contained in Appendix B.

Due to Remark 3.2.1, equation (2.7) is equivalent to

\[
U(r_0) = (d_1 + g_2(d_1, d_3; \epsilon) \ln r_0) V_1^0 + \frac{1}{r_0} g_2(d_1, d_3; \epsilon) V_2^0 + d_3 V_3(r_0) + g_4(d_1, d_3; \epsilon) V_4(r_0).
\]

(2.8)

We remark that we will consider the fiber \( \widetilde{W}_{cu}^\epsilon(\epsilon) \big|_{r=r_1} \) for each fixed \( r_1 \in [0, r_0] \) as a two-dimensional submanifold of \( \mathbb{R}^4 \).

### 3.3 Far-field dynamics and matching with the core

We have constructed the core manifold on bounded intervals. We now turn our attention to the far field. We recall (2.3), the forced complex Ginzburg–Landau
equation,

$$\partial_r U = \mathcal{A}(1/r)U + \mathcal{F}(U, \epsilon^2),$$  \hspace{1cm} (3.1)$$

where

$$\mathcal{A}(\kappa) = \begin{pmatrix} 0 & I \\ C_1 & -\kappa I \end{pmatrix}, \quad \mathcal{F}((\bar{U}, \bar{V})^T; \nu) = \begin{pmatrix} 0 \\ \nu C_2 \bar{U} + |\bar{U}|^2 C_3 \bar{U} \end{pmatrix}.$$  

We augment (3.1) near $r = \infty$ with $\kappa = 1/r$. The resulting vector field

$$\begin{pmatrix} U \\ \kappa \end{pmatrix}_r = \begin{pmatrix} \mathcal{A}(\kappa)U + \mathcal{F}(U; \epsilon^2) \\ -\kappa^2 \end{pmatrix}$$  \hspace{1cm} (3.2)$$

is autonomous with linearization about the fixed point $(U, \kappa) = (0, 0)$ given by the system $V_r = \mathcal{A}(0)V$ and $\rho_r = 0$. As discussed in Section 3.2.1, the equation for $V$ has two center directions, one unstable direction, and one stable direction given by the subspaces $E^c_{r+}, E^u_{r+},$ and $E^s_{r+}$ from (2.5), respectively. Taking the additional $\kappa$-direction into account, we therefore expect to find a four-dimensional center-stable manifold and a three-dimensional center manifold near $U = 0$.

In this section, we prove the existence of these manifolds. As illustrated in Figure 3.4, we can write the center-stable manifold $W^{cs}_{r+}(\epsilon)$ as the stable foliation $\{\mathcal{F}^s(p, \kappa)\}_{p \in W^c_{r+}(\epsilon)|_{\kappa=1/r}}$ with base points in the center manifold $W^c_{r+}(\epsilon) \subset W^{cs}_{r+}(\epsilon)$: a trajectory with initial data $(q, \kappa)$ converges exponentially to zero as $r \to \infty$ if, and only if, its associated base point with initial data $(p, \kappa)$ on the center manifold does. The far-field stable manifold, consisting by definition of all solutions for which $U(r) \to 0$ as $r \to \infty$, is therefore given by the union of the stable fibers associated with decaying solutions on the center manifold. The remaining steps for finding lo-
calized solutions are therefore to (i) derive an expansion for the vector field restricted to the center manifold, (ii) analyze the flow on the center manifold, and (iii) match the resulting far-field stable manifold with the core manifold. In this section, we will carry out steps (i) and (iii), anticipating the results for step (ii) which will be carried out in Section 3.4.

Throughout this section, all invariant manifolds will be considered as subsets of $\mathbb{R}^4 \times \mathbb{R}$; we will consider their restrictions to $\kappa = \kappa_1$, for each fixed $\kappa_1$, as submanifolds of $\mathbb{R}^4$.

### 3.3.1 Existence of a center-stable manifold

The existence of center-stable and center manifolds are standard; see, for example, [61]. However, since we will need specific properties of these manifolds, we show here briefly how the results of [61] apply. First, we control the nonlinear terms via a cutoff function: let $\chi(z)$ be a smooth cutoff function with $\chi(z) = 1$ for $z \leq 1$ and
\chi(z) = 0 \text{ for } z \geq 2 \text{ and define, for } \rho \text{ small enough, the modified vector field }

\begin{pmatrix}
\vec{U} \\
\vec{V} \\
\kappa
\end{pmatrix}_r =

\begin{pmatrix}
0 & I & 0 \\
C_1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}

\begin{pmatrix}
\vec{U} \\
\vec{V} \\
\kappa
\end{pmatrix}

+ \begin{pmatrix}
0 \\
-\kappa \vec{V} + \mathcal{F}_2(U, \epsilon^2) \chi \left( \frac{|U|^2}{\rho^2} \right) \chi \left( \frac{\epsilon^4}{\rho^2} \right) \chi \left( \frac{|\kappa|^2}{\rho^2} \right) \\
-\kappa^2 \chi \left( \frac{|\kappa|^2}{\rho^2} \right)
\end{pmatrix}, \quad (3.3)

which then coincides with (3.2) for \(|U|, |\kappa|, \epsilon^2 \leq \rho\) and also satisfies the hypotheses of the invariant-manifold theorems in [61]. Let \(\tilde{S}_r(U_0, \kappa_0)\) represent the solution to (3.3) at time \(r\) starting from \((U, \kappa)(0) = (U_0, \kappa_0)\). It remains to show that these manifolds satisfy certain properties which we will need later on.

**Proposition 3.3.1.** Fix \(\ell \geq 2\). Then there exist \(\epsilon_0 > 0, \rho_0 > 0\) so that, for all \(0 \leq \epsilon \leq \epsilon_0\) and \(0 < \rho \leq \rho_0\) equation (3.2) possesses a flow-invariant four-dimensional center-stable manifold \(W_{cs}^{r+}(\epsilon)\) near the equilibrium \((U, \kappa, \epsilon) = 0\). The manifold \(W_{cs}^{r+}(\epsilon)\) is \(C^\ell\), depends \(C^\ell\) on \(\epsilon^2\) and contains all solutions to (3.2) with \(\sup_{r \geq 0} \{|U(r)|, |\kappa(r)|, |\epsilon|\} \leq \rho\). Furthermore, \(W_{cs}^{r+}(\epsilon)\) satisfies the following properties:

(i) there exists a smooth, monotonically decreasing function \(\tilde{\kappa}_\rho(r)\) with \(|\tilde{\kappa}_\rho(r)| \leq 2\rho\) so that for every \(0 \leq \epsilon \leq \epsilon_0\),

\[U_*(r)|_{r \geq 1/\rho} := \{(U, \kappa) = (0, \tilde{\kappa}_\rho(r))\}|_{|\kappa| \leq \rho} \in W_{cs}^{r+}(\epsilon), \quad \text{and} \quad \tilde{\kappa}_\rho(r)|_{r \geq 1/\rho} = 1/r.\]

In particular, \(0 \in W_{cs}^{r+}(\epsilon)|_{\kappa = 1/r}\); and

(ii) \(T_{U_*(r)}W_{cs}^{r+}(0)|_{\kappa = 1/r} = \text{span} \{V_1^0, V_2^0, V_4(r)\}\) for all \(0 \leq \kappa \leq \rho\), where we recall
from (2.6) that the functions \(\{V_j(r)\}_{j=1}^{4}\) are the solutions to \(V_r = A(\kappa)V\):

\[
V_1(r) = \begin{pmatrix} \tilde{U}_0 \\ 0 \end{pmatrix}, \quad V_2(r) = \begin{pmatrix} \tilde{U}_0 \ln |r| \\ \tilde{U}_0 \end{pmatrix}, \\
V_3(r) = \begin{pmatrix} \tilde{U}_1 \ln(\gamma r) \\ \tilde{U}_1 I_0(\gamma r) \end{pmatrix}, \quad V_4(r) = \begin{pmatrix} \tilde{U}_1 K_0(\gamma r) \\ -\tilde{U}_1 K_1(\gamma r) \end{pmatrix},
\]

(3.4)

and where \(V^0_1 := \begin{pmatrix} \tilde{U}_0 \\ 0 \end{pmatrix}, V^0_2 := \begin{pmatrix} 0 \\ \tilde{U}_0 \end{pmatrix}\) were defined in Remark 3.2.1 so that

\[\text{span}\{V_1(r), V_2(r)\} = \text{span}\{V^0_1, V^0_2\} = E^c_{r+}\]

with \(V^0_1\) and \(V^0_2\) independent of \(r\).

**Proof.** Let \(\rho_0\) be such that (3.3) satisfies the hypotheses of [61, Theorem 5.3]. Then, for every \(0 < \rho \leq \rho_0\) there exists a smooth global center-stable manifold near the equilibrium \(U = 0\), denoted \(\widetilde{W}^{cs}_{r+}(\epsilon)\), with

\[
\widetilde{W}^{cs}_{r+}(\epsilon) := \left\{ (U_0, \kappa_0) \in \mathbb{R}^5 : \sup_{r \geq 0} |P^{\mu}_{r+}(\kappa_0)(U_0, \kappa_0)| < \infty \right\},
\]

where the operators \(P^{j}_{r+}\) are the complementary projections onto the subspaces \(E^{j}_{r+}\) for \(j \in \{s, u, c\}\). For every \(\epsilon \leq \rho\) define \(W^{cs}_{r+}(\epsilon) := \widetilde{W}^{cs}_{r+}(\epsilon) \cap \{(U, \kappa) : |U| \leq \rho, |\kappa| \leq \rho\}\) to obtain a center-stable manifold for (3.2). It remains to verify properties (i) and (ii).

(i) Direct substitution shows that \(U \equiv 0\) solves the evolution equation in \(U\) for the modified equation (3.3); it remains to solve \(\kappa_r = -\kappa^2 \chi(|\kappa|^2/\rho^2)\) for \(\tilde{\kappa}_\rho(r)\). First intersect with the set \(\tilde{\kappa}_\rho(r) \leq \rho\) to show \(\tilde{\kappa}_\rho(r)|_{r \geq 1/\rho} = 1/r\). Then observe that \(\tilde{\kappa}_\rho(r)\) is monotone since \(\chi(\cdot)\) is nonnegative. Smoothness follows from the smoothness of \(\chi(\cdot)\). Monotonicity and \(\chi(2) = 0\) together show \(|\tilde{\kappa}_\rho(r)| \leq 2\rho\) for all \(r\). Boundedness of \(\tilde{\kappa}_\rho(r)\) shows \(U_*(r) \in \widetilde{W}^{cs}_{r+}(\epsilon)\).
(ii) We solve the linearization of (3.3) about $U_*(r)$ with $\epsilon = 0$

\[
\begin{pmatrix}
\tilde{U} \\
\tilde{V}
\end{pmatrix}
= \begin{pmatrix}
0 & I \\
C_1 & -\tilde{\kappa}_\rho(r)\chi(|\tilde{\kappa}_\rho(r)|^2/\rho^2)
\end{pmatrix}
\begin{pmatrix}
U \\
V
\end{pmatrix}
\]

(3.5a)

\[
\tilde{\kappa}_r = -2\left[\tilde{\kappa}_\rho(r)\chi(|\tilde{\kappa}_\rho(r)|^2/\rho^2) + \tilde{\kappa}_\rho^3(r)/\rho^2\chi'(|\tilde{\kappa}_\rho(r)|^2/\rho^2)\right] \tilde{\kappa}.
\]

(3.5b)

Since $\tilde{\kappa}(r)$ decouples from the rest of the system we set $\tilde{\kappa}(r) = 0$ and solve (3.5a).

Since $\tilde{\kappa}_\rho(r)$ is positive and monotonically decreasing we solve forward until $\tilde{\kappa}_\rho(r) \leq \rho$ so that $\chi(\cdot) = 1$. Now (3.5a) reduces to $V_r = \tilde{A}(m, \kappa)V$ solved by span $\{V_j(r)\}_{j=1}^4$.

Observing that $V_3(r)$ grows exponentially as $r \to \infty$ completes the proof. 

3.3.2 Existence of a center manifold

Next we show the existence of a center manifold $W^c_{r+} \subset W^{cs}_{r+}$ for (3.2) with $U_*(r) \in W^c_{r+}$.

**Proposition 3.3.2.** Fix $\ell \geq 2$. Then there exist $0 < \epsilon_1 \leq \epsilon_0$ and $0 < \rho_1 \leq \rho_0$ so that, for every $0 \leq \epsilon \leq \epsilon_1$ and $0 < \rho \leq \rho_1$, equation (3.2) possesses a flow-invariant three-dimensional $C^\ell$-center manifold $W^c_{r+}(\epsilon)$ near the equilibrium $(U, \kappa, \epsilon) = 0$, which contains all solutions with $\sup_{r \in \mathbb{R}}\{|U(r)|, |\kappa(r)|, |\epsilon|\} \leq \rho$. The center manifold depends $C^\ell$ on $\epsilon^2$ and has the following additional properties:

(i) for all $0 \leq \epsilon \leq \epsilon_1$, $W^c_{r+}(\epsilon) \subset W^{cs}_{r+}(\epsilon)$;

(ii) for all $0 \leq \epsilon \leq \epsilon_1$, $U_*(r) \in W^c_{r+}(\epsilon)$;

(iii) $T_{U_*(r)}W^c_{r+}(0)|_{\kappa=1/r} = E^c_{r+}$ for $r \geq 1/\rho$; and

(iv) the flow on $W^c_{r+}(\epsilon)$ respects the actions
\textbf{Remark 3.3.3.} We emphasize that property (iii) states in particular that the tangent space $T_{U^*_s(r)}W_{r^+}^c(0)$ is independent of $\kappa = 1/r$ and hence $E_{r^+}^c$ is invariant under the linearization about $U_s(r)$.

Before we prove Proposition 3.3.2, we first state and prove the following result, which will be used in the proof of property (iii).

\textbf{Lemma 3.3.4.} There exist functions $a_j(r), b_j(r), j \in \{2, 3, 4\}$, so that all solutions to (3.5a) are given by linear combinations of $\{\tilde{V}_j\}_{j=1}^4$ where

$$\tilde{V}_1(r) := V_1 = \begin{pmatrix} \tilde{U}_0 \\ 0 \end{pmatrix}, \quad \tilde{V}_2(r) := \begin{pmatrix} a_2(r)\tilde{U}_0 \\ b_2(r)\tilde{U}_0 \end{pmatrix},$$

$$\tilde{V}_3(r) := \begin{pmatrix} a_3(r)\tilde{U}_1 \\ b_3(r)\tilde{U}_1 \end{pmatrix}, \quad \tilde{V}_4(r) := \begin{pmatrix} a_4(r)\tilde{U}_1 \\ b_4(r)\tilde{U}_1 \end{pmatrix}.$$  

Moreover, $\tilde{V}_2(r)$ grows at most exponentially with rate $m/4$ as $r \to \pm\infty$, whereas $\tilde{V}_3(r)$ and $\tilde{V}_4(r)$ grow exponentially with rate at least $m/2$ as $r \to \infty$ and $r \to -\infty$, respectively.

\textbf{Proof.} By construction $\tilde{V}_3$ and $\tilde{V}_4$ are linearly independent of $\tilde{V}_1$ and $\tilde{V}_2$ since $\tilde{U}_0$ and $\tilde{U}_1$ are. It remains to show that

(i) $\tilde{V}_1, \tilde{V}_2$ are linearly independent of each other with $\tilde{V}_2$ bounded by weak exponential growth for all $r \in \mathbb{R}$; and
(ii) \( \tilde{V}_3 \) and \( \tilde{V}_4 \) are linearly independent with each solution growing exponentially quickly either for \( r \to +\infty \) or \( r \to -\infty \).

(i) We begin with \( \tilde{V}_1 \) and \( \tilde{V}_2 \). \( \tilde{V}_1 \) is a solution to (3.5) for every \( r \) since \( C_3(m)\tilde{U}_0 = 0 \) and is clearly bounded for all \( r \in \mathbb{R} \). Substituting \( \tilde{V}_2(r) \) into (3.5) we find that \( a_2(r), b_2(r) \) satisfy

\[
\begin{align*}
\partial_r a_2 &= b_2(r) \\
\partial_r b_2 &= -\tilde{\kappa}(r) \chi \left( |\tilde{\kappa}(r)|^2 / \rho^2 \right) b_2(r).
\end{align*}
\]

Restricting to the set \( \tilde{\kappa}(r) \leq \rho \), we explicitly solve to find \( b(r) = 1/r \) and \( a(r) = \ln(r) \). Solving backwards we find \( a_2(r), b_2(r) \) are smooth functions with weak exponential growth. In particular, \( 0 \leq \tilde{\kappa}(r) \leq 2\rho \) for all \( r \in \mathbb{R} \) means \( |b_2(r)| \leq |b_0|e^{2\rho|r|} \) and \( |a_2(r)| \leq |a_0|+|cb_0|e^{2\rho|r|} \). It is straightforward to observe \( \tilde{V}_2(r) \) is linearly independent of \( V_1(r) \) provided \( r < \infty \).

(ii) Next to show independence of \( \tilde{V}_3(r) \) and \( \tilde{V}_4(r) \). We substitute \( \tilde{V}_{3/4} \) into (3.5) using \( C_3\tilde{U}_1 = m^2\tilde{U}_1 \) and augmenting with \( \tilde{\kappa}_\rho(r) = -\kappa_\rho^2 \chi \left( \frac{\kappa_\rho^2}{\rho^2} \right) \) to make the system autonomous. Then

\[
\begin{pmatrix}
a_j \\
b_j \\
\tilde{\kappa}_\rho
\end{pmatrix}
\begin{pmatrix}
a_j \\
b_j \\
\tilde{\kappa}_\rho
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 \\
m^2 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a_j \\
b_j \\
\tilde{\kappa}_\rho
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\chi \left( \frac{|\tilde{\kappa}_\rho|^2}{\rho^2} \right) \\
0 & 0 & -\kappa_\rho \chi \left( \frac{\kappa_\rho^2}{\rho^2} \right)
\end{pmatrix}
\begin{pmatrix}
a_j \\
b_j \\
\tilde{\kappa}_\rho
\end{pmatrix}
\]

\[
:= \mathcal{A} \begin{pmatrix}
u_j \\
\tilde{\kappa}_\rho
\end{pmatrix}
+ B_\rho(u_j, \tilde{\kappa}_\rho),
\]

with \( u_j = (a_j, b_j)^T \).
The matrix $A$ has eigenvalues $\lambda = -m, m, 0$ with associated eigenvectors $e^s := (1, -m, 0)^T$, $e^u := (1, m, 0)^T$ and $e^c := (0, 0, 1)^T$, respectively. From Lemma 3.3.1(a) we have $\kappa_\rho(r)$ forms a heteroclinic connection in three dimensional space between $p_- := (0, 0, c)$ and $p_+ := (0, 0, 0)$ with $0 < c \leq 2\rho$. We linearize (3.6) about $p_-$ and $p_+$

\[
\begin{pmatrix} v_j \\ \pi_\rho \end{pmatrix}_r = (A + DB_\rho(p_\pm)) \begin{pmatrix} v_j \\ \pi_\rho \end{pmatrix}
\]

using the observations: (1) $||B_\rho|| = 2\rho \ll 1$ and (2) $DB_\rho(0, 0, C) = O(C)$. Then, by [31, Thm. 1.8] for $\rho$ small enough, the invariant subspaces of $A + DB_\rho(p_\pm)$ are $E_{p_+}^s = e^s$, $E_{p+}^u = e^u$ and $E_{p-}^s = e^s + O(\rho)$, $E_{p-}^u = e^u + O(\rho)$ respectively. Note $E_{p\pm}^c = e^c$ for both linearizations since $\pi_\rho$ decouples from the rest of the system.

We then use the variation of constants formula to define $W_{p-}^u$ and $W_{p+}^u$, the standard unstable manifold of $p_-$ and stable manifold of $p_+$. A standard contraction mapping argument shows that $W_{p-}^u|_{r=0} = e^u + O(\rho)$ and $W_{p+}^s|_{r=0} = e^s + O(\rho)$. We omit the details; see Appendix B for a similar argument. Thus, $u_3 = W_{p-}^u$ and $u_4 = W_{p+}^s$ are linearly independent solutions to (3.6), provided $\rho$ is small enough, since $e^s$ and $e^u$ are. It then follows from the stable and unstable manifold theorems that $u_3(r)$ grows exponentially for $r \to \infty$ while $u_4(r)$ grows exponentially for $r \to -\infty$.

We are now ready to prove Proposition 3.3.2. We recall that $\tilde{S}_\epsilon(U_0, \kappa_0)$ represents the solution to (3.3) at time $r$ starting from $(U, \kappa)(0) = (U_0, \kappa_0)$. We also recall that we need to verify the following properties:

(i) for all $0 \leq \epsilon \leq \epsilon_1$, $W_{r+}^c(\epsilon) \subset W_{r+}^{cs}(\epsilon)$;
(ii) for all $0 \leq \epsilon \leq \epsilon_1$, $U_*(r) \in \mathcal{W}^c_{r+}(\epsilon)$;

(iii) $T_{U_*(r)}\mathcal{W}^c_{r+}(0)|_{\kappa=1/r} = E^c_{r+}$ for $r \geq 1/\rho$; and

(iv) the flow on $\mathcal{W}^c_{r+}(\epsilon)$ respects the $\mathbb{Z}_2$ and reverser symmetries.

**Proof of Proposition 3.3.2** As in the proof of Proposition 3.3.1, we apply [61, Theorem 2.1] to (3.3) near $(U,\kappa) = 0$ for $\rho$ small enough to show the existence of a smooth global center manifold $\tilde{W}^c_{r+}(\epsilon)$ with

$$\tilde{W}^c_{r+}(\epsilon) := \left\{ (U_0,\kappa_0)^T \in \mathbb{R}^5 : \sup_{r \in \mathbb{R}} |P^{SU}_{r+}\tilde{S}^{\epsilon}_r(U_0,\kappa_0)| < \infty \right\}.$$  

We then define $\mathcal{W}^c_{r+}(\epsilon) := \tilde{W}^c_{r+}(\epsilon) \cap \{ (U,\kappa) : |U| \leq \rho, |\kappa| \leq \rho \}$ to obtain a center manifold for (3.2). It remains to verify properties (i)-(iv).

(i) Comparing the definitions of the center-stable and center manifolds constructed in [61] shows that $\tilde{W}^c_{r+}(\epsilon) \subset \tilde{W}^{cs}_{r+}(\epsilon)$. The containment remains true after intersection with $\{ (U,\kappa) : |U| \leq \rho, |\kappa| \leq \rho \}$.

(ii) By the same argument as in Lemma 3.3.1(a) we see that $U_*(r)$ remains bounded for all $r \in \mathbb{R}$.

(iii) We again linearize (3.3) about $U_*(r)$ with $\epsilon = 0$ to obtain (3.5). Again, since $\bar{\kappa}(r)$ decouples from the rest of the system we set $\bar{\kappa}(r) = 0$ and solve (3.5a). We then apply Lemma 3.3.4 to obtain $T_{U_*(r)}\tilde{W}^c_{r+}(0)|_{\kappa=1/r} = \text{span} \{ \tilde{V}_1, \tilde{V}_2 \}$. Intersection with $|\bar{\kappa}| < \rho$ shows $T_{U_*} \mathcal{W}^c_{r+}(0) = \text{span} \{ V_1, V_2 \}$, since we know from the proof of Lemma 3.3.4 that $\tilde{V}_2(r)|_{|\bar{\kappa}| \leq \rho} = V_2$.

(iv) Equation (3.2) respects the $\mathbb{Z}_2$ symmetry $(\tilde{U},\tilde{U}_r,\kappa, r) \mapsto (-\tilde{U}, -\tilde{U}_r, \kappa, r)$ and the reverser $(\tilde{U},\tilde{U}_r,\kappa, r) \mapsto (\tilde{U}, -\tilde{U}_r, -\kappa, -r)$. The cutoff $\chi(|z|)$ is symmetric in $z$ so
that (3.3) respects these actions as well. Then

\[(\tilde{U}(r), \tilde{V}(r), \kappa(r)) \in \tilde{W}_{r+}^c(\epsilon) \iff (-\tilde{U}(r), -\tilde{V}(r), \kappa(r)) \in \tilde{W}_{r+}^c(\epsilon)\]

since \(|U| = | - U|\) and

\[(\tilde{U}(r), \tilde{V}(r), \kappa(r)) \in \tilde{W}_{r+}^c(\epsilon) \iff (\tilde{U}(-r), -\tilde{V}(-r), -\kappa(-r)) \in \tilde{W}_{r+}^c(\epsilon)\]

since the hyperbolic projections of solutions in \(\tilde{W}_{r+}^c(\epsilon)\) are bounded in both forward and backward \(r\). Together this implies that (3.3) respects the symmetries on \(\tilde{W}_{r+}^c(\epsilon)\). These symmetries are preserved after intersection with \(\{(U, \kappa) : |U| \leq \rho, |\kappa| \leq \rho\}\), again because \(|U| = | - U|\) and \(|\kappa| = | - \kappa|\).

**3.3.3 Strong stable foliations**

In this section we show \(W_{r+}^{cs}(\epsilon)\) is given as the union of strong stable fibers over base points in \(W_{r+}^c(\epsilon)\). We refer to [12] for background on stable foliations and to Figure 3.4 for an illustration. Let \(S_r^s(U_0, \kappa_0)\) represent the solution to (3.2)

\[
\begin{pmatrix}
\tilde{U} \\
\tilde{V} \\
\kappa
\end{pmatrix}
= 
\begin{pmatrix}
0 & I & 0 \\
C_1 & -\kappa I & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{U} \\
\tilde{V} \\
\kappa
\end{pmatrix}
+ 
\begin{pmatrix}
0 \\
\epsilon^2 C_2 \tilde{U} + |\tilde{U}|^2 C_3 \tilde{U} \\
-\kappa^2
\end{pmatrix}
\]

at time \(r\) with initial data \((U, \kappa)(0) = (U_0, \kappa_0)\). We recall from (2.5) and Remark 3.2.1 the definitions

\[
V_1^0 = 
\begin{pmatrix}
\tilde{U}_1 \\
0
\end{pmatrix}, 
V_2^0 = 
\begin{pmatrix}
0 \\
\tilde{U}_0
\end{pmatrix}, 
E_r^s = 
\begin{pmatrix}
\tilde{U}_1 \\
-m\tilde{U}_1
\end{pmatrix}, 
E_r^u = 
\begin{pmatrix}
\tilde{U}_1 \\
m\tilde{U}_1
\end{pmatrix}.
\]
Lemma 3.3.5. Fix \( \ell \geq 1 \) and a decay rate \( \nu \) with \( 0 < \nu < m \). Define \( \epsilon_1, \rho_1 \) as in Lemma 3.3.2 so that the manifolds \( W_{r+}^c(\epsilon)\big|_{\kappa=1/r} \) and \( W_{r+}^c(\epsilon)\big|_{\kappa=1/r} \) exist. For every \( \epsilon \leq \epsilon_1 \), any \( \kappa = 1/r \leq \rho_1 \), and each \( p \in W_{r+}^c(\epsilon)\big|_{\kappa=1/r} \), there exists a one-dimensional strong stable fiber \( F^s_\epsilon(p, \kappa) \) in \( \mathbb{R}^4 \) so that the following are true:

(i) \( p \in F^s_\epsilon(p, \kappa) \) for all \( p \in W_{r+}^c(\epsilon)\big|_{\kappa=1/r} \);

(ii) \( W_{r+}^c(\epsilon)\big|_{\kappa=1/r} = \bigcup_{p \in W_{r+}^c(\epsilon)\big|_{\kappa=1/r}} F^s_\epsilon(p, \kappa) \);

(iii) \( F^s_\epsilon(\cdot, \kappa) \) depends \( \mathcal{C}^\ell \) on \( \epsilon^2 \) and \( \kappa \);

(iv) for every \( \epsilon, \kappa \), there exists a \( \mathcal{C}^\ell \times \mathcal{C}^\ell \) function \( \phi^s_\epsilon(\cdot, \kappa, \cdot) : W_{r+}^c(\epsilon)\big|_{\kappa=1/r} \times \mathbb{R}V_4(r) \to \mathbb{R}V_0^0 \oplus \mathbb{R}V_2^0 \oplus \mathbb{R}V_3(r) \) so that \( F^s_\epsilon(p, \kappa) = \text{graph(} \phi^s_\epsilon(p, \kappa, \cdot) \text{)} \);

(v) \( |S^s_\epsilon(q_1, \kappa - S^s_\epsilon(q_2, \kappa)| = O(e^{-\nu t}) \) for \( r \geq 0 \) and all \( q_1, q_2 \in F^s_\epsilon(p, \kappa) \); and

(vi) \( T_{\mathcal{F}^s_\epsilon(1/r, U_\star(r))} = V_4(r) \).

Proof. Recall that equation (3.3) is the autonomous vector field (3.2) with an appropriate cutoff applied near \( (U, \kappa) = (0, 0) \). For every \( \epsilon \) and \( \rho \) small enough, we can apply [12, Theorem 4.3] to (3.3): we therefore know the existence of stable fibers \( \tilde{F}^s_\epsilon(p') \), with \( p' = (p, \kappa) \in W_{r+}^c(\epsilon) \), which satisfy

(i') \( p' \in \tilde{F}^s_\epsilon(p') \) for all \( p' \in W_{r+}^c(\epsilon) \);

(ii') \( W_{r+}^c(\epsilon) = \bigcup_{p' \in W_{r+}^c(\epsilon)} \tilde{F}^s_\epsilon(p') \);

(iii') \( \tilde{F}^s_\epsilon(\cdot) \) depends \( \mathcal{C}^\ell \) on \( \epsilon^2 \).
(iv') for every $\epsilon$ there exists a $C^\ell \times C^\ell$ function $\tilde{\phi}_s^\epsilon : W^c_{r+}(\epsilon) \times E^s_{r+} \to (\mathbb{R} \oplus E^cu_{r+})$ so that $\tilde{F}_s^\epsilon(p') = \text{graph}(\tilde{\phi}_s^\epsilon(p', \cdot))$;

(v') $|S^s_r(q'_1) - S^s_r(q'_2)| = O(e^{-\nu t})$ for $r \geq 0$ and all $q'_1, q'_2 \in \tilde{F}_s^\epsilon(p')$; and

(vi') $T_{\tilde{F}_0^s}(0, 0) = E^s_{r+}$.

It remains to show that conditions (i'–vi') imply (i–vi).

First, we show that each fiber is completely contained within the Poincare section $\kappa = 1/r$. We then define $\{F_s^\epsilon(\cdot, \kappa)\} := \{\tilde{F}_s^\epsilon(\cdot)\}|_{\kappa=1/r}$ for every $\kappa = 1/r$ small enough; conditions (i'–v') are then equivalent to (i–v). We argue by contradiction: assume that there exists a fiber $\tilde{F}_s^\epsilon(p_0)$ with base point $p_0 = (x_0, \kappa_0)$ and furthermore assume that there exists some $p_1 = (x_1, \kappa_1) \in \tilde{F}_s^\epsilon(p_0)$ with $\kappa_1 \neq \kappa_0$. We use the evolution equation for $\kappa$ to see that $|S^s_r(p_0) - S^s_r(p_1)| \geq |\kappa_0(r) - \kappa_1(r)| \geq C/r^2$ for some $C > 0$ as $r \to \infty$ in contradiction to condition (v').

Next we show that condition (vi') extends to (vi) for $\kappa > 0$. First, we have $U^s_*(r) = \{(U, \kappa) = (0, 1/r)\} \in W^c_{r+}(\epsilon)$ for every $\epsilon$ from Lemma 3.3.1(a). Next, we again linearize about $U^s_*(r)$ with $\epsilon = 0$ to get (3.5). We solve forward in $\kappa$ until $\kappa < \rho$ so that (3.5a) reduces to $V_r = \mathcal{A}(\kappa)V$, which is solved by linear combinations of $V_j(r)$ for $j \in \{1, 2, 3, 4\}$ where

$$V_1(r) = V_1^0, \quad V_2(r) = \begin{pmatrix} \ln |r| \tilde{U}_0 \\ \frac{1}{r} \tilde{U}_0 \end{pmatrix},$$

$$V_3(r) = I_0(mr) \begin{pmatrix} \tilde{U}_1 \\ m \tilde{U}_1 I_1(mr) I_0(mr) \end{pmatrix}, \quad V_4(r) = K_0(mr) \begin{pmatrix} \tilde{U}_1 \\ -m \tilde{U}_1 K_1(mr) K_0(mr) \end{pmatrix}.$$ 

We define $\tilde{V}_j(r) := V_j(r)/|V_j(r)|$ and observe that $\tilde{V}_3(r)$ and $\tilde{V}_4(mr)$ converge to unit
vectors in $E_r^u$ and $E_r^s$, respectively, as $r \to \infty$. We define the normalized tangent vector $a(r)\tilde{V}(r) := T\phi_0^s(U_*(r), \cdot)|_{\kappa=1/r}$ where

$$\tilde{V}(r) = (a_1 + a_2 \ln |r|)V_1^0 + \frac{a_2}{r}V_2^0 + a_3I_0(mr)\tilde{V}_3(r) + a_4K_0(mr)\tilde{V}_4(r)$$

is a solution to the linear flow (3.5a). By condition (vi') we must have

$$a(r)\tilde{V}(r) \to E_r^s as r \to \infty.$$

Taking $a(r) = \frac{1}{K_0(mr)}$ we find

$$T\phi_0^s(U_*(r), \cdot)|_{\kappa=1/r} = \frac{a_1 + a_2 \ln |r|}{K_0(mr)}V_1^0 + \frac{a_2}{rK_0(mr)}V_2^0 + a_3\frac{I_0(mr)}{K_0(mr)}\tilde{V}_3(r) + a_4\tilde{V}_4(r).$$

Using the asymptotic expansion for $K_0(mr)$ in the limit $r \to \infty$ given in Table 3.1 we see that $a_1 = a_2 = a_3 = 0$ and $a_4 = 1$. The proof now follows by uniqueness.

### 3.3.4 Parametrization of $W_{r+}^c(\epsilon)$, $W_{r+}^{cs}(\epsilon)$, and $F^s_\epsilon$

We use the properties of $W_{r+}^c(\epsilon)$, $W_{r+}^{cs}(\epsilon)$, and $F^s_\epsilon$ listed in Sections 3.3.1-3.3.3 to parametrize each of these manifolds as graphs. Each parametrization will be performed near $U_*(r)$. We first collect the relevant results, where $0 < \rho_1 \leq \rho_0 \ll 1$:

(i) For each $\epsilon$, $U_*(r) := \{(U, \kappa) = (0, 1/r)\} \in W_{r+}^c(\epsilon) \subset W_{r+}^{cs}(\epsilon)$

[Propositions 3.3.1(i) and 3.3.2(i-ii)];

(ii) $T_{U_*(r)}W_{r+}^{cs}(0)|_{\kappa=1/r} = \text{span} \{V_1^0, V_2^0, V_4(r)\}$ for all $0 \leq \kappa \leq \rho_0$

[Proposition 3.3.1(ii)];

(iii) $T_{U_*(r)}W_{r+}^c(0)|_{\kappa=1/r} = \text{span} \{V_1^0, V_2^0\}$ for all $0 \leq \kappa \leq \rho_1$ [Proposition 3.3.2(iii)];
(iv) \( T_{U_\kappa(r)} \mathcal{F}_{0}^\kappa(\cdot, \kappa) = V_4(r) \) [Lemma 3.3.5(vi)].

We begin with \( \mathcal{W}^\kappa_{r+}(\epsilon) \). For every \( \kappa \leq \rho_0 \), \( \mathcal{W}^\kappa_{r+}(\epsilon) \big|_{\kappa=1/r} \) can be written as graph of a function \( \hat{h}_3(\cdot; \kappa, \epsilon) : \mathbb{R}V_1^0 \oplus \mathbb{R}V_2^0 \oplus \mathbb{R}V_4(r) \to \mathbb{R}V_3(r) \) so that

\[
\mathcal{W}^\kappa_{r+}(\epsilon) \big|_{\kappa=1/r} = \{ d_1 V_1^0 + d_2 V_2^0 + \hat{h}_3(d_1, d_2, d_4; \kappa, \epsilon)V_3(r) + d_4 V_4(r) : d_1, d_2, d_4 \text{ small} \}.
\]

(3.7a)

Property (i) implies \( \hat{h}_3(0, 0, 0; \kappa, \epsilon^2) = 0 \) for all \( \epsilon \) and property (ii) implies

\[
\hat{h}_3(d_1, d_2, d_4; \kappa, 0) = O_\kappa(|d_1|^2 + |d_2|^2 + |d_4|^2).
\]

Hence,

\[
\hat{h}_3(d_1, d_2, d_4; \kappa, \epsilon) = O_\kappa \left( (|d_1| + |d_2| + |d_4|)(|d_1| + |d_2| + |d_4| + \epsilon^2) \right),
\]

(3.7b)

where we recall that the Landau symbol \( O_\kappa(\cdot) \) is interpreted in the usual sense except that the subscript \( \kappa \) means that the bounding constant and the region where the estimate is valid may depend on \( \kappa \).

Similarly, for every \( \kappa \leq \rho_1 \) we can write \( \mathcal{W}^\kappa_{r+}(\epsilon) \big|_{\kappa=1/r} \) as

\[
\mathcal{W}^\kappa_{r+}(\epsilon) \big|_{\kappa=1/r} = \left\{ \begin{array}{c} d_1 V_1^0 + d_2 V_2^0 + \hat{h}_3(d_1, d_2; \kappa, \epsilon)V_3(r) + \tilde{h}_4(d_1, d_2; \kappa, \epsilon)V_4(r) + \tilde{h}_4(d_1, d_2; \kappa, \epsilon)V_4(r) \mid d_1, d_2 \text{ small} \end{array} \right\}
\]

(3.8a)
where \( \tilde{h}_j(\cdot; \kappa, \epsilon) : V_1^0 \oplus V_2^0 \to V_j(r) \). Property (i) shows \( \tilde{h}_j(0, 0; \kappa, \epsilon^2) = 0 \) for all \( \epsilon \) and property (iii) implies \( \tilde{h}_j(d_1, d_2; \kappa, 0) = O_\kappa(|d_1|^2 + |d_2|^2) \). Therefore,

\[
\tilde{h}_j(d_1, d_2; \kappa, \epsilon) = O_\kappa \left( (|d_1| + |d_2|)(|d_1| + |d_2| + \epsilon^2) \right), \quad j \in \{3, 4\}. \tag{3.8b}
\]

Since \( \mathcal{W}_r^c(\epsilon) \subset \mathcal{W}_r^{cs}(\epsilon) \), we have in particular

\[
\tilde{h}_3(d_1, d_2; \kappa, \epsilon) = \hat{h}_3(d_1, d_2, \tilde{h}_4(d_1, d_2; \kappa, \epsilon); \kappa, \epsilon).
\]

Lastly, we can parametrize the fibers \( \mathcal{F}_r^s(p, \kappa) \) using appropriate functions \( h_j(\cdot, \cdot; \kappa, \epsilon) : \mathcal{W}_{r_+}^c(\epsilon) \big|_{\kappa=1/r} \oplus \mathbb{R}V_4(r) \to \mathbb{R}V_j(r) \) for \( j \in \{1, 2, 3\} \) so that

\[
\mathcal{F}_r^s(p, \kappa) = \left\{ \begin{array}{l}
\quad p + h_1(p, \overline{d}_s; \kappa, \epsilon)V_1^0 + h_2(p, \overline{d}_s; \kappa, \epsilon)V_2^0 \\
\quad + h_3(p, \overline{d}_s; \kappa, \epsilon)V_3(r) + \overline{d}_sV_4(r) \quad | \overline{d}_s \text{ small}
\end{array} \right\}. \tag{3.9a}
\]

In the parametrizations of \( \mathcal{W}_{r_+}^{cs}(\epsilon) \) and \( \mathcal{W}_{r_+}^c(\epsilon) \), the coordinates \( d_j \) were taken relative to \( U = 0 \). In contrast, in our parametrization of \( \mathcal{F}_r^s \), the offset is measured from the base point \( p \) so that \( h_j(p, 0; \kappa, \epsilon) = 0 \); see Figure 3.5. From condition (iv) we also know that \( h_j(0, \overline{d}_s; \kappa, 0) = O_\kappa(|\overline{d}_s|^2) \). Taking these properties together we arrive at the expansion

\[
h_j(p, \overline{d}_s; \kappa, \epsilon) = O_\kappa \left( |\overline{d}_s|(|p| + |\overline{d}_s| + \epsilon^2) \right) \tag{3.9b}
\]

of the function \( h_j(\cdot, \cdot; \kappa, \epsilon) \).

Our goal now is to write the fibers \( \mathcal{F}_r^s \) relative to \( U_*(r) \). Let \( P_1^c, P_2^c, P_4^u(r), \) and \( P^s(r) \) be the complementary projection operators onto the subspaces \( V_1^0, V_2^0, V_3(r), \) and \( V_4(r) \) respectively. We define \( A := P_1^c p, B := P_2^c p \) as the center-manifold coordinates of \( p \) and compute \( P^u(r)p \) and \( P^s(r)p \) using (3.8). Lastly, we use (3.9) to
Figure 3.5: Parametrization of a strong stable fiber within $\mathcal{W}_{r+}^{cs}\big|_{1/r}$ for some $\kappa \leq \rho_1$ fixed.

compute the foliation expansion

$$\mathcal{F}_\epsilon^s(p, \kappa) = \{d_1^c V_1^0 + d_2^c V_2^0 + d^u V_3(r) + d^s V_4(r)\}$$  (3.10a)

where

$$d_1^c := P_1^c p + h_1(p, \bar{d}_s; \kappa, \epsilon) = A + O_\kappa (|\bar{d}_s|(|A| + |B| + |\bar{d}_s| + \epsilon^2))$$

$$d_2^c := P_2^c p + h_2(p, \bar{d}_s; \kappa, \epsilon) = B + O_\kappa (|\bar{d}_s|(|A| + |B| + |\bar{d}_s| + \epsilon^2))$$

$$d^u := P^u(r)p + h_3(p, \bar{d}_s; \kappa, \epsilon) = O_\kappa ((|A| + |B| + |\bar{d}_s|)(|A| + |B| + |\bar{d}_s| + \epsilon^2))$$

$$d^s := P^s(r)p + \bar{d}_s = O_\kappa ((|A| + |B|)(|A| + |B| + \epsilon^2)) + \bar{d}_s.$$  (3.10b)

Since $\mathcal{F}_\epsilon^s(p, \kappa) \subset \mathcal{W}_{r+}^{cs}(\epsilon)\big|_{1/r}$, we could also have used (3.7) to determine

$$d^u = \tilde{h}_3(d_1^c, d_2^c, d^s; \kappa, \epsilon)$$

once the other three components had been computed. The result is the same.
3.3.5 Reduction of the vector field to the center manifold

Next we derive a convenient expansion for the vector field (3.2)

\[
\begin{pmatrix}
\tilde{U} \\
\tilde{V} \\
\kappa
\end{pmatrix}' =
\begin{pmatrix}
0 & I & 0 \\
C_1 & -\kappa I & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{U} \\
\tilde{V} \\
\kappa
\end{pmatrix} +
\begin{pmatrix}
0 \\
\epsilon^2 C_2 \tilde{U} + |\tilde{U}|^2 C_3 \tilde{U} \\
-\kappa^2
\end{pmatrix}
\]

restricted to the center manifold. In Proposition 3.3.2(iv) we showed that the flow on the center manifold respects the \(\mathbb{Z}_2\) symmetry \((\tilde{U}, \tilde{U}_r, \kappa, r) \mapsto (-\tilde{U}, -\tilde{U}_r, \kappa, r)\) and the reverser symmetry \((\tilde{U}, \tilde{U}_r, \kappa, r) \mapsto (\tilde{U}, -\tilde{U}_r, -\kappa, -r)\). We use these symmetries in the proof of the following lemma.

**Lemma 3.3.6.** Using the coordinates \(P^c_{r_+}U = AV_1^0 + BV_2^0\) from the center-manifold parametrization (3.8), with \(A = d_1\) and \(B = d_2\), the vector field (3.2) restricted to \(W^c_{r_+}(\epsilon)\) can be written as

\[
\begin{align*}
A_r &= B + R_A(A, B, \kappa; \epsilon) \\
B_r &= -\kappa B + \epsilon^2 A + \epsilon^0 A^3 + R_B(A, B, \kappa; \epsilon) \\
\kappa_r &= -\kappa^2
\end{align*}
\]

where \(\text{sgn}(\epsilon^0) = \text{sgn}(\mu_m - \beta \omega)\). The remainder terms \(R_A\) and \(R_B\) satisfy

\[
\begin{align*}
R_A(A, B, \kappa; \epsilon) &= O\left(|A|^2 + |B|^2 + |A|^2 |B|\right) \\
R_B(A, B, \kappa; \epsilon) &= O\left(|\epsilon^2 \kappa^2 A| + (\epsilon^2 + \kappa^2)|A|^3 + |A|^5 \right. \\
&\quad \left. + (\epsilon^2 + \kappa^2 + A^2)|\kappa B| + |AB^2| + |B|^3\right).
\end{align*}
\]

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Proof.} The derivation of (3.11) follows from a modification of the proof of [45, Lemma 3.9 in Chapter 3]. We define $u^c := AV_1^0 + BV_2^0$ and $\psi(u^c, \kappa; \epsilon) : E^c \times \mathbb{R} \times \mathbb{R} \to E^s \oplus E^u$ so that $W^c_{r+} = \{(u^c, \kappa, \psi(u^c, \kappa; \epsilon)); |u^c|, |\kappa|, \epsilon^2 \leq \rho_1\}$. By equation (3.8), $\psi(A, B, \kappa; \epsilon) = O(\kappa ((A + B)(A + B + \epsilon^2)))$. We recall from Section 3.2.1 that $\mathcal{F}(U; \epsilon) = O(\epsilon^2 |U| + U^2)$. By Remark 3.3.3, $E^c(r) = E^c$ is independent of $r$ and is invariant under the non-autonomous linearization $V_r = \mathcal{A}(\kappa)V$. Therefore, the reduced vector field on $W^c_{r+}(\epsilon)$, projected onto $E^c_{r+}$, is

$$
\partial_r u^c = \mathcal{A}(\kappa)|_{E^c} u^c + P^c \mathcal{F}(u^c + \psi(u^c, \kappa; \epsilon); \epsilon),
$$

with $\widetilde{\mathcal{A}}(\kappa)|_{E^c} = \begin{pmatrix} 0 & 1 \\ 0 & -\kappa \end{pmatrix}$, $P^c := P^c_1 + P^c_2$ (3.12)

and where $\widetilde{\mathcal{A}}(\kappa)$ was defined in (3.2), $P^c_j$ were defined in Section 3.3.4. Equation (3.12) shows, in particular, that

$$
A_r = B + A\gamma_A(\kappa; \epsilon^2) + B\gamma_B(\kappa; \epsilon^2) + \widetilde{\mathcal{R}}_A(A, B, \kappa; \epsilon) \quad (3.13)
$$

with $\gamma_A(\kappa; 0) = \gamma_B(\kappa; 0) = 0$. We next use the reverser action and $\mathbb{Z}_2$ symmetry; by Proposition 3.3.2(iv), the remainder terms $\mathcal{R}_A$ and $\mathcal{R}_B$ must respect both these symmetries. The $\mathbb{Z}_2$ symmetry shows that $\widetilde{\mathcal{R}}_A(A, 0, 0; 0) = O(A^3)$. This shows that the leading-order terms in $\mathcal{R}_A$ are given by (3.11b).

For $\mathcal{R}_B$ we start with the remainder terms from [45, Lemma 3.9 in Chapter 3]. In that lemma, a new variable $\widetilde{B} := B + \mathcal{R}_A(A, B, \kappa; \epsilon)$ was introduced. However, because $\mathcal{R}_A$ is higher order in $B$ and because it respects the $\mathbb{Z}_2$ and reverser symmetries, no new terms are introduced into $\mathcal{R}_B$ through undoing this transformation.
Therefore, $B_r = -\kappa B + \tilde{R}_B(A, B, \kappa; \epsilon)$ with

$$
\tilde{R}_B(A, B, \kappa; \epsilon) = \gamma_1(\kappa; \epsilon^2) A + \gamma_2(\kappa; \epsilon^2) A^2 + \gamma_3(\kappa; \epsilon^2) A^3
+ O((\epsilon^2 + \kappa^2 + |A|)|\kappa B| + |AB^2| + |A|^4 + |B|^4),
$$

(3.14)

and $\gamma_1(\kappa; 0) = 0$ and $\gamma_2(\kappa; 0) \equiv \gamma_2(0; 0)$. In the derivation of (3.14), the reverser symmetry was enforced. We now drop all terms which do not also respect the $\mathbb{Z}_2$ symmetry and obtain $B_r = -\kappa B + c_0^1 \epsilon^2 A + c_0^3 A^3 + R_B(A, B, \kappa; \epsilon)$ with $R_B$ given in (3.11c). Although the term $B^3$ does not respect the reverser symmetry, it is a higher-order term relative to $AB^2$ and serves to bound non-algebraic terms in $B$.

To determine the sign of $c_0^1$ and $c_0^3$, we use a weakly nonlinear analysis near the curve $\Gamma_0 : \{ (\mu_m, \gamma_m) : \gamma_m^2 = \mu_m^2 + \omega^2 \}$ as in [8, (A.7) pp. 698-699]. The computations are similar for the planar radial case, so we omit the details. 

3.3.6 Matching core and far-field stable manifolds

Finally, we use the results of the preceding sections to prove the following result.

**Theorem 3.2.** Fix $\mu > \alpha \omega$ and $\mu < \beta \omega$ so that $c_0^3 < 0$ in the vector field (3.11) on the center manifold and let $\gamma = \sqrt{\mu^2 + \omega^2 - \epsilon^2}$. Then there is an $\epsilon_0 > 0$ so that (2.2) has a nontrivial stationary localized radial solution of amplitude $O(\epsilon)$ for each $\epsilon \in (0, \epsilon_0]$.

We will prove Theorem 3.2 by showing that there exists a nontrivial intersection of the core manifold $\tilde{W}^c_{-\epsilon}(\epsilon)$ and the far-field stable manifold $\tilde{W}^{s+}_{+\epsilon}(\epsilon)$. Recall that the core manifold consists of all solutions which remain bounded as $r \to 0$, while the far-field stable manifold is the set of all solutions that decay to zero as $r \to \infty$. 

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Any nontrivial solution laying in the intersection of these two sets is, by definition, a localized solution to (2.2). The core manifold was constructed in Section 3.2.2 on bounded intervals \( r \in [0, r_0] \), whereas the far-field stable manifold relies on an analysis of the far-field center-stable and center manifolds; the existence proofs of these far-field manifolds are only valid for \( r \geq 1/\rho_1 \). In the proof of Theorem 3.2 we will choose \( r_0 \) large enough so that \( 1/\rho_1 < r_0 \), which ensures that both \( \mathcal{W}_{r_+}(\epsilon) \) and \( \mathcal{W}_{cs}^{r_+}(\epsilon) \) exist at the matching point \( r = r_0 \).

In order to complete the construction of \( \tilde{\mathcal{W}}_+(\epsilon) \), it remains to find solutions on the center manifold that decay to zero as \( r \to \infty \). The following result, which we will prove in Section 3.4 using the blow-up coordinates of [38], characterizes such solutions.

**Proposition 3.3.7** (proven in Section 3.4). Fix \( c^0_3 < 0, \hat{\eta}_0 > 0 \) and \( r_0 > 1/\rho_1 \), where \( \rho_1 \leq \rho_0 \) was chosen in the proof of Proposition 3.3.2. Then there exists an \( a_0 > 0 \) so that for every fixed \( \delta_0 > 0 \) there is an \( \epsilon_0 > 0 \) so that for each \( \hat{\eta} \in [0, \hat{\eta}_0] \) and each \( \epsilon \) with \( 0 < \epsilon \leq \epsilon_0 \) there exists an initial condition of the form

\[
A(r_0) = \epsilon \left( a_0 + O(\delta_0^2 + 1/r_0^2) \right) + \epsilon^2 O_{\delta_0}(1 + |\hat{\eta}| \ln \epsilon)
\]

\[
B(r_0) = -\epsilon^2 \hat{\eta}(a_0 + O(\delta_0^2 + 1/r_0^2)) + \epsilon^3 O_{\delta_0, r_0} \left( (1 + |\hat{\eta}|^2) \ln \epsilon + |\hat{\eta}| \right)
\]

(3.15)

so that the corresponding solution of (3.11) decays to zero exponentially as \( r \to \infty \).

The Landau symbol \( O_{\delta_0, r_0} \) is interpreted in the usual sense, except that the subscript \( (\delta_0, r_0) \) means that the bounding constant and region of validity may depend on \( \delta_0 \) and \( r_0 \).

We can now complete the proof of Theorem 3.2. As shown in (3.10), the strong
stable fiber associated with solutions (3.15) can be parametrized as

\[ \mathcal{F}_\epsilon^s(1/r_0, p) = \{ d_1^a V_1^0 + d_2^c V_2^0 + d_3^a V_3(r_0) + d_4^a V_4(r_0) \} \] (3.16a)

where, due to (3.10b) and (3.15), we have

\[
\begin{align*}
d_1^c &= A + O_{r_0} (|\mathcal{d}_s|(|A| + |B| + |\mathcal{d}_s| + \epsilon^2)) \\
&= \epsilon (a_0 + O(\delta_0^2 + 1/r_0^2)) + O_{\delta_0, r_0} (\epsilon^2 (1 + |\tilde{\eta}| \ln \epsilon) + |\mathcal{d}_s| (\epsilon + \epsilon^2 |\tilde{\eta}| + |\mathcal{d}_s|)) \\
d_2^c &= B + O_{r_0} (|\mathcal{d}_s|(|A| + |B| + |\mathcal{d}_s| + \epsilon^2)) \\
&= -\epsilon^2 \tilde{\eta} (a_0 + O(\delta_0^2 + 1/r_0^2)) + O_{\delta_0, r_0} (\epsilon^3 ((1 + |\tilde{\eta}|^2) \ln \epsilon + |\tilde{\eta}|) + |\mathcal{d}_s| (\epsilon + \epsilon^2 |\tilde{\eta}| + |\mathcal{d}_s|)) \\
d_3^a &= O_{r_0} (|A| + |B|)(|A| + |B| + |\mathcal{d}_s| + \epsilon^2)) \\
&= O_{\delta_0, r_0} (\epsilon^2 + \epsilon^4 |\tilde{\eta}|^2 + |\mathcal{d}_s|^2) \\
d_4^a &= O_{r_0} (|A| + |B|)(|A| + |B| + \epsilon^2) + \mathcal{d}_s \\
&= O_{\delta_0, r_0} (\epsilon^2 + \epsilon^4 |\tilde{\eta}|^2) + \mathcal{d}_s. \tag{3.16b}
\end{align*}
\]

Proof of Theorem 3.2  We find a nontrivial solution contained in the intersection \( \tilde{W}_{cu}^c(\epsilon) \cap \tilde{W}_+^c(\epsilon) \) by matching the coefficients of each manifold at \( r = r_0 \) in the directions \( V_1^0, V_2^0, V_3(r_0), \) and \( V_4(r_0) \). The manifold \( \tilde{W}_+^c(\epsilon) \big|_{r=r_0} \) is parametrized by (3.16), whilst the manifold \( \tilde{W}_{cu}^c(\epsilon) \big|_{r=r_0} \) is parametrized by (2.8),

\[
\tilde{W}_{cu}^c(\epsilon) \big|_{r=r_0} = \left( d_1 + \ln r_0 g_2(d_1, d_3; \epsilon) \right) V_1^0 + \frac{1}{r_0} g_2(d_1, d_3; \epsilon) V_2^0 + d_3 V_3(r_0) + g_4(d_1, d_3; \epsilon) V_4(r_0),
\]

where \( (g_2, g_4)(d_1, d_3; \epsilon) = O_{r_0}(\epsilon^2 |d| + |d|^3) \) and \( d = (d_1, d_3) \). Collecting the expansion
of $\tilde{W}_-^{cu}(\epsilon)|_{r=r_0}$ on the left-hand side and of $\tilde{W}_+^{b}(\epsilon)|_{r=r_0}$ on the right-hand side, we get the system of equations

$$V_1^0 : \quad d_1 + O_{r_0} (\epsilon^2 |d| + |d|^3) = \epsilon \left( a_0 + O(\delta_0^2 + 1/r_0^2) \right)$$

$$+ O_{\delta_0, r_0} \left( \epsilon^2 (|\tilde{\eta}| \ln \epsilon + 1) + |\tilde{d}_s| (\epsilon + \epsilon^2 |\tilde{\eta}| + |\tilde{d}_s|) \right)$$

$$V_2^0 : \quad O_{r_0} (\epsilon^2 |d| + |d|^3) = -\epsilon^2 \tilde{\eta} (a_0 + O(\delta_0^2 + 1/r_0^2))$$

$$+ O_{\delta_0, r_0} (\epsilon^3 ((1 + |\tilde{\eta}|^2) \ln \epsilon + |\tilde{\eta}|) + |\tilde{d}_s| (\epsilon + \epsilon^2 |\tilde{\eta}| + |\tilde{d}_s|))$$

$$V_3(r_0) : \quad d_3 = O_{\delta_0, r_0} (\epsilon^2 + \epsilon^4 |\tilde{\eta}|^2 + |\tilde{d}_s|^2)$$

$$V_4(r_0) : \quad O_{r_0} (\epsilon^2 |d| + |d|^3) = O_{\delta_0, r_0} (\epsilon^2 + \epsilon^4 |\tilde{\eta}|^2) + \tilde{a}_s.$$

We solve the first, third, and fourth equation for $(d_1, d_3, \tilde{d}_s)$ near zero as functions of $(\epsilon, \tilde{\eta})$ near zero by finding zeros of $F(d_1, d_3, \tilde{d}_s; \epsilon, \tilde{\eta})$ for all $\epsilon < \epsilon_0$ and $|\tilde{\eta}| < \tilde{\eta}_0$, where

$$F(d_1, d_3, \tilde{d}_s; \epsilon, \tilde{\eta}) = \begin{pmatrix}
    d_1 + O_{\delta_0, 1/r_0} (\epsilon^2 |d| + |d|^3) - \epsilon (a_0 + O(\delta_0^2 + 1/r_0^2)) \\
    + O_{\delta_0, 1/r_0} (\epsilon^2 (|\tilde{\eta}| \ln \epsilon + 1) + |\tilde{d}_s| (\epsilon + \epsilon^2 |\tilde{\eta}| + |\tilde{d}_s|)) \\
    d_3 + O_{\delta_0, r_0} (\epsilon^2 + \epsilon^4 |\tilde{\eta}|^2 + |\tilde{d}_s|^2) \\
    O_{\delta_0, r_0} (\epsilon^2 |d| + |d|^3 + \epsilon^2 + \epsilon^4 |\tilde{\eta}|^2) - \tilde{d}_s
  \end{pmatrix}.$$  

Note that $F(0; 0) = 0$ and that $DF(0; 0)$ is invertible for all sufficiently small $\delta_0$ and sufficiently large $r_0$ since $a_0 \neq 0$: hence, we can apply the implicit function theorem near the origin and, afterwards, match orders in $\epsilon$ to find the expansions

$$d_1 = \epsilon \left( a_0 + O(\delta_0^2 + 1/r_0^2) + O_{\delta_0, r_0} (\epsilon \ln \epsilon) \right), \quad d_3 = O_{\delta_0, r_0} (\epsilon^2), \quad \tilde{d}_s = O_{\delta_0, r_0} (\epsilon^2).$$

Using these expansions, we see that the remaining equation for the $V_2^0$ coordinate becomes $\epsilon^2 G(\tilde{\eta}; \epsilon) = 0$ where

$$G(\tilde{\eta}; \epsilon) = \tilde{\eta} \left( a_0 + O(\delta_0^2 + 1/r_0^2) \right) + O_{\delta_0, r_0} (\epsilon \ln \epsilon + |\tilde{\eta}|^2 \epsilon^2 \ln \epsilon).$$

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We again apply the implicit function theorem to find zeros of $G(\cdot; \epsilon)$ and, matching orders in $\epsilon$, we arrive at $\hat{\eta} = O_{\delta_0, r_0}(\epsilon \ln \epsilon)$.

3.4 Dynamics on the center manifold

We complete the proof of Theorem 3.2 by proving Proposition 3.3.7. To do so, it will be convenient to transform the vector field (3.11) on the center manifold. We introduce $\tilde{B} := B + \mathcal{R}_A(A, B, \kappa; \epsilon)$ and note that we can invert this transformation by the implicit function theorem for $(A, B, \kappa, \epsilon)$ near zero so that $B = \tilde{B} + \tilde{\mathcal{R}}_A(A, \tilde{B}, \kappa; \epsilon)$, where the remainder term satisfies

$$
\tilde{\mathcal{R}}_A(A, \tilde{B}, \kappa; \epsilon) = O \left( |A| (\epsilon^2 + |A|^2) + |\tilde{B}| (\epsilon^2 + |A| + |\tilde{B}|) \right).
$$

The vector field (3.11) written in the $(A, \tilde{B})$ coordinates becomes

$$
\begin{align*}
A_r &= \tilde{B} \\
\tilde{B}_r &= -\kappa \tilde{B} + \epsilon^2 A + c_2^0 |A|^3 + \tilde{\mathcal{R}}_B(A, \tilde{B}, \kappa; \epsilon) \\
\kappa_r &= -\kappa^2,
\end{align*}
$$

where

$$
\tilde{\mathcal{R}}_B(A, \tilde{B}, \kappa; \epsilon) := \mathcal{R}_B \left( A, \tilde{B} + \tilde{\mathcal{R}}_A(A, \tilde{B}, \kappa; \epsilon), \kappa; \epsilon \right)
= O \left( \epsilon^2 \kappa^2 |A| + (\epsilon^2 + \kappa^2) |A|^3 + |A|^5 \\
+ (\epsilon^2 + \kappa^2 + |A|^2) |\kappa||\tilde{B}| + |A||\tilde{B}|^2 + |\tilde{B}|^3 \right).
$$
We now state the following proposition, whose proof will occupy the remainder of this section and which implies Proposition 3.3.7.

**Proposition 3.4.1.** Fix \( c_3^0 < 0 \), \( \hat{\eta}_0 > 0 \) and \( r_0 > 1/\rho_1 \), where \( \rho_1 \leq \rho_0 \) was chosen in the proof of Proposition 3.3.2. Then there exists an \( a_0 > 0 \) so that, for every fixed \( \delta_0 > 0 \), there is an \( \epsilon_0 > 0 \) so that for each \( \hat{\eta} \in [0, \hat{\eta}_0] \) and each \( \epsilon \) with \( 0 < \epsilon \leq \epsilon_0 \) there exists an initial condition of the form

\[
A(r_0) = \epsilon \left( a_0 + O(\delta_0^2 + 1/r_0^2) \right) + \epsilon^2 O_\delta_0 \left( |\hat{\eta}| \ln \epsilon + 1 \right)
\]

\[
\tilde{B}(r_0) = -\epsilon^2 \hat{\eta} \left( a_0 + O(\delta_0^2 + 1/r_0^2) \right) + \epsilon^3 O_{\delta_0, r_0} \left( (1 + |\hat{\eta}|^2) \ln \epsilon + |\hat{\eta}| \right) \tag{4.2}
\]

so that the corresponding solution of (4.1) decays to zero exponentially as \( r \to \infty \).

**Proof of Proposition 3.3.7** Assuming Proposition 3.4.1, we need to derive the expansion for \( B \):

\[
B(r_0) = \tilde{B}(r_0) + \mathcal{R}_A (A(r_0), \tilde{B}(r_0), 1/r_0; \epsilon^2)
\]

\[
= \tilde{B}(r_0) + O \left( (\epsilon^2 + A^2)|A| + (\epsilon^2 + |A| + |\tilde{B}|)|\tilde{B}| \right)
\]

\[
= \tilde{B}(r_0) + O(\epsilon^3),
\]

and Proposition 3.3.7 follows.

It remains to prove Proposition 3.4.1. With an abuse of notation, we write \( B \) in place of \( \tilde{B} \) from now on. We construct solutions to the vector field (4.1) that decay exponentially. We append the evolution equation for the parameter \( \epsilon \), given by \( \epsilon_r = 0 \), and use the geometric blow-up coordinates of [38] in our analysis.

Before giving the details, we discuss the intuition behind our specific choice of blow-up coordinates. We make the ansatz that decaying solutions on the far-field
center manifold are of the form $A(r) = cA_2(\epsilon r)$ for some function $A_2(s)$. Using this scaling and $s = \epsilon r$ we find that equation (4.1) at $\epsilon = 0$ reduces to the real Ginzburg–Landau equation

$$\partial_{ss} A_2 + \frac{\partial_s A_2}{s} = A_2 + c_3^0 A_2^3$$  \hspace{1cm} (4.3)

with $\text{sgn}(c_0^3) = \text{sgn}(\mu - \omega \beta)$. We will see in Section 3.4.2 that (4.3) has non-trivial bounded solutions if, and only if, $c_3^0 < 0$, which explains why we imposed this hypothesis in Proposition 3.4.1 and Theorem 3.2. Solutions to (4.3) have dominant exponential behavior with rate $\pm 1$ as $s \to \infty$. Exponentially decaying solutions of the form $A_2 \sim \exp(\pm s)$ can be expanded for $s \in [\delta_0, \infty)$, hence for $r \geq \delta_0/\epsilon$. Thus, at the matching point $r = r_0$ we will lose control over bounds on the solution $A_2$ as $\epsilon \to 0$. This widening gap is represented as the space between the two Poincare sections in Figure 3.6. We therefore introduce two coordinate transformations: one to handle the exponential behavior of solutions for $r \geq \delta_0/\epsilon$ and another to maintain control over a parametrization of such solutions for $r_0 \leq r \leq \delta_0/\epsilon$.

First consider $r \geq \delta_0/\epsilon$. We introduce $z_2 := \partial_s A_2/A_2$ and observe that, heuristically, solutions of the form $A_2 \sim \exp(\pm s)$, correspond to $z_2 \sim \pm 1$ as $s \to \infty$. Therefore, intuitively, the set of exponentially decaying solutions in the original coordinates corresponds to the center-stable manifold of $(A_2, z_2) = (0, -1)$. We will show that the center-stable manifold of $(A_2, z_2) = (0, -1)$ is two-dimensional for every fixed $\epsilon$. Following [38], we call the $(A_2, z_2)$ coordinates the “rescaling chart” coordinates.

Next, consider $r_0 \leq r \leq \delta_0/\epsilon$. We expect the solution behavior in this intermediate region to be, in general, algebraic. For example, radially symmetric localized solutions to the planar Swift–Hohenberg equation solutions in the interme-
The transition and rescaling chart are blow-up coordinates of the center-manifold coordinates, fixed points of which capture algebraically and exponentially behaving solutions, respectively.

diate region are $O(r^\pm \frac{1}{2})$ [38]. Heuristically, intermediate algebraic solution behavior serves to mediate between bounded behavior near the core and exponential behavior in the tail. In order to capture the transitional algebraic behavior, we introduce the blow-up coordinates $A_1 := rA(r)$, $z_1 := r\partial_r A/A$. Then solutions of the form $A =: \frac{1}{r}A_1 \sim r^\rho$ as $r \to 0$ correspond with $z_1 \sim \rho$ so that algebraic solutions correspond with $(A_1, z_1) = (0, \rho)$ in the limit $r \to 0$. Again following [38], we call the $(A_1, z_1)$ coordinates the “transition chart” coordinates.

This concludes the discussion of the rationale behind our choice of blow-up coordinates $(A_2, z_2)$ and $(A_1, z_1)$.

### 3.4.1 Rescaling and transition charts

We first define $z := B/A = A_r/A$. Next, we augment (4.1) by the evolution equation for $\epsilon$, given by $\epsilon_r = 0$. We then blow-up the vector field and all four variables in two different directions. First, we blow-up the vector field (4.1) in the $\kappa$ direction using...
the coordinates

\[ A_1 := \frac{A}{\kappa}, \quad z_1 := \frac{z}{\kappa}, \quad \kappa_1 := \kappa, \quad \epsilon_1 := \frac{\epsilon}{\kappa}, \quad \tau := \ln r, \] (4.4)

called the “transition chart” coordinates. Using the rescaled time \( e^\tau := r \), we obtain

\[ \partial_\tau A_1 = A_1(z_1 + 1) \]
\[ \partial_\tau z_1 = \epsilon_1^2 + \epsilon_3^0 A_1^2 - z_1^2 + \kappa_1^2 R_1(A_1, z_1, \epsilon_1) \]
\[ \partial_\tau \kappa_1 = -\kappa_1 \]
\[ \partial_\tau \epsilon_1 = \epsilon_1 \] (4.5)

with \( R_1(A_1, z_1, \epsilon_1) = O(\epsilon_1^2 + A_1^2 + z_1) \). The vector field (4.5) has the fixed point \( P_1 = (0, 0, 0, 0) \). The linearization of (4.5) about \( P_1 \) is

\[
\begin{pmatrix}
\tilde{A}_1 \\
\tilde{z}_1 \\
\tilde{\kappa}_1 \\
\tilde{\epsilon}_1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_1 \\
\tilde{z}_1 \\
\tilde{\kappa}_1 \\
\tilde{\epsilon}_1
\end{pmatrix}
\] (4.6)

with eigenvalues \( \{1, 0, -1, 1\} \). Therefore, the strong stable manifold \( W^s_{\tau} \) of \( P_1 \) is two-dimensional and given to leading order near \( P_1 \) by the \((A_1, \epsilon_1)\) plane.

Next, we blow-up the vector field (4.1) in the \( \epsilon \)-direction using the coordinates

\[ A_2 := \frac{A}{\epsilon}, \quad z_2 := \frac{z}{\epsilon}, \quad \kappa_2 := \frac{\kappa}{\epsilon}, \quad \epsilon_2 := \epsilon, \quad s := \epsilon r = \frac{1}{\kappa_2}, \] (4.7)

called the “rescaling chart” coordinates. We remark that solutions in the coordinate
systems (4.4) and (4.7) are related through the relationships

\[ A_1 = sA_2, \quad z_1 = s z_2, \quad \kappa_1 \epsilon_1 = \epsilon_2, \quad \epsilon_1 = \epsilon e^\tau = \frac{1}{\kappa_2} = s. \]  

(4.8)

Using (4.7) and the rescaled time \( s := \epsilon r \), we obtain

\[
\begin{align*}
\partial_s A_2 &= z_2 A_2 \\
\partial_s z_2 &= -\kappa_2 z_2 + \epsilon_3^0 A_2^2 - z_2^2 + 1 + \epsilon_2^2 \mathcal{R}_2(A_2, z_2, \kappa_2) \\
\partial_s \kappa_2 &= -\kappa_2^2 \\
\partial_s \epsilon_2 &= 0
\end{align*}
\]

(4.9)

with \( \mathcal{R}_2(A_2, z_2, \kappa_2) = O(\kappa_2^2 + A_2^2 + \kappa_2 z_2) \). We first set \( \epsilon = \epsilon_2 = 0 \). Then the vector field (4.9)\( \epsilon_2=0 \) has the two fixed points \( P_2^\pm = (0, \pm 1, 0, 0) \). We will show at the end of this section that \( P_2^- \) is associated with exponentially decaying solutions to (4.1) in the original coordinates. The linearization of (4.9) about \( P_2^- \) is

\[
\begin{pmatrix}
\ddot{A}_2 \\
\ddot{z}_2 \\
\ddot{\kappa}_2 \\
\ddot{\epsilon}_2
\end{pmatrix}_s =
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\ddot{A}_2 \\
\ddot{z}_2 \\
\ddot{\kappa}_2 \\
\ddot{\epsilon}_2
\end{pmatrix}.
\]

(4.10)

The linearization (4.10) has eigenvalues \( \{-1, 2, 0, 0\} \) with associated eigenvectors

\[
v_1 = (1, 0, 0, 0)^T, \quad v_2 = (0, 1, 0, 0)^T, \quad v_3 = (0, -1, 2, 0)^T, \quad v_4 = (0, 0, 0, 1)^T.
\]

For \( \epsilon = 0 \), the two-dimensional center-stable manifold \( \mathcal{W}_{s+}^{cs} \) of \( P_2^- \) is given to leading order by span \{\( v_1, v_3 \)\}. We remark that \( \epsilon_2(s) = \epsilon_2 = \epsilon \) is constant for all \( s \); therefore, the \( v_4 \) direction is neutral and not contained in \( \mathcal{W}_{s+}^{cs} \), to leading order.
For all $\epsilon$ small enough, the fixed points $P^\pm_2(\epsilon)$ and all invariant manifolds $W^u_{r_\pm}(\epsilon)$, $W^{cs}_{s_\pm}(\epsilon)$ persist and depend smoothly on $\epsilon$. For simplicity of notation, all fixed points and invariant manifolds are evaluated at $\epsilon = 0$ unless we explicitly indicate their dependence on $\epsilon$.

We conclude this section by arguing that a small-amplitude solution $A(r) = O(\epsilon)$ in the original center-manifold coordinates decays exponentially for $\epsilon > 0$ if, and only if, it is contained in the center-stable manifold $W^{cs}_{s_+}(\epsilon)$ of $P^-_2(\epsilon)$. Indeed, fix $\epsilon > 0$ and let $A(r)$ be a solution such that $(A_2, z_2, \kappa_2, \epsilon_2) \to P^-_2(\epsilon)$ as $r \to \infty$. The evolution equation for $A_2(s)$ in (4.9) and a standard fixed-point argument show that $A_2(s) = A_0 e^{\int_{s_0}^s z_2(\rho)d\rho}$ for some $s_0 \gg 1$ with $z_2(s) \to -1$ as $s \to \infty$. It is straightforward to show that, for $s_0$ large enough, there exists a $0 < C < \infty$ and a $\delta \ll 1$ so that $\sup_{s \geq s_0} |A_2(s)| \leq Ce^{(-1+\delta)s}$. Transforming back into the original coordinates

$$A(r) = \epsilon A_2(\epsilon r),$$

we see that $|A(r)| \leq Ce^{-\epsilon r}$ as $r \to \infty$. Conversely, fix $\epsilon > 0$ and consider only solutions for $r \gg 1/\epsilon$. Then, using a fixed-point argument in an appropriate exponentially weighted space, one can show that $A(r) = \epsilon r^{-1/2} e^{-\epsilon r} (1 + O(1/r))$ for the nonlinear problem (4.1), and it is straightforward to show that the corresponding solution in the rescaling chart coordinates converges to $P^-_2(\epsilon)$ as $s \to \infty$. 

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3.4.2 Singular connecting orbit between transition and rescaling charts

We first set $\epsilon = 0$ so that $\epsilon_2 = 0$. Inspecting (4.8) shows that $\kappa_1 = 0$ as well. The rescaling-chart vector field (4.9) then reduces to

$$\begin{align*}
\partial_s A_2 &= z_2 A_2 \\
\partial_s z_2 &= -\kappa_2 z_2 + c_3^0 A_2^2 - z_2^2 + 1 \\
\partial_s \kappa_2 &= -\kappa_2^3
\end{align*}$$

and the transition chart vector field (4.5) reduces to

$$\begin{align*}
\partial_{\tau} A_1 &= A_1 (z_1 + 1) \\
\partial_{\tau} z_1 &= \epsilon_1^2 + c_3^0 A_1^2 - z_1^2 \\
\partial_{\tau} \epsilon_1 &= \epsilon_1.
\end{align*}$$

Note that the equations for $\kappa_2$ and $\epsilon_1$ decouple from the rest of the system. Using $\kappa_2(s) = 1/s$ we can rewrite (4.11) as a second-order equation with $a := A_2$

$$a_{ss} + \frac{a_s}{s} = a + c_3^0 a^3, \quad s > 0. \tag{4.13}$$

We have the following result.

**Lemma 3.4.2.** For $c_3^0 < 0$, (4.13) has a unique monotonically decreasing, nontrivial, bounded solution $q_0(s)$. Furthermore, there exist constants $a_0, a_2, c_1 > 0$ such that
the solution $q_0(s)$ satisfies

$$q_0(s) = a_0 - a_2 s^2 + O(s^4 \ln s) \quad \text{as } s \to 0, \text{ and}$$

$$q_0(s) = K_0(s) \left( c_1 + O(e^{-2s}) \right) \quad \text{as } s \to \infty.$$

Lastly, the linearization of (4.13) about $q_0(s)$ does not have a nontrivial solution that is bounded uniformly on $\mathbb{R}^+$. For $c_3^0 > 0$, the only bounded solution to (4.13) is $a(s) \equiv 0$.

**Proof.** First consider $c_3^0 < 0$. Without loss of generality, let $c_3^0 = -1$, otherwise rescale $a$. Then equation (4.13) reduces to

$$a_{ss} + a_s/s = a(1 - a^2). \quad (4.14)$$

Equation (4.14) arises in many applications and has been well studied; it is related, for example, to the nonlinear Schrödinger equation. In two dimensions, the existence of a unique ground state solution to (4.14) is due to [39], a result which was later made more general in [34]. The nonexistence of a nontrivial bounded solution of the linearization is shown in [10, Lemma 2.1]. The asymptotics of $q_0(s)$ follow from the variation-of-constants formula and a standard fixed-point argument in each limit; the sign of $a_2$ then follows from monotonicity. See Appendices C and D for the derivation of the asymptotic expansion in the limits $s \to 0$ and $s \to \infty$, respectively.

Next consider $c_3^0 > 0$. We multiply (4.13) by $sa(s)$ and integrate over $s \in (0, \infty)$ as in [36, Lemma 4]; integration by parts shows that the only possible localized solution is $a(s) \equiv 0$. 

\[ \blacksquare \]
We now show that \( q_0(s) \) gives a connecting orbit between the fixed points \( P_1 \) and \( P_2^- \).

**Lemma 3.4.3.** Assume that \( c^0_3 < 0 \) so that Lemma 3.4.2 is true and set \( \epsilon = \epsilon_2 = \kappa_1 = 0 \). Then \( Q_0 \), given in the transition and rescaling chart coordinates by

\[
Q_1^0(\tau) = \left\{ (A_1, z_1, \epsilon_1)(\tau) = \left( e^{\tau} q_0(e^{\tau}), e^{\tau} \frac{q_0(e^{\tau})}{q_0(e^{\tau})}, e^{\tau} \right) \right\}
\]

and

\[
Q_2^0(s) = \left\{ (A_2, z_2, \kappa_2)(s) = \left( q_0(s), \frac{q_0(s)}{q_0(s)}, \frac{1}{s} \right) \right\}
\]

respectively, forms a connecting orbit between \( P_1 \) of (4.12) and \( P_2^- \) of (4.11), which lies in the intersection \( W^u_{r^-} \cap W^{cs}_{s^+} \). Moreover, the intersection \( W^u_{r^-} \cap W^{cs}_{s^+} \) is transverse.

**Proof.** The proof is essentially the same as in [38, Lemma 2.4]. Using the asymptotic expansions for \( q_0(s) \) given in Lemma 3.4.2, it is easy to check that \( Q_1^0(\tau) \) converges to \( P_1 \) exponentially as \( \tau \to -\infty \) and that \( Q_2^0(s) \) converges to \( P_2^- \) as \( s \to \infty \). The details are in Appendix sections C.3 and D.3, respectively. Since \( Q_1^0(\tau) \) and \( Q_2^0(s) \) satisfy (4.12) and (4.11), respectively, we have that \( Q_0 \in W^u_{r^-} \cap W^{cs}_{s^+} \).

To show that the intersection of \( W^u_\tau \) and \( W^{cs}_\tau \) along \( Q_0 \) is transverse, we invoke the nondegeneracy condition in Lemma 3.4.2 and argue by contradiction. Assume that the intersection \( W^u_{r^-} \cap W^{cs}_{s^+} \) is not transverse, then there exists a nonzero solution \( \widehat{Q} \in T_{Q_1^0(\tau)} W^u_{r^-} \cap T_{Q_2^0(s)} W^{cs}_{s^+} \). Let \( \widehat{Q}_1(\tau) \) and \( \widehat{Q}_2(s) \) denote the solution \( \widehat{Q} \) written in the transition and rescaling chart coordinates, respectively; we write \( \widehat{Q}_2(s) =: (\widehat{A}_2, \widehat{z}_2, \widehat{\kappa}_2)(s) \). Linearizing (4.9) about \( Q_2^0(s) \), we find that \( \widehat{\kappa}_2 = 0 \) and that \( \widehat{A}_2 \) and
\( \hat{z}_2 \) satisfy
\[
\partial_s \hat{A}_2 = \frac{q'_0}{q_0} \hat{A}_2 + q_0 \hat{z}_2
\]
\[
\partial_s \hat{z}_2 = -\frac{1}{s} \hat{z}_2 + 2 c_3^0 q_0 \hat{A}_2 - 2 \frac{q'_0}{q_0} \hat{z}_2. \tag{4.16}
\]

Letting \( \hat{a} := \hat{A}_2 \), a straightforward computation shows that (4.16) is equivalent to
\[
\hat{a}_{ss} + \frac{\hat{a}_s}{s} = \hat{a}(1 + 3c_3^0 q_0^2), \tag{4.17}
\]
the linearization of (4.13) about \( q_0(s) \). In particular, \( \hat{A}_2(s) \) is a nonzero bounded solution of the linearization of (4.14) about \( q_0(s) \), in contradiction to Lemma 3.4.2.

3.4.3 Formal Scaling Argument

We begin with a formal argument to build intuition about the expected solution scaling. We first transform \( z_1 \mapsto \tilde{z}_1 \) so that \( \tilde{z}_1 \equiv 0 \) along \( Q_1^0(\tau) \). To leading order, vector field (4.5) is given by
\[
\partial_\tau A_1 = A_1
\]
\[
\partial_\tau \tilde{z}_1 = -\tilde{z}_1^2
\]
\[
\partial_\tau \kappa_1 = -\kappa_1
\]
\[
\partial_\tau \epsilon_1 = \epsilon_1. \tag{4.18}
\]

We explicitly solve \( \epsilon_1 = \delta_0 e^\tau \), \( \kappa_1 = \epsilon/\delta_0 e^{-\tau} \), with \( \tau = 0 \) at the Poincare section \( \epsilon_1 = \delta_0 \). As discussed in Lemma 3.4.3, the intersection \( W^u_{\tau_-} \cap W^s_{\tau_+} \) is transverse.
Since the transformation into the $\tilde{z}_1$ coordinates puts $\mathcal{W}_{u}^{c_{s}}_{\tau^-}$ on the $A_1$ axis for every fixed $\epsilon = \delta_0$, this means that the intersection $\mathcal{W}_{c_{s}}^{c_{s}}_{s+} \cap \{\tilde{z}_1\}$ is transverse. Thus, we can parametrize $\mathcal{W}_{c_{s}}^{c_{s}}_{s+}$ near $Q_1^0(0)$ in the $\tilde{z}_1$ direction; see Figure 3.7b. We let $\eta \ll 1$ be the resulting parametrization so that the initial data is $(A_1, \tilde{z}_1)\big|_{\tau=0} = (a, -\eta)$, where $a = O(1)$ is the $A_1$ component of $Q_1^0(0)$. Then (4.18) is solved by

$$A_1(\tau) = ae^\tau, \quad \tilde{z}_1(\tau) = \frac{\eta}{1 + \eta^\tau}.$$ 

Letting $\delta_0 = r_0 = a = 1$, we find that $\kappa_1 = 1/r_0$ at time $\tau_s = \ln \epsilon$ so that

$$A_1(\tau_s) = \epsilon, \quad \tilde{z}_1(\tau_s) = \eta.$$ 

In the original $A, B$ coordinates we therefore have $P_c\tilde{W}_{s}^c(\epsilon)\big|_{r=r_0} = \epsilon V_1^0 + \eta V_2^0$. We match with $\tilde{W}_{cu}^{-}(\epsilon)\big|_{r=r_0}$ within the center manifold, where, to leading order,

$$P_c\tilde{W}_{cu}^{-}(\epsilon)\big|_{r=r_0} = \{d_1 V_1^0 + O(d_1 (\epsilon^2 + d_1^2)) V_2^0 : d_1 \in \mathbb{R}, \text{ small}\}. \quad (4.19)$$ 

The intersection $P_c\tilde{W}_{cu}^{-}(\epsilon)\big|_{r=r_0} \cap P_c\tilde{W}_{+}^{-}(\epsilon)\big|_{r=r_0}$ satisfies

$$V_1^0 : \quad \epsilon = d_1$$

$$V_2^0 : \quad \eta \epsilon = O(d_1 (\epsilon^2 + d_1^2)).$$

We find that $d_1 = O(\epsilon)$ and $\eta = O(\epsilon^2)$. 

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3.4.4 The dynamics near $P_1$ in the transition chart coordinates

Our goal is to use Lemma 3.4.3 to trace $\mathcal{W}_{s+}^{*}(\epsilon)$ backwards in time to the equilibrium $P_1$ and beyond for all $\epsilon < \epsilon_0$. We will find it convenient to first transform the vector field (4.5)

\[
\begin{align*}
\partial_\tau A_1 &= A_1(z_1 + 1) \\
\partial_\tau z_1 &= -z_1^2 + \epsilon_1^2 + \epsilon_3^0 A_1^2 + \kappa_1^2 O(\epsilon_1^2 + A_1^2 + z_1) \\
\partial_\tau \kappa_1 &= -\kappa_1 \\
\partial_\tau \epsilon_1 &= \epsilon_1 
\end{align*}
\]

in the transition chart coordinates into a more convenient form by straightening the center, stable, and unstable manifolds near $P_1 = (0, 0, 0, 0)$ as well as the strong stable and strong unstable fibers. We have already discussed center manifolds, center-(un)stable manifolds, and (un)stable foliations in Sections 3.3.1-3.3.3. We now state the stable manifold theorem for completeness. The unstable manifold theorem is completely analogous.

**Theorem 3.3** (Stable manifold theorem). [53, Theorem 9.3] Consider

\[ \dot{x} = f(x). \]

Suppose $f \in C^k$, $k \geq 1$, has a fixed point $x_0$ with corresponding Jacobian matrix $A$. Then, if $\alpha > 0$ and $A + \alpha I$ is hyperbolic, there is a neighborhood $U(x_0) = x_0 + U$ and a function $h^{+,\alpha} \in C^k(E^{+,\alpha} \cap U, E^{-,\alpha})$ such that

\[ M^{+,\alpha}(x_0) \cap U(x_0) = \left\{ x_0 + a + h^{+,\alpha}(a) \mid a \in E^{+,\alpha} \cap U \right\} \]
where $E^{+,-\alpha}$ is the space spanned by all generalized eigenvectors for matrix $A$ whose associated eigenvalues have real part less than $-\alpha$ (greater than $\alpha$). Both $h^{+,-\alpha}$ and its Jacobian matrix vanish at 0, that is, $M^{+,-\alpha}(x_0)$ is tangent to its linear counterpart $E^{+,-\alpha}$ at $x_0$. Furthermore, we have $M^{+,-\alpha_2}(x_0) \subseteq M^{+,-\alpha_1}(x_0)$ whenever $\alpha_a \leq \alpha_2$, and $M^{+,-\alpha_2}(x_0) = M^{+,-\alpha_1}(x_0)$ whenever $E^{+,-\alpha_2} = E^{+,-\alpha_1}$.

We use the stable manifold theorem, as well as properties of center manifolds, center-(un)stable manifolds, and (un)stable foliations to prove

**Lemma 3.4.4.** There is a smooth change of coordinates of the form

$$\tilde{z}_1 = z_1 + O(z_1 \kappa_1^2 + A_1^2 + \epsilon_1^2)$$  \hspace{1cm} (4.21)

that transforms equation (4.20) near $P_1$ into

$$\partial_\tau A_1 = A_1(1 + O(|\tilde{z}_1| + \kappa_1^2 + A_1^2 + \epsilon_1^2))$$

$$\partial_\tau \tilde{z}_1 = -\tilde{z}_1^2 + \kappa_1^2 O(A_1^2 + \epsilon_1^2)$$

$$\partial_\tau \kappa_1 = -\kappa_1$$

$$\partial_\tau \epsilon_1 = \epsilon_1.$$  \hspace{1cm} (4.22)

The inverse transformation is given by

$$z_1 = \tilde{z}_1 + O(\tilde{z}_1 \kappa_1^2 + A_1^2 + \epsilon_1^2).$$  \hspace{1cm} (4.23)

**Proof.** We first note that the center-stable and center-unstable manifolds of $P_1$ are
given by $\mathcal{W}_{\tau-}^{cs} = \{A_1 = \epsilon_1 = 0\}$ and $\mathcal{W}_{\tau-}^{cu}(\epsilon) = \{\kappa_1 = 0\}$, respectively. Furthermore,

$$\mathcal{W}_{\tau-}^{s} \subset \mathcal{W}_{\tau-}^{cs}(\epsilon) = \{A_1 = z_1 = \epsilon_1 = 0\} \text{ and } \mathcal{W}_{\tau-}^{c} := \mathcal{W}_{\tau-}^{cs}(\epsilon) \cap \mathcal{W}_{\tau-}^{cu}(\epsilon) = \{A_1 = \kappa_1 = \epsilon_1 = 0\}.$$

We also know that solutions on the center manifold are given by

$$z_1 = z_1^*(\tau) := \frac{1}{c_1 + \tau} \quad \text{with } c_1 = 1/z_1^*(0).$$

We first show that there exists a smooth change of coordinates of the form

$$\tilde{z}_1 = z_1 + O(A_1^2 + \epsilon_1^2) \quad (4.24)$$

which transforms the evolution of $z_1$ in equation (4.20) near $P_1$ into

$$\partial_\tau \tilde{z}_1 = -\tilde{z}_1^2 + \kappa_1^2 O(A_1^2 + \epsilon_1^2 + \tilde{z}_1). \quad (4.25)$$

To achieve this, we straighten out the strong unstable fibers within the center-unstable manifold: this guarantees that the evolution of $\tilde{z}_1$ must be of the form (4.25) so that, within the center-unstable manifold, $\tilde{z}_1$ evolves independently of $A_1$ and $\epsilon_1$. It remains to show that this transformation is of the form (4.24). Let $(A_1, z_1, \epsilon_1) = (0, \tilde{z}_1, 0)$ be a point on the center manifold within the center-unstable manifold. Then, for every $A_1, \tilde{z}_1, \epsilon_1 \ll 1$, the strong unstable fiber associated with base point $z_1 = \tilde{z}_1$ can be written as a graph

$$z_1 = z_1(\tilde{z}_1; A_1, \epsilon_1) = \tilde{z}_1 + O((|A_1| + |\epsilon_1|)(|A_1| + |\epsilon_1| + |\tilde{z}_1|)). \quad (4.26)$$
Next we claim that the graph (4.26) is actually of the form

$$z_1 = z_1(\bar{z}_1; A_1, \epsilon_1) = \bar{z}_1 + O(A_1^2 + \epsilon_1^2). \tag{4.27}$$

Letting $z_1$ be the linearized $z_1$ coordinate, we have that $|z_1(\tau) - \bar{z}_1^*(\tau)| = O(e^{\nu \tau})$ for some $\nu > 0$ in the linearized strong unstable fiber with $\tau \leq 0$. Solving the linearized vector field

$$\partial_\tau \bar{z}_1 = -2\bar{z}_1^*(\tau)\bar{z}_1$$

for $\bar{z}_1$, we have $\bar{z}_1 = c_2/(c_1 + \tau)^2$ so that $c_2 = 0$. The expansion for the strong unstable fiber which has tangent space $\bar{z}_1 = 0$ is given by (4.27). By the implicit function theorem we invert (4.27) near $(A_1, z_1, \epsilon_1; \bar{z}_1) = (0, 0, 0; 0)$ to get (4.24).

Next we straighten the strong stable fibers within the center-stable manifold. We let $(\bar{z}_1, \kappa_1) = (\tilde{z}_1, 0)$ be a point on the center manifold within the center-stable manifold after transformation (4.24). A completely analogous argument shows that the strong stable fibers are given by graphs

$$\tilde{z}_1 = \tilde{z}_1(\bar{z}_1; \kappa_1) = \bar{z}_1(1 + O(\kappa_1^2)). \tag{4.28}$$

By combining (4.27) and (4.28) we obtain the inverse transformation (4.23). By inverting (4.28) near $(\bar{z}_1, \kappa_1; \tilde{z}_1) = (0, 0; 0)$ we find

$$\tilde{z}_1 = \bar{z}_1(1 + O(\kappa_1^2)). \tag{4.29}$$

The evolution of $\tilde{z}_1$ is independent of $\kappa_1$ within the center-stable manifold. It is also still independent of $A_1$ and $\epsilon_1$ within the center-unstable manifold since (4.28) has no effect when $\kappa_1 = 0$. Therefore, the evolution of $\tilde{z}_1$ must be of the form given in
(4.22). Composing equations (4.24) and (4.29) gives the transformation (4.21).

3.4.5 Passage through transition chart coordinates

We use the transversality of \( q_0(s) \) stated in Lemma 3.4.3 to parametrize \( \mathcal{W}_{s+}^{cs}(\epsilon) \) near \( P_1(\epsilon) \) in the coordinates of (4.21). Throughout this section we use a tilde to denote any object which has been transformed into the coordinates (4.21). We refer to Figure 3.7 for a visualization.

**Lemma 3.4.5.** Assume \( c_3^0 < 0 \) so that Lemma 3.4.3 is true. For each sufficiently small \( \delta_0 > 0 \), there exists constants \( \eta_0, \epsilon_0 > 0 \) such that, for all \( 0 \leq \epsilon \leq \epsilon_0 \), the following is true. Define the Poincare section \( \Sigma_{\delta_0} := \{ \epsilon_1 = \delta_0 \} \) and let \( \tilde{\mathcal{W}}_{s+}^{cs}(\epsilon) \) denote the center-stable manifold transformed into the coordinates of (4.21). Then

\[
\tilde{\mathcal{W}}_{s+}^{cs}(\epsilon) \cap \Sigma_{\delta_0} = \left\{ (A_1, \tilde{z}_1, \kappa_1, \epsilon_1) = (\delta_0 q_0(\delta_0) + O(|\eta| + \epsilon), -\eta, \epsilon/\delta_0, \delta_0) : \eta \in (-\eta_0, \eta_0) \right\} \quad (4.30)
\]
near $\tilde{Q}_1^0(\tau)$, the connecting orbit transformed into the coordinates of (4.21).

**Proof.** First consider $\epsilon = 0$. Lemma 3.4.3 shows that the connecting orbit $Q_1^0(\tau)$ is contained in the unstable manifold of $P_1$ so that, after transformation into the coordinates of (4.21), the $\tilde{z}_1$ component of $Q_1^0(\tau)$ is zero for all $\tau$. Therefore, the connecting orbit in the transformed coordinates is given by

$$\tilde{Q}_1^0(\tau) \cap \Sigma_{\delta_0} = \{(A_1, \tilde{z}_1, \kappa_1, \epsilon_1) = (\delta_0 q_0(\delta_0), 0, 0, \delta_0)\}.$$

In the transformed coordinates, the strong unstable manifold $\tilde{W}_\tau^u\tau$ of $P_1$ is the $A_1$-axis. Using the transversality of the intersection $\tilde{W}_s^\text{cs} \cap \tilde{W}_\tau^u\tau$ we can parametrize $\tilde{Q}_1^0 \cap \Sigma_{\delta_0}$ by $\eta$, a small offset in the $\tilde{z}_1$ direction, so that

$$\tilde{W}_s^\text{cs}(0) \cap \Sigma_{\delta_0} = \{(A_1, \tilde{z}_1, \kappa_1, \epsilon_1) = (\delta_0 q_0(\delta_0) + O(|\eta|), -\eta, 0, \delta_0)\}.$$

The fixed points, invariant manifolds, and connecting orbit are all smooth in $\epsilon$ so that for $0 \leq \epsilon \leq \epsilon_0$ the parametrization is given by (4.30).

Lastly, we propagate the initial data (4.30) under vector field (4.22) backwards until the matching point $\kappa_1 = 1/r_0$.

**Lemma 3.4.6.** Assume $c_3^0 < 0$ so that Lemma 3.4.5 is true. Also fix $\tilde{\eta}_0 > 0$ and $r_0 > 1/\rho_1$, where $\rho_1$ was determined in the proof of Proposition 3.3.2. Then, for each fixed $\delta_0 > 0$ there is an $\epsilon_0 > 0$ so that for all $\eta$ of the form $\eta = \epsilon \tilde{\eta}$ with $\tilde{\eta} \in [0, \tilde{\eta}_0]$ and all $0 < \epsilon \leq \epsilon_0$, we can solve (4.22) with initial data given by (4.30) at time $\tau = 0$. 

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back to $\tau_*=\ln \frac{\epsilon r_0}{\delta_0}$. The associated solution at $\tau=\tau_*$ is given by

$$A_1(\tau_*) = \epsilon r_0 \left(a_0 + O(\delta_0^2 + 1/r_0^2)\right) + r_0 O_{\delta_0,r_0}(\epsilon^2 \ln \epsilon |\hat{\eta}| + \epsilon^2)$$

$$\tilde{z}_1(\tau_*) = -\epsilon \hat{\eta} + O_{\delta_0,r_0}(\epsilon^2 \ln \epsilon)$$

$$\epsilon_1(\tau_*) = \epsilon r_0$$

$$\kappa_1(\tau_*) = 1/r_0.$$  \hfill(4.31)

In the original coordinates the solution becomes

$$A(\tau_*) = \epsilon \left(a_0 + O(\delta_0^2 + 1/r_0^2)\right) + O_{\delta_0}(\epsilon^2 \ln \epsilon + \epsilon^2)$$

$$B(\tau_*) = -\epsilon^2 \hat{\eta} \left(a_0 + O(\delta_0^2 + 1/r_0^2)\right) + O_{\delta_0,r_0}\left(\epsilon^2 (|\hat{\eta}|^2 \epsilon \ln \epsilon + \epsilon |\hat{\eta}| + \epsilon \ln \epsilon)\right),$$  \hfill(4.32)

where the Landau symbol $O_{\delta_0,r_0}$ means that the bounding constant and the region of validity may depend on $\delta_0$ and $r_0$.

**Proof.** We set $\tau = 0$ at $\epsilon_1 = \delta_0$, which is possible because the vector field (4.22) is autonomous. A formal analysis shows that, as a first approximation,

$$A_1(\tau) \sim e^{\tau}, \quad \eta = O(\epsilon^2), \quad \text{and} \quad z_1(\tau_*) = O(\eta).$$

We therefore define $\eta =: \epsilon \hat{\eta}$, $\tilde{z}_1 =: \epsilon \tilde{z}_1$, and $A_1 =: e^{\tau} \hat{A}_1$ and consider the fixed point system

$$e^{\tau} \hat{A}_1(\tau) = [\delta_0 q_0(\delta_0) + \epsilon O(|\hat{\eta}| + 1)] e^{\tau} e^{\int_0^\tau O(e^{\tilde{z}_1} + \kappa_1^2 e^{2\sigma} \hat{A}_1^2 + \epsilon_1^2) d\sigma}$$

$$\epsilon \tilde{z}_1(\tau) = -\epsilon \hat{\eta} + \int_0^\tau \left[-e^{2\hat{z}_1^2} + O \left(\kappa_1^2 (\hat{A}_1^2 e^{2\sigma} + \epsilon_1^2)\right)\right] d\sigma$$  \hfill(4.33)

in the $(\hat{A}_1, \hat{z}_1)$ coordinates for $\tau \in [\tau_*,0]$. Due to the variation of parameters formula, discussed in Appendix A.2, a pair of smooth functions $(A_1, z_1) = (e^{\tau} \hat{A}_1, \epsilon \tilde{z}_1)$ satisfies
(4.22) with initial data given by (4.30) if, and only if, \((\hat{A}_1, \tilde{z}_1)\) is a fixed point of (4.33). We show that (4.33) has a unique fixed point.

We observe that \(\kappa_1\) and \(\epsilon_1\) decouple from the rest of the system with \(\kappa_1 = \epsilon/\delta_0 e^{-\tau}\) and \(\epsilon_1 = \delta_0 e^\tau\). We substitute these expressions into (4.33) and explicitly integrate the \(O(\kappa_1^2 + \epsilon_1^2)\) terms in the equation for \(\hat{A}_1\). Then (4.33) is equivalent to the system

\[
\hat{A}_1(\tau) = \left[ \delta_0 q_0(\delta_0) + \epsilon O(|\hat{\eta}| + 1) \right] e^{O(1/\rho_0^2 + \delta_0^2 + \epsilon^2 r_0^2 + \delta_0^2 e^{2\tau})} + \int_0^\tau \left[ -\hat{z}_1 + O\left( \hat{A}_1^2/\delta_0^2 + 1 \right) \right] d\sigma.
\]

Using a standard contraction mapping principal argument on the space of continuous functions with \(\tau \in [\tau_*, 0]\), one can show that (4.34) has a unique fixed point. The details are in Appendix E. By uniqueness, the fixed point can be written

\[
\hat{A}_1(\tau) = \left[ \delta_0 q_0(\delta_0) + \epsilon O(|\hat{\eta}| + 1) \right] e^{O(1/\rho_0^2 + \delta_0^2 + \epsilon^2 r_0^2 + \delta_0^2 e^{2\tau})} = \left[ \delta_0 q_0(\delta_0) + \epsilon O(|\hat{\eta}| + 1) \right] \left[ 1 + O_{\rho_0, r_0, \epsilon}(\delta_0^2 + \epsilon |\hat{\eta}| \tau + \delta_0^2 e^{2\tau}) \right]
\]

\[
\hat{z}_1(\tau) = -\hat{\eta} + \epsilon \tau O(|\hat{\eta}|^2 + 1).
\]

We transform back into \((A_1, \tilde{z}_1)\) and evaluate at \(\tau = \tau_*\) to get

\[
A_1(\tau_*) = \frac{\epsilon r_0}{\delta_0} \left[ \delta_0 q_0(\delta_0) + O_{\delta_0, r_0}(\epsilon |\hat{\eta}| + \epsilon + |\hat{\eta}|^2 \ln \epsilon + \epsilon^2) \right]
\]

\[
= \epsilon r_0 \left( a_0 + O(\delta_0^2 + 1/r_0^2) \right) + r_0 O_{\delta_0, r_0}(\epsilon^2 \ln |\hat{\eta}| + \epsilon^2)
\]

\[
\tilde{z}_1(\tau_*) = -\epsilon \hat{\eta} + \epsilon^2 \ln \frac{\epsilon r_0}{\delta_0} O(|\hat{\eta}|^2 + 1)
\]

\[
= -\epsilon \hat{\eta} + O_{\delta_0, r_0}(\epsilon^2 \ln \epsilon),
\]
which proves (4.31). We invert the transformation $\tilde{z}_1$ from Lemma 3.4.4 to recover

$$z_1 = \tilde{z}_1(\tau_*) + O\left(\kappa_1^2(\tau_*) + A_1^2(\tau_*) + \epsilon_1^2(\tau_*)\right)$$

$$= -\epsilon\hat{\eta} \left(1 + O(1/r_0^2)\right) + O_{\delta, r_0}(\epsilon^2 \ln \epsilon).$$

Finally, we write $(A_1, z_1)$ in the original $(A, B)$ coordinates by inverting the transition chart transformation (4.4). Then $A(\tau_*) = A_1(\tau_*)/r_0$ and $B(\tau_*) = A_1(\tau_*)z_1(\tau_*)/r_0^2$,

which are given by (4.32).

This concludes the proof of Proposition 3.3.7.

### 3.4.6 Singular connecting orbit: $A \equiv 0$

We briefly discuss a second connecting orbit between the fixed points $P_1$ and $P_2^-$. The schematic is shown in Figure 3.8. We will see that this second orbit cannot be used to construct localized solutions. Let $A = A_1 = A_2 = 0$ so that the transition chart vector field (4.5) becomes

$$\partial_\tau A_1 = 0$$

$$\partial_\tau z_1 = \epsilon_1^2 - z_1^2 + \kappa_1^2 R_1(A_1, z_1, \epsilon_1)$$

$$\partial_\tau \kappa_1 = -\kappa_1$$

$$\partial_\tau \epsilon_1 = \epsilon_1$$

(4.35)
The manifold $\mathcal{W}_{s+}^\epsilon(\epsilon)$ for fixed $\epsilon < \epsilon_0$.

Figure 3.8: Parametrization of $\mathcal{W}_{s+}^\epsilon$ in the Poincare section $\Sigma_{\delta_0}$ near the connecting orbit with $A_1 = A_2 \equiv 0$.

and the rescaling chart vector field (4.9) becomes

\[
\begin{align*}
\partial_s A_2 &= 0 \\
\partial_s z_2 &= -\kappa_2 z_2 - z_2^2 + 1 + \epsilon_2^2 R_2(A_2, z_2, \kappa_2) \\
\partial_s \kappa_2 &= -\kappa_2^2 \\
\partial_s \epsilon_2 &= 0.
\end{align*}
\]

(4.36)

The variables $\kappa_1(\tau) = \kappa_0 e^{-\tau}$, $\epsilon_1(\tau) = \epsilon_0 e^\tau$, $\kappa_2(s) = 1/s$, and $\epsilon_2(s) = \epsilon$ decouple from the rest of the system and can be explicitly solved. Letting $\tau = 0$ at $\epsilon_1 = \delta_0$, we have $\epsilon_1 = \delta_0 e^\tau$ and $\kappa_1 = \epsilon/\delta_0 e^{-\tau}$.

First set $\kappa_1 = \epsilon_2 = \epsilon = 0$ and define $a(s)$ so that $z_2 =: a/s$. Then equation (4.36) is equivalent to

\[
a_{ss} + \frac{a_s}{s} - a = 0
\]

(4.37)

the zeroth order modified Bessel equation, solved by a linear combination of the
Table 3.2: The asymptotic behavior of the modified Bessel functions for small argument \( z \ll 1 \) and large argument \( z \gg 1 \) quoted from [1, (9.6.10)-(9.6.13), (9.7.1)-(9.7.2)], respectively, where \( \gamma = \lim_{n \to \infty} \left( \sum_{j=1}^{n} \frac{1}{n} - \ln n \right) \) is the Euler-Mascheroni constant.

zeroth order modified Bessel functions \( a(s) = c_1 K_0(s) + c_2 I_0(s) \). The asymptotic expansion of these functions for both large and small argument is displayed in Table 3.2.

<table>
<thead>
<tr>
<th>( z \to 0 )</th>
<th>( z \to \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_0(z) )</td>
<td>( 1 + O(z^4) )</td>
</tr>
<tr>
<td>( I_1(z) )</td>
<td>( \frac{z}{2} + O(z^3) )</td>
</tr>
<tr>
<td>( K_0(z) )</td>
<td>(- \ln \left( \frac{z}{2} \right) - \gamma + O(z^2 \ln z) )</td>
</tr>
<tr>
<td>( K_1(z) )</td>
<td>( \frac{1}{z} + O(z \ln z) )</td>
</tr>
</tbody>
</table>

It can then be shown that the orbit \( \tilde{Q}_0 \) given in the transition and rescaling coordinates by

\[
\tilde{Q}_1^0(\tau) := \left\{ (A_1, z_1, \kappa_1, \epsilon_1) = \left(0, -e^\tau \frac{K_1(e^\tau)}{K_0(e^\tau)}, 0, e^\tau \right) \right\}
\]

and

\[
\tilde{Q}_2^0(s) := \left\{ (A_2, z_2, \kappa_2, \epsilon_2) = \left(0, -\frac{K_1(s)}{K_0(s)} \frac{1}{s}, 0 \right) \right\},
\]

respectively, forms a connecting orbit between \((A_1, z_1) = (0, 0)\) and \((A_2, z_2) = (0, -1)\) since

\[
z_1(\tau) \sim -\frac{1 + O(\tau e^{2\tau})}{-\tau + \ln 2 - \gamma + O(\tau e^{2\tau})} = \frac{1}{\tau} \left( 1 + O(1/\tau) \right) \quad \text{as } \tau \to -\infty
\]

and

\[
z_2(s) \sim \frac{1 + O(1/s)}{1 + O(s)} \quad \text{as } s \to \infty.
\]

In fact, the same computation shows that \( z_1(\tau) \) actually approaches zero algebraically from \( z_1 < 0 \), for \( \ln \tau \ll 1 \).
Moreover, $\mathcal{W}_{s_+}^{cs} \cap \{A_1 \equiv 0\}$ transversely near $\tilde{Q}_0^0(\tau)$ for $\epsilon = 0$. Suppose not. Then, since the plane $\{A_2 \equiv 0\}$, equivalent to $\{A_1 \equiv 0\}$, is invariant under the flow, the non-transversality holds for all $s \in \mathbb{R}$. But this is a contradiction: as discussed in Section 3.4.1, near $(A_2, z_2) = (0, -1)$ the tangent space of the manifold $\tilde{\mathcal{W}}_{s_+}^{cs}$ is given by span $\{v_1, v_3\}$ where

$$v_1 = (1, 0, 0, 0)^T \quad \text{and} \quad v_3 = (0, -1, 2, 0)^T.$$  

The transversality then holds for all $\epsilon < \epsilon_0$ small enough by continuity. Since the intersection $\mathcal{W}_{s_+}^{cs} \cap \{A_1 \equiv 0\}$ is transverse, we can parametrize $\mathcal{W}_{s_+}^{cs}$ in the $A_1$ direction near $\tilde{Q}_1^0(0)$; see Figure 3.8b. We let $\eta \ll 1$ be the resulting parametrization so that the initial data is $(A_1, z_1)|_{\tau=0} = (\eta, -a)$, where $a = O(1)$ is the $z_1$ component of $\tilde{Q}_1^0(0)$. We also have $a > 0$ provided we make $\delta_0$ small enough.

The remainder of this argument is formal, as in Section 3.4.3. To leading order, vector field (4.5) is given by

$$\partial_\tau A_1 = A_1(1 + z_1)$$
$$\partial_\tau z_1 = -z_1^2,$$  \hspace{1cm} (4.38)

solved by

$$A_1(\tau) = \eta(1 - a\tau)e^\tau, \quad z_1(\tau) = \frac{-a}{1 - a\tau}.$$  

Letting $\delta_0 = r_0 = a = 1$ we find that $\kappa_1 = 1/r_0$ at time $\tau_* = \ln \epsilon$ so that

$$A_1(\tau_*) = \eta\epsilon(1 - \ln \epsilon), \quad z_1(\tau_*) = \frac{-1}{1 - \ln \epsilon}.$$
In the original $A,B$ coordinates we therefore have

$$P^c\tilde{W}_+^s(\epsilon)|_{r=r_0} = \eta\epsilon(1 - \ln \epsilon)V_1^0 - \eta\epsilon V_2^0.$$  

We match with $\tilde{W}_{cu}^-\epsilon)|_{r=r_0}$ within the center manifold, where, to leading order,

$$P^c\tilde{W}_{cu}^-\epsilon)|_{r=r_0} = \{d_1V_1^0 + \mathcal{O}(d_1(\epsilon^2 + d_1^2))V_2^0 : d_1 \in \mathbb{R}, \text{ small}\}. \quad (4.39)$$

The intersection $P^c\tilde{W}_{cu}^-\epsilon)|_{r=r_0} \cap P^c\tilde{W}_+^s\epsilon)|_{r=r_0}$ satisfies

$$V_1^0 : \quad \eta\epsilon(1 - \ln \epsilon) = d_1$$

$$V_2^0 : \quad \eta\epsilon = \mathcal{O}(d_1(\epsilon^2 + d_1^2)).$$

We find that

$$\mathcal{O}(\epsilon^2 + d_1^2) = 1/(1 - \ln \epsilon)$$

and

$$1 = \mathcal{O}\left(\epsilon^2(1 - \ln \epsilon)(1 + \eta^2(1 - \ln \epsilon)^2)\right)$$

so that $d_1 = \mathcal{O}\left((1 - \ln \epsilon)^{-1/2}\right)$ and $\eta = \mathcal{O}\left(\frac{1}{\epsilon}(1 - \ln \epsilon)^{-3/2}\right)$. We observe that $\eta \to \infty$ as $\epsilon \to 0$, which means that a non-trivial intersection $P^c\tilde{W}_{cu}^-\epsilon)|_{r=r_0} \cap P^c\tilde{W}_+^s\epsilon)|_{r=r_0}$ cannot be parametrized near $\tilde{Q}_1^0$ uniformly in $\epsilon$. The formal analysis motivates the following conjecture.

**Conjecture 3.4.7.** For $\mathcal{W}_{cs}^+(\epsilon)$ parametrized near $\tilde{Q}_1^0(\tau)$ in the transition chart coordinates, the only uniformly bounded intersection with $\tilde{W}_{cu}^-(\epsilon)$ as $\epsilon \to 0$ is $d_1 = \eta = 0.$
3.5 Conclusion

In this chapter, we have rigorously shown that standard oscillons with monotone tails exist as solutions to the steady state planar radial forced complex Ginzburg–Landau equation

$$0 = (1 + i\alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma \bar{u}$$

equation near the bifurcation curve

$$\Gamma_0 := \{ (\mu, \gamma) : \gamma = \sqrt{\mu^2 + \omega^2} \}.$$
Chapter Four

Reciprocal oscillons
4.1 Introduction

In this chapter, we prove the existence of small-amplitude reciprocal oscillons with monotone tails (see Figure 4.1 for a representation in one spatial dimension) in the stationary planar radial forced complex Ginzburg–Landau (CGL) equation

\[ 0 = (1 + i \alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i \omega)u - (1 + i \beta)|u|^2u + \gamma \bar{u} \]  

near onset. Such solutions can be written as a localized offset from the nontrivial homogeneous upper branch solution

\[ u(r) = u_{\text{unif}}^+ + \tilde{u}(r), \]

where \( \tilde{u}(r) \to 0 \) as \( r \to \infty \) and where \( u_{\text{unif}}^\pm \) are described by equations (2.8)-(2.10) in Section 2.2.1. We briefly review the key properties of \( u_{\text{unif}}^\pm \). The nontrivial uniform solutions bifurcate from a saddle node on the parameter space curve

\[ \Gamma_b := \{(\mu, \gamma) : (1 + \beta^2)\gamma^2 = (\omega + \beta \mu)^2\}. \]  

On \( \Gamma_b \), \( |u_{\text{unif}}^+|^2 = |u_{\text{unif}}^-|^2 = R_0^2 \) with

\[ R_0^2 := (R^\pm)^2 = \frac{\omega \beta - \mu}{1 + \beta^2}. \]
Then, with $\gamma = \gamma_b + \epsilon^2$ (where $\gamma_b$ is defined so that $(\mu, \gamma_b(\mu)) \in \Gamma_b$), the nontrivial uniform solutions persist and can be expanded

$$
\begin{pmatrix}
v^+ \\
w^+
\end{pmatrix} = 
\begin{pmatrix}
v_0 \\
w_0
\end{pmatrix} + \epsilon \begin{pmatrix}
v^+_1 \\
w^+_1
\end{pmatrix} + \epsilon^2 \begin{pmatrix}
v^+_2 \\
w^+_2
\end{pmatrix} + \ldots
$$

where

$$u_{\text{unif}}^\pm := v^\pm + iw^\pm \text{ with } v^\pm, w^\pm \in \mathbb{R},$$

$$
\begin{pmatrix}
v_0 \\
w_0
\end{pmatrix} = \begin{pmatrix}
\eta_b \\
1
\end{pmatrix} \Upsilon_0, \text{ with } \Upsilon_0 := \frac{R_0}{\sqrt{1 + \eta_b^2}},
$$

$$\eta_b := \beta + \text{sgn}(\omega + \beta \mu)\sqrt{1 + \beta^2}, \quad (1.4)$$

and

$$
\begin{pmatrix}
v^+_{1} \\
w^+_{1}
\end{pmatrix} = \begin{pmatrix}
\xi_b \\
1
\end{pmatrix} \Upsilon_1, \text{ with } \xi_b := \frac{\eta \omega + (1 - \beta \eta_b)R_0^2}{\omega - (\beta + \eta_b)R_0^2},
$$

$$\Upsilon_1 := \text{sgn} [\xi_b \eta_b + 1] \sqrt{\frac{\eta_b}{(\xi_b \eta_b + 1)(\xi_b - \eta_b)}}. \quad (1.5)$$

As was argued in Section 2.2.2, summarized from [8], reciprocal oscillons may bifurcate for the one-dimensional CGL into the region $\gamma > \gamma_b$ provided that also $z(\omega - \omega_z) < 0$, where\(^1\)

$$z := \alpha(1 - \beta^2) - 2\beta \quad \text{and} \quad \omega_z := \frac{\mu(1 - \beta^2 + 2\alpha \beta)}{z}.$$

The argument relies on a spatial eigenvalue analysis: reciprocal oscillons with mono-

\(^1\)We remark that [8] defines $-z$ instead of $z$. Hence, all inequalities involving $z$ are reversed in [8] from the inequalities discussed in this section.
Figure 4.2: Shown is the \((\mu, \gamma)\)-plane with \(\alpha, \beta,\) and \(\omega\) fixed. The inlays are the spatial eigenvalues associated with the linearization of equation (2.2) about \(u = u_{\text{unif}}^+\) plotted in the complex plane. The shaded region is the expected existence region of reciprocal oscillons; reciprocal oscillons are seen numerically to terminate at the dotted line in a stationary 1D reciprocal front solution.

Tone tails are expected to bifurcate into the region where all four spatial eigenvalues associated with the linearization of (1.1) about \(u = u_{\text{unif}}^+\) are purely real; on \(\Gamma_b\), two of these eigenvalues are located at the origin of the complex plane. We remark that \(\mu < \beta \omega\) is another necessary condition; it ensures that \(u_{\text{unif}}^+\) exists. See Figure 4.2 for an illustration.

In this chapter, we show that this result also holds for the planar CGL. We now state our main result.

**Theorem 4.1.** Fix \(z(\omega - \omega_z) < 0, \mu < \beta \omega, \omega \neq -\beta \mu\) and let \(\gamma = \gamma_b + \epsilon^2\). Then there is an \(\epsilon_0 > 0\) so that (1.1) has a nontrivial stationary solution of the form

\[
u(r) = u_{\text{unif}}^+ + \tilde{u}(r)
\]

where \(\tilde{u}(r)\) is localized and radially symmetric with amplitude \(O(\epsilon)\) for each \(\epsilon \in (0, \epsilon_0)\).

The condition \(\omega \neq -\beta \mu\) ensures that the leading order term in an appropriate center manifold reduction of (1.1) near \(\epsilon = 0\) is quadratic.
The idea for the proof is exactly analogous to standard oscillons, and we refer the reader to Section 3.1.1 for an overview. This chapter is organized as follows. In Section 4.2, we construct the core manifold. In Section 4.3, we establish the existence and expansions of the far-field center-stable and center manifolds, as well as the stable foliation. We also derive the reduction of equation (1.1) to the center manifold. Finally, we carry out the matching analysis between the core and far-field manifolds. In Section 4.4, we analyze the flow on the center manifold using geometric blow-up techniques, in order to find the slowly decaying solution $A_*(r)$.

### 4.2 Bounded solutions near the core

In this section, we construct the set of all bounded solutions of the planar, radial, stationary forced Ginzburg–Landau equation

$$
0 = (1 + i\alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma\bar{u} \quad (2.1)
$$
on $r \in [0, r_0]$ with $r_0 < \infty$ fixed. We expand these solutions near the upper branch nontrivial uniform solution $u^+_{\text{unif}}$, in contrast to our work on standard oscillons in which this set was constructed near the trivial solution $u = 0$. 

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4.2.1 The linearization about $u = u_{\text{unif}}^\pm$

We let $\gamma = \gamma_b + \epsilon^2$ and

$$
\begin{pmatrix}
v(r) \\
w(r)
\end{pmatrix}
= \begin{pmatrix}
v^+ \\
w^+
\end{pmatrix}
+ \epsilon \begin{pmatrix}
\tilde{v}(r) \\
\tilde{w}(r)
\end{pmatrix}
= \begin{pmatrix}
v_0 \\
w_0
\end{pmatrix}
+ \epsilon \begin{pmatrix}
v_1^+ \\
w_1^+
\end{pmatrix}
+ \tilde{\Omega}(r)
+ O(\epsilon^2), \quad (2.2)
$$

$\tilde{U} = (\tilde{v}, \tilde{w})^T \in \mathbb{R}^2$ and rescale the radial variable $r$ via $r \mapsto r/\sqrt{\alpha^2 + 1}$. In these coordinates, equation (2.1) becomes

$$
0 = \tilde{U}_{rr} + \frac{1}{r} \tilde{U}_r - C_1 \tilde{U} + O\left(\epsilon \tilde{U} + |\tilde{U}|^2\right) \quad (2.3)
$$

where the entries of $C_1$

$$C_1 := \begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix}
$$

are given by

$$
C_{11} = \mu - \alpha \omega - \gamma_b + 2(1 + \alpha \beta)R_0^2 + \alpha (\beta \mu + \omega) \frac{R_0^2}{\gamma_b}
$$

$$
C_{12} = \omega + \alpha \mu + \alpha \gamma_b - 2(\beta - \alpha)R_0^2 + (\omega + \beta \mu) \frac{R_0^2}{\gamma_b}
$$

$$
C_{21} = -\omega - \alpha \mu + \alpha \gamma_b + 2(\beta - \alpha)R_0^2 + (\omega + \beta \mu) \frac{R_0^2}{\gamma_b}
$$

$$
C_{22} = \mu - \alpha \omega + \gamma_b + 2(1 + \alpha \beta)R_0^2 - \alpha (\beta \mu + \omega) \frac{R_0^2}{\gamma_b}.
$$
In computing $C_1$ we used that

\[ v_0^2 - w_0^2 = R_0^2 \cos 2\phi_0 = R_0^2 \left( \frac{R_0^2 + \mu}{\gamma_b} \right) \quad \text{and} \]
\[ 2v_0w_0 = R_0^2 \sin 2\phi_0 = R_0^2 \left( \frac{\omega - \beta R_0^2}{\gamma_b} \right). \quad (2.4) \]

The matrix $C_1$ has eigenvalues $\lambda_0 = 0$ and $\lambda_1 = m^2 := 2(\mu - \omega + 2(1 + \alpha \beta)R_0^2)$ with associated eigenvectors $\tilde{U}_0$ and $\tilde{U}_1$, respectively. We fix the parameters so that $z(\omega - \omega_z) > 0$, and hence $m^2 > 0$.

Equation (2.3) can also be written as the first-order system

\[ \partial_r U = \mathcal{A}(1/r)U + \mathcal{F}(U; \epsilon) \quad (2.5) \]

in $U = (\tilde{U}, \tilde{V})^T \in \mathbb{R}^4$, where

\[
\mathcal{A}(\kappa) := \begin{pmatrix} 0 & I \\ C_1 & -\kappa I \end{pmatrix}
\]

\[
\mathcal{F}((\tilde{U}, \tilde{V})^T; \nu) = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} ((\tilde{U}, \tilde{V})^T; \nu) := \begin{pmatrix} 0 \\ 0 \left( \nu \tilde{U} + |\tilde{U}|^2 \right) \end{pmatrix}.
\]

We linearize (2.5) at $\epsilon = 0$ about $U \equiv 0$ and obtain

\[ V_r = \mathcal{A}(1/r)V. \quad (2.6) \]

Exactly as with standard oscillons, the eigenvalues of the matrix $\mathcal{A}(0)$ are $\nu^c = 0$ (with multiplicity two), $\nu^u = m$, and $\nu^s = -m$ with associated generalized
\[ I_0(z) \rightarrow 1 + O(z^2) \quad z \rightarrow 0 \]
\[ K_0(z) \rightarrow -\ln \left( \frac{1}{2} \right) I_0(z) - \gamma + O(z^2) \quad z \rightarrow \infty \]

\[
\begin{array}{c|c|c}
   & z \rightarrow 0 & z \rightarrow \infty \\
\hline
I_0(z) & 1 + O(z^2) & \sqrt{\frac{2}{\pi z}} e^{z} (1 + O \left( \frac{1}{z} \right)) \\
K_0(z) & -\ln \left( \frac{1}{2} \right) I_0(z) - \gamma + O(z^2) & \sqrt{\frac{2}{\pi z}} e^{-z} (1 + O \left( \frac{1}{z} \right)) \\
\end{array}
\]

\textbf{Table 4.1:} The asymptotic behavior of the zeroth-order modified Bessel functions for small and large arguments quoted from [1, (9.6.12)-(9.6.13), (9.7.1)-(9.7.2)], respectively, where \( \gamma = \lim_{n \to \infty} \left( \sum_{j=1}^{n} \frac{1}{n} - \ln n \right) \) is the Euler-Mascheroni constant.

Eigenspaces

\[
E_r^c = \text{span} \left\{ \begin{pmatrix} \tilde{U}_0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \tilde{U}_0 \end{pmatrix} \right\},
\]

\[
E_r^u = \text{span} \left\{ \begin{pmatrix} \tilde{U}_1 \\ m\tilde{U}_1 \end{pmatrix} \right\}, \quad E_r^s = \text{span} \left\{ \begin{pmatrix} \tilde{U}_1 \\ -m\tilde{U}_1 \end{pmatrix} \right\}.
\]

(2.7)

Four linearly independent solutions \( \{V_j(r)\}_{j=1}^{4} \) of the linearization (2.6) are given by

\[
V_1 = \begin{pmatrix} \tilde{U}_0 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \tilde{U}_0 \ln r \\ \frac{1}{\tilde{U}_0} \end{pmatrix},
\]

\[
V_3 = \begin{pmatrix} \tilde{U}_1 I_0(mr) \\ \tilde{U}_1 m I_1(mr) \end{pmatrix}, \quad V_4 = \begin{pmatrix} \tilde{U}_1 K_0(mr) \\ -\tilde{U}_1 m K_1(mr) \end{pmatrix},
\]

(2.8)

where \( I_0(z) \) and \( K_0(z) \) are the zeroth-order modified Bessel functions [1, §9.6]. The asymptotic behaviors of \( I_0(z) \) and \( K_0(z) \) will be important in the analysis and are displayed in Table 4.1. Note that only \( V_1(r) \) and \( V_3(r) \) are bounded as \( r \to 0 \).

For \( r \) large enough, we will see that the far-field stable and unstable manifolds will remain close, in an appropriate sense, to the stable and unstable subspaces \( E_r^s \) and \( E_r^u \), respectively. We will also see that the space \( E_r^c \) gives appropriate center-manifold coordinates. Finally, we note that normalized solution vectors \( V_3(r) \) or \( V_3(r) \)
and $V_4(r)/|V_4(r)|$ converge to unit vectors in $E^u_{r+}$ and $E^s_{r+}$, respectively. Furthermore, the center subspace $E^c_{r+}$ is actually invariant under the linearization (2.6), as discussed in the following remark.

**Remark 4.2.1.** We have $\text{span}\{V_1(r), V_2(r)\} = E^c_{r+}$ for all $r$ since $c_1V_1(r) + c_2V_2(r) = (c_1 + c_2 \ln r)V_0^0 + \frac{e_2}{r}V_2^0$ for all $c_1, c_2 \in \mathbb{R}$, where $V_0^0 := V_1$ and $V_2^0 := (0, \tilde{U_0})^T$.

### 4.2.2 Construction of the core manifold

We now construct the set of small-amplitude solutions to (2.5) that stay bounded on intervals of the form $[0, r_0]$ with $0 < r_0 < \infty$ fixed. We call this set the “core manifold” and denote it by $\tilde{W}^{cu}(\epsilon)$. We define $P_{r-}(r_0)$ to be the projection onto $\text{span}\{V_1(r_0), V_3(r_0)\}$ along $\text{span}\{V_2(r_0), V_4(r_0)\}$.

**Definition 4.2.2.** Throughout this chapter, we say that $f(x) = O_{r_0}(x)$ if, for each fixed $r_0$, there are positive constants $C = C(r_0)$ and $\delta = \delta(r_0)$ such that $|f(x)| \leq C|x|$ for all $x$ with $|x| < \delta$. An analogous convention holds for all other Landau symbols used in this chapter.

**Lemma 4.2.3.** Fix $0 < r_0 < \infty$ and let $d = (d_1, d_3) \in \mathbb{R}^2$. Then there exist constants $\rho_1, \rho_2, \epsilon_0 > 0$ so that, for $\epsilon \leq \epsilon_0$,

$$
\tilde{W}^{cu}(\epsilon) := \left\{ (U(r), r) \mid \begin{array}{l}
(U(r)) \text{ satisfies (2.5) for } r \in [0, r_0] \text{ with } \\
\sup_{0 \leq r \leq r_0} |U(r)| < \rho_1, \ |P_{r-}(r_0)U(r_0)| < \rho_2
\end{array} \right\}
$$

is a smooth three-dimensional submanifold of $\mathbb{R}^5$. Moreover, there are smooth functions $(g_2, g_4)(d_1, d_3, \epsilon)$ with $(g_2, g_4)(d_1, d_3; \epsilon) = O_{r_0}(\epsilon^2|d| + |d|^3)$ so that $U \in \tilde{W}^{cu}(\epsilon)$.
if, and only if,

\[ U(r_0) = d_1V_1(r_0) + g_2(d_1, d_3; \epsilon)V_2(r_0) + d_3V_3(r_0) + g_4(d_1, d_3; \epsilon)V_4(r_0) \]  

(2.9)

with \(|d| = |(d_1, d_3)| < \rho_2|.

**Proof.** The proof follows from a standard application of the variation-of-constants formula on a bounded interval completely analogously to the standard oscillon case, Lemma 3.2.3. The details for standard oscillons are in Appendix B. 

Due to Remark 4.2.1, equation (2.9) is equivalent to

\[ U(r_0) = (d_1 + g_2(d_1, d_3; \epsilon) \ln r_0) V_1^0 + \frac{1}{r_0} g_2(d_1, d_3; \epsilon)V_2^0 \]

\[ + d_3V_3(r_0) + g_4(d_1, d_3; \epsilon)V_4(r_0). \]  

(2.10)

We remark that we will consider the fiber \( \mathcal{W}^c_-(\epsilon)|_{r=r_1} \) for each fixed \( r_1 \in [0, r_0] \) as a two-dimensional submanifold of \( \mathbb{R}^4 \).

### 4.3 Far-field dynamics and matching with the core

We augment (2.5) near \( r = \infty \) with \( \kappa = 1/r \). The resulting vector field

\[
\left( \begin{array}{c}
U \\
\kappa
\end{array} \right) = 
\left( \begin{array}{c}
\mathcal{A}(\kappa)U + \mathcal{F}(U; \epsilon) \\
-\kappa^2
\end{array} \right)
\]  

(3.1)

is autonomous with linearization about the fixed point \((U, \kappa) = (0, 0)\) given by the system \( V_r = \mathcal{A}(0)V \) and \( \rho_r = 0 \). As discussed in Section 4.2.1, the equation for
V has two center directions, one unstable direction, and one stable direction given by the subspaces $E_c^{r+}$, $E_u^{r+}$, and $E_s^{r+}$ from (2.7), respectively. Taking the additional $\kappa$-direction into account, we therefore expect to find a four-dimensional center-stable and a three-dimensional center manifold near $U = 0$.

In this section, we prove the existence of these manifolds. We can write the center-stable manifold $\mathcal{W}_{cs}^r(\epsilon)$ as the stable foliation $\{F^s_\epsilon(p, \kappa)\}_{p \in \mathcal{W}_{cs}^r(\epsilon)}$ with base points in the center manifold $\mathcal{W}_{cs}^r(\epsilon)$: a trajectory with initial data $(q, \kappa)$ converges exponentially to zero as $r \to \infty$ if, and only if, its associated base point with initial data $(p, \kappa)$ on the center manifold does (see Figure 3.4 from Chapter 3). The far-field stable manifold, consisting by definition of all solutions for which $U(r) \to 0$ as $r \to \infty$, is therefore given by the union of the stable fibers associated with decaying solutions on the center manifold. The remaining steps for finding localized solutions are therefore to (i) derive an expansion for the vector field restricted to the center manifold, (ii) analyze the flow on the center manifold, and (iii) match the resulting far-field stable manifold with the core manifold. In this section, we will carry out steps (i) and (iii), anticipating the results for step (ii) which will be carried out in Section 4.4.

Throughout this section, all invariant manifolds will be considered as subsets of $\mathbb{R}^4 \times \mathbb{R}$; we will consider their restrictions to $\kappa = \kappa_1$, for each fixed $\kappa_1$, as submanifolds of $\mathbb{R}^4$.

4.3.1 Existence of a center-stable manifold

The existence of center-stable and center manifolds are standard; see, for example, [61]. However, since we will need specific properties of these manifolds, we show
here briefly how the results of [61] apply. First, we control the nonlinear terms via a cutoff function: let $\chi(z)$ be a smooth cutoff function with $\chi(z) = 1$ for $z \leq 1$ and $\chi(z) = 0$ for $z \geq 2$ and define, for $\rho$ small enough, the modified vector field

$$
\begin{pmatrix}
\tilde{U} \\
\tilde{V} \\
\kappa
\end{pmatrix}_r = 
\begin{pmatrix}
0 & I & 0 \\
C_1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{U} \\
\tilde{V} \\
\kappa
\end{pmatrix}
+ 
\begin{pmatrix}
0 \\
(-\kappa \tilde{V} + \mathcal{F}_2(U, \epsilon^2))\chi\left(\frac{|\kappa|^2}{\rho^2}\right)\chi\left(\frac{c_1^2}{\rho^2}\right)\chi\left(\frac{|\kappa|^2}{\rho^2}\right) \\
-\kappa^2 \chi\left(\frac{|\kappa|^2}{\rho^2}\right)
\end{pmatrix},
\tag{3.2}
$$

which then coincides with (3.1) for $|U|, |\kappa|, \epsilon \leq \rho$ and also satisfies the hypotheses of the invariant-manifold theorems in [61]. Let $\tilde{S}_r(U_0, \kappa_0)$ represent the solution to (3.2) at time $r$ starting from $(U, \kappa)(0) = (U_0, \kappa_0)$. It remains to show that these manifolds satisfy certain properties which we will need later on.

**Proposition 4.3.1.** Fix $\ell \geq 2$. Then there exist $\epsilon_0 > 0$, $\rho_0 > 0$ so that, for all $0 \leq \epsilon \leq \epsilon_0$ and $0 < \rho \leq \rho_0$, equation (3.1) possesses a flow-invariant four-dimensional center-stable manifold $W_{cs}^{r+}(\epsilon)$ near the equilibrium $(U, \kappa, \epsilon) = 0$. The manifold $W_{cs}^{r+}(\epsilon)$ is $C^\ell$, depends $C^\ell$ on $\epsilon2$ and contains all solutions to (3.1) with $\sup_{r \geq 0}\{|U(r)|, |\kappa(r)|, |\epsilon|\} \leq \rho$. Furthermore, $W_{cs}^{r+}(\epsilon)$ satisfies the following properties:

(i) there exists a smooth, monotonically decreasing function $\tilde{\kappa}_\rho(r)$ with $|\tilde{\kappa}_\rho(r)| \leq 2\rho$ so that for every $0 \leq \epsilon \leq \epsilon_0$,

$$
U_*(r)|_{r \geq 1/\rho} := \{(U, \kappa) \in \mathcal{W}_{cs}^{r+}(\epsilon) : |\kappa| \leq \rho, \tilde{\kappa}_\rho(r) \leq 1/r\}.
$$
In particular, $0 \in \mathcal{W}_{r+}^{cs}(\epsilon)\big|_{\kappa=1/r}$; and

(ii) $T_{U}(r)\mathcal{W}_{r+}^{cs}(0)\big|_{\kappa=1/r} = \text{span} \{V_{1}^{0}, V_{2}^{0}, V_{4}(r)\}$ for all $0 \leq \kappa \leq \rho$, where we recall from (2.8) that the functions $\{V_{j}(r)\}_{j=1}^{4}$ are the solutions to $V_{r} = A(\kappa)V$:

\[
V_{1}(r) = \begin{pmatrix} \tilde{U}_{0} \\ 0 \end{pmatrix}, \quad V_{2}(r) = \begin{pmatrix} \tilde{U}_{0} \ln |r| \\ \tilde{U}_{0} \frac{1}{r} \end{pmatrix},
\]

\[
V_{3}(r) = \begin{pmatrix} \tilde{U}_{1} I_{0}(mr) \\ \tilde{U}_{1} m I_{1}(mr) \end{pmatrix}, \quad V_{4}(r) = \begin{pmatrix} \tilde{U}_{1} K_{0}(mr) \\ -\tilde{U}_{1} m K_{1}(mr) \end{pmatrix}, \quad (3.3)
\]

and where $V_{1}^{0} := \begin{pmatrix} \tilde{U}_{0} \\ 0 \end{pmatrix}$, $V_{2}^{0} := \begin{pmatrix} 0 \\ \tilde{U}_{0} \end{pmatrix}$ were defined in Remark 4.2.1 so that $\text{span} \{V_{1}(r), V_{2}(r)\} = \text{span} \{V_{1}^{0}, V_{2}^{0}\} = E_{r+}^{c}$ with $V_{1}^{0}$ and $V_{2}^{0}$ independent of $r$.

**Proof.** Proof follows exactly as for the proof of Proposition 3.3.1, using the linearization of (3.2) about $U_{*}(r)$ with $\epsilon = 0$:

\[
\begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} \bigg|_{r} = \begin{pmatrix} 0 & I \\ C_{1} & -\tilde{\kappa}_{r}(r) \chi(|\tilde{\kappa}_{r}(r)|^{2}/\rho^{2}) \end{pmatrix} \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} \quad (3.4a)
\]

\[
\tilde{\kappa}_{r} = -2 \left[ \tilde{\kappa}_{r}(r) \chi(|\tilde{\kappa}_{r}(r)|^{2}/\rho^{2}) + \frac{\tilde{\kappa}_{r}^{3}(r)}{\rho^{2}} \frac{\chi'(|\tilde{\kappa}_{r}(r)|^{2}/\rho^{2})}{\rho^{2}} \right] \tilde{\kappa}. \quad (3.4b)
\]


4.3.2 Existence of a center manifold

Next we show the existence of a center manifold $\mathcal{W}_{r+}^{c} \subset \mathcal{W}_{r+}^{cs}$ for (3.1) with $U_{*}(r) \in \mathcal{W}_{r+}^{c}$. 

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Proposition 4.3.2. Fix $\ell \geq 2$. Then there exist $0 < \epsilon_1 \leq \epsilon_0$ and $0 < \rho_1 \leq \rho_0$ so that, for every $0 \leq \epsilon \leq \epsilon_1$ and $0 < \rho \leq \rho_1$, equation (3.1) possesses a flow-invariant three-dimensional $C^\ell$-center manifold $W_{r+}^c(\epsilon)$ near the equilibrium $(U, \kappa, \epsilon) = 0$, which contains all solutions with $\sup_{r \in \mathbb{R}} \{|U(r)|, |\kappa(r)|, |\epsilon|\} \leq \rho$. The center manifold depends $C^\ell$ on $\epsilon^2$ and has the following additional properties:

(i) for all $0 \leq \epsilon \leq \epsilon_1$, $W_{r+}^c(\epsilon) \subset W_{r+}^{cs}(\epsilon)$;

(ii) for all $0 \leq \epsilon \leq \epsilon_1$, $U_*(r) \in W_{r+}^c(\epsilon)$;

(iii) $T_{U_*}(r)W_{r+}^c(0)|_{\kappa=1/r} = E_{r+}^c$ for $r \geq 1/\rho$; and

(iv) the flow on $W_{r+}^c(\epsilon)$ respects the reverser action $(\tilde{U}, \tilde{V}, \kappa, r) \mapsto (\tilde{U}, -\tilde{V}, -\kappa, -r)$.

Remark 4.3.3. We emphasize that property (iii) states in particular that the tangent space $T_{U_*}(\kappa)W_{r+}^c(0)$ is independent of $\kappa = 1/r$ and hence $E_{r+}^c$ is invariant under the linearization about $U_*(r)$.

Proof. The proof follows exactly as for Proposition 3.3.2 except that the flow on the center manifold no longer respects the $\mathbb{Z}_2$ symmetry $(\tilde{U}, \tilde{V}, \kappa, r) \mapsto (-\tilde{U}, -\tilde{V}, \kappa, r)$.

4.3.3 Strong stable foliations

In this section we show $W_{r+}^{cs}(\epsilon)$ is given as the union of strong stable fibers over base points in $W_{r+}^c(\epsilon)$. Let $S^r(U_0, \kappa_0)$ represent the solution to (3.1) at time $r$ with initial data $(U, \kappa)(0) = (U_0, \kappa_0)$. 

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Lemma 4.3.4. Fix $\ell \geq 1$ and a decay rate $\nu$ with $0 < \nu < m$. Define $\epsilon_1, \rho_1$ as in Lemma 4.3.2 so that the manifolds $W^{cs}_{r+}(\epsilon)|_{\kappa=1/r}$ and $W^{c}_{r+}(\epsilon)|_{\kappa=1/r}$ exist. For every $\epsilon \leq \epsilon_1$, any $\kappa = 1/r \leq \rho_1$, and each $p \in W^{c}_{r+}(\epsilon)|_{\kappa=1/r}$, there exists a one-dimensional strong stable fiber $F^s_{\epsilon}(p, \kappa)$ in $\mathbb{R}^4$ so that the following are true:

(i) $p \in F^s_{\epsilon}(p, \kappa)$ for all $p \in W^{c}_{r+}(\epsilon)|_{\kappa=1/r}$;

(ii) $W^{cs}_{r+}(\epsilon)|_{\kappa=1/r} = \bigcup_{p \in W^{c}_{r+}(\epsilon)|_{\kappa=1/r}} F^s_{\epsilon}(p, \kappa)$;

(iii) $F^s_{\epsilon}(\cdot, \kappa)$ depends $C^\ell$ on $\epsilon^2$ and $\kappa$;

(iv) for every $\epsilon, \kappa$, there exists a $C^\ell \times C^\ell$ function $\phi^s_{\epsilon}(\cdot, \kappa, \cdot) : W^{c}_{r+}(\epsilon)|_{\kappa=1/r} \times \mathbb{R}V_4(r) \to \mathbb{R}V^0_1 \oplus \mathbb{R}V^0_2 \oplus \mathbb{R}V^0_3(r)$ so that $F^s_{\epsilon}(p, \kappa) = \text{graph}(\phi^s_{\epsilon}(p, \kappa, \cdot))$;

(v) $|S^s_r(q_1, \kappa) - S^s_r(q_2, \kappa)| = O(e^{-\nu t})$ for $r \geq 0$ and all $q_1, q_2 \in F^s_{\epsilon}(p, \kappa)$; and

(vi) $T F^s_0(1/r, U_*(r)) = V_4(r)$.

Proof. The proof follows exactly the same as for Lemma 3.3.5.

4.3.4 Parametrization of $W^{cs}_{r+}(\epsilon)$, $W^{c}_{r+}(\epsilon)$, and $F^s_{\epsilon}$

We use the properties of $W^{c}_{r+}(\epsilon)$, $W^{cs}_{r+}(\epsilon)$, and $F^s_{\epsilon}$ listed in Sections 4.3.1-4.3.3 to parametrize each of these manifolds as graphs. Each parametrization will be performed near $U_*(r)$. In particular, we can parametrize the fibers $F^s_{\epsilon}(p, \kappa)$ using appropriate functions $h_j(\cdot, \cdot; \kappa, \epsilon) : W^{c}_{r+}(\epsilon)|_{\kappa=1/r} \oplus \mathbb{R}V_4(r) \to \mathbb{R}V_j(r)$ for $j \in \{1, 2, 3\}$ so that
\[ F^s_\epsilon(p, \kappa) = \begin{cases} 
\overline{d}_s(p, \kappa, \epsilon) V^0_1 + h_2(p, \overline{d}_s; \kappa, \epsilon) V^0_2 \\
+ h_3(p, \overline{d}_s; \kappa, \epsilon) V_3(r) + \overline{d}_s V_4(r) 
\text{small} 
\end{cases} \] (3.5)

where \( p \) is a point on the center manifold and \( \overline{d}_s \) parametrizes the stable fiber. To write the fibers \( F^s_\epsilon \) relative to \( U_*(r) \) instead of relative to \( p \), let \( P^c_1, P^c_2, P^u(r) \), and \( P^s(r) \) be the complementary projection operators onto the subspaces \( V^0_1, V^0_2, V_3(r) \), and \( V_4(r) \) respectively and define \( A := P^c_1 p, B := P^c_2 p \) as the center-manifold coordinates of \( p \). Then the foliation can be expanded

\[ F^s_\epsilon(p, \kappa) = \{ d^c_1 V^0_1 + d^c_2 V^0_2 + d^u V_3(r) + d^s V_4(r) \} \] (3.6a)

where

\[
\begin{align*}
    d^c_1 &:= P^c_1 p + h_1(p, \overline{d}_s; \kappa, \epsilon) = A + O_\kappa \left( |\overline{d}_s|(|A| + |B| + |\overline{d}_s| + \epsilon) \right) \\
    d^c_2 &:= P^c_2 p + h_2(p, \overline{d}_s; \kappa, \epsilon) = B + O_\kappa \left( |\overline{d}_s|(|A| + |B| + |\overline{d}_s| + \epsilon) \right) \\
    d^u &:= P^u(r)p + h_3(p, \overline{d}_s; \kappa, \epsilon) = O_\kappa \left( (|A| + |B| + |\overline{d}_s|)(|A| + |B| + |\overline{d}_s| + \epsilon) \right) \\
    d^s &:= P^s(r)p + \overline{d}_s = O_\kappa \left( (|A| + |B|)(|A| + |B| + \epsilon) \right) + \overline{d}_s. \quad (3.6b)
\end{align*}
\]

The derivation of the expansion (3.6b) follows exactly as in Section 3.3.4.

### 4.3.5 Reduction of the vector field to the center manifold

Next we derive a convenient expansion for the vector field (3.1) restricted to the center manifold. In Proposition 4.3.2(iv) we showed that the flow on the center manifold respects the reverser symmetry \( (\overline{U}, \overline{U}_r, \kappa, r) \mapsto (\overline{U}, -\overline{U}_r, -\kappa, -r) \). We use
this symmetry in the proof of the following lemma.

**Lemma 4.3.5.** Using the coordinates $P_{r+}U = AV_1^0 + BV_2^0$, with $A = d_1$ and $B = d_2$, the vector field (3.1) restricted to $W_{r+}^c(\epsilon)$ can be written as

\[
A_r = B
\]

\[
B_r = -\kappa B + \epsilon A + c_2^0 A^2 + \mathcal{R}_B(A, B, \kappa; \epsilon)
\]

\[
\kappa_r = -\kappa^2
\]

where $\text{sgn}(c_2^0) = \text{sgn}(b_b/a_b)$ with

\[
a_b := 1 + \alpha \xi_b + \alpha \eta_b - \eta_b \xi_b,
\]

\[
b_b := -\frac{\Upsilon_0(1 + \eta_b^2)}{\Upsilon_1^2}
\]

and where $\eta_b$, $\xi_b$, $\Upsilon_0$, and $\Upsilon_1$ were defined in equations (1.4) and (1.5)

\[
R_0^2 = \frac{\omega \beta - \mu}{1 + \beta^2}, \quad \Upsilon_0 := \frac{R_0}{\sqrt{1 + \eta_b^2}}, \quad \Upsilon_1 := \text{sgn}[\xi_b \eta_b + 1] \sqrt{\frac{\eta_b}{(\xi_b \eta_b + 1)(\xi_b - \eta_b)}}
\]

\[
\eta_b := \beta + \text{sgn}(\omega + \beta \mu) \sqrt{1 + \beta^2}, \quad \xi_b := \frac{\eta_b \omega + (1 - \beta \eta_b)R_0^2}{\omega - (\beta + \eta_b)R_0^2}
\]

The remainder term $\mathcal{R}_B$ satisfies

\[
\mathcal{R}_B(A, B, \kappa; \epsilon) = O((\epsilon + \kappa^2)|\epsilon A| + (\epsilon + \kappa^2)A^2 + |A^3| + (\epsilon + \kappa^2 + A^2 + B^2)|\kappa B|
\]

\[
+ |AB^2| + \epsilon^2 B^2 + B^4).
\]

**Remark 4.3.6.** It is straightforward to observe that the coefficient $c_2^0 \neq 0$ for all $\alpha, \beta, \gamma, \mu, \omega < \infty$, provided $R_0^2 \neq 0, \omega/\beta$ since $\Upsilon_0 = 0$, $\xi_b = \eta_b$ only for $R_0 = 0$ and $\xi_b = -1/\eta_b$ only for $R_0^2 = \omega/\beta$. Furthermore, $R_0^2 = 0 \iff \mu = \omega \beta$ and
\[ R_0^2 = \omega / \beta \iff \omega = -\beta \mu. \]

**Proof.** Define

\[
W = \begin{pmatrix}
\tilde{U}_0 & 0 & \tilde{U}_1 & \tilde{U}_1 \\
0 & \tilde{U}_0 & m\tilde{U}_1 & -m\tilde{U}_1
\end{pmatrix}
\text{ and } \Psi = \begin{pmatrix} A \\ B \\ \psi^u \\ \psi^s \end{pmatrix}.
\] (3.8)

Project onto the first two components of

\[ \Psi' = W^{-1} \left( A(\kappa)W\Psi + \mathcal{F}(W\Psi) \right) \] (3.9)

using the expansion \( \psi^j = \psi^j(A, B, \kappa, \epsilon) = O((|A| + |B|)(|A| + |B| + |\kappa| + \epsilon)) \) for the flow-invariant hyperbolic projection of the center manifold. Then (3.7) is obtained by enforcing that the resulting vector field respects the reverser symmetry.

The value of \( c_0^2 \) as a function of the parameters is given in [8, (C.18) pp. 702-705].

**Remark 4.3.7.** We alternatively could have proved Lemma 4.3.5 in the same way as Lemma 3.3.6 by modifying [45, Lemma 3.9 in Chapter 3]. This would have resulted in higher order terms in the evolution equation for \( A \). Then, in order to prove Proposition 4.3.8 below, we would need to prove the analogous statement to Proposition 3.4.1, which was used in the proof of Proposition 3.3.7.

### 4.3.6 Matching core and far-field stable manifolds

Finally, we use the results of the preceding sections to prove the following result.
Theorem 4.2. Fix $z(\omega - \omega_z) < 0$, $\mu < \beta \omega$, $\omega \neq -\beta \mu$, and let $\gamma = \gamma_b + \epsilon^2$. Then there is an $\epsilon_0 > 0$ so that (2.3) has a nontrivial stationary localized radial solution of amplitude $O(\epsilon)$ for each $\epsilon \in (0, \epsilon_0]$.

We will prove Theorem 4.2 by showing that there exists a nontrivial intersection of the core manifold $\tilde{W}^{cu}(\epsilon)$ and the far-field stable manifold $\tilde{W}_s^{\pm}(\epsilon)$. Recall that the core manifold consists of all solutions which remain bounded as $r \to 0$, while the far-field stable manifold is the set of all solutions that decay to zero as $r \to \infty$. Any nontrivial solution laying in the intersection of these two sets is, by definition, a localized solution to (2.3). The core manifold was constructed in Section 4.2.2 on bounded intervals $r \in [0, r_0]$, whereas the far-field stable manifold relies on an analysis of the far-field center-stable and center manifolds; the existence proofs of these far-field manifolds are only valid for $r \geq 1/\rho_1$. In the proof of Theorem 4.2 we will choose $r_0$ large enough so that $1/\rho_1 < r_0$, which ensures that both $W_{r \pm}^{c}(\epsilon)$ and $W_{r_0}^{cs}(\epsilon)$ exist at the matching point $r = r_0$.

In order to complete the construction of $\tilde{W}_s^{\pm}(\epsilon)$, it remains to find solutions on the center manifold that decay to zero as $r \to \infty$. The following result, which we will prove in Section 4.4 using the blow-up coordinates of [38], characterizes such solutions.

**Proposition 4.3.8** (proven in Section 4.4). Assume $c_2^0 \neq 0$. Fix $\hat{\eta}_0 > 0$ and $r_0 > 1/\rho_1$, where $\rho_1 \leq \rho_0$ was chosen in the proof of Proposition 4.3.2. Then there exists an $a_0 > 0$ so that for every fixed $\delta_0 > 0$ there is an $\epsilon_0 > 0$ so that for each $\hat{\eta} \in [0, \hat{\eta}_0]$
and each $\epsilon$ with $0 < \epsilon \leq \epsilon_0$ there exists an initial condition of the form

$$
A(r_0) = \epsilon \left( a_0 + O(\delta_0^2 + 1/r_0^2) \right) + O_{\delta_0,r_0} \left( \epsilon^{3/2}(|\hat{\eta}| \ln \sqrt{\epsilon} + 1) \right)
$$

$$
B(r_0) = -\epsilon^{3/2}\hat{\eta} \left( a_0 + O(\delta_0^2 + 1/r_0^2) \right)
$$

$$
+ O_{\delta_0,r_0} \left( \epsilon^{3/2}(|\hat{\eta}|^2 \sqrt{\epsilon} \ln \sqrt{\epsilon} + \sqrt{\epsilon} |\hat{\eta}| + \sqrt{\epsilon} \ln \sqrt{\epsilon}) \right),
$$

(3.10)

so that the corresponding solution of (3.7) decays to zero exponentially as $r \to \infty$.

The Landau symbol $O_{\delta_0,r_0}$ is interpreted in the usual sense, except that the subscript $(\delta_0, r_0)$ means that the bounding constant and region of validity may depend on $\delta_0$ and $r_0$.

We can now complete the proof of Theorem 4.2. As shown in (3.6), the strong stable fiber associated with solutions (3.10) can be parametrized as

$$
\mathcal{F}^s_{\epsilon}(1/r_0, p) = \{ d_1^sV_1^0 + d_2^sV_2^0 + d_3^sV_3(r_0) + d_4^sV_4(r_0) \}
$$

(3.11a)
where, due to (3.6b) and (3.10), we have

\[ d_1 = A + O_{r_0} \left( |\tilde{d}_s|(|A| + |B| + |\tilde{d}_s| + \varepsilon) \right) \]
\[ = \varepsilon \left( a_0 + O(\delta^3_0 + 1/r^2_0) \right) + O_{\delta_0, r_0} \left( \varepsilon^{3/2}(|\tilde{\eta}| \ln \sqrt{\varepsilon} + 1) + |\tilde{d}_s|(\varepsilon + \varepsilon^{3/2}|\tilde{\eta}| + |\tilde{d}_s|) \right) \]
\[ d_2 = B + O_{r_0} \left( |\tilde{d}_s|(|A| + |B| + |\tilde{d}_s| + \varepsilon) \right) \]
\[ = - \varepsilon^{3/2}\tilde{\eta} \left( a_0 + O(\delta^3_0 + 1/r^2_0) \right) \]
\[ + O_{\delta_0, r_0} \left( \varepsilon^2(|\tilde{\eta}|^2 \ln \sqrt{\varepsilon} + |\tilde{\eta}| \ln \sqrt{\varepsilon}) + |\tilde{d}_s|(\varepsilon + \varepsilon^{3/2}|\tilde{\eta}| + |\tilde{d}_s|) \right) \]
\[ d^n = O_{r_0} \left( (|A| + |B| + |\tilde{d}_s|)(|A| + |B| + |\tilde{d}_s| + \varepsilon) \right) \]
\[ = O_{\delta_0, r_0} \left( \varepsilon^2 + \varepsilon^3|\tilde{\eta}|^2 + |\tilde{d}_s|^2 \right) \]
\[ d^s = O_{r_0} \left( (|A| + |B|)(|A| + |B| + \varepsilon) \right) + \tilde{d}_s \]
\[ = O_{\delta_0, r_0} \left( \varepsilon^2 + \varepsilon^3|\tilde{\eta}|^2 \right) + \tilde{d}_s. \]  

(3.11b)

**Proof of Theorem 4.2** We find a nontrivial solution contained in the intersection \( \tilde{W}_{cu}^-(\varepsilon) \cap \tilde{W}^s_+ (\varepsilon) \) by matching the coefficients of each manifold at \( r = r_0 \) in the directions \( V^0_1, V^0_2, V_3(r_0), \) and \( V_4(r_0) \). The manifold \( \tilde{W}^s_+(\varepsilon)|_{r=r_0} \) is parametrized by (3.11), whilst the manifold \( \tilde{W}_{cu}^-(\varepsilon)|_{r=r_0} \) is parametrized by (2.10),

\[ \tilde{W}^-_{cu}(\varepsilon)|_{r=r_0} \]
\[ = (d_1 + \ln r_0 g_2(d_1, d_3; \varepsilon)) V^0_1 + \frac{1}{r_0} g_2(d_1, d_3; \varepsilon) V^0_2 + d_3 V_3(r_0) + g_4(d_1, d_3; \varepsilon) V_4(r_0), \]

where \((g_2, g_4)(d_1, d_3; \varepsilon) = O_{r_0}(\varepsilon|d| + |d|^2)\) and \(d = (d_1, d_3)\). Collecting the expansion of \( \tilde{W}^-_{cu}(\varepsilon)|_{r=r_0} \) on the left-hand side and of \( \tilde{W}^s_+(\varepsilon)|_{r=r_0} \) on the right-hand side, we get
Using these expansions, we see that the remaining equation for the near the origin and, afterwards, match orders in \( \epsilon \) sufficiently large.

We solve the first, third, and fourth equation for \((\epsilon, \hat{\eta})\) near zero by finding zeros of \( F_1, F_2, F_3, F_4 \) for all \( \epsilon < 0 \) and \( |\hat{\eta}| < \hat{\eta}_0 \), where

\[
F(d_1, d_3, \overline{d}_s; \epsilon, \hat{\eta}) = \begin{pmatrix}
    d_1 + O_{\delta_0, r_0} (\epsilon |d| + |d|^2) - \epsilon (a_0 + O(\delta_0^2 + 1/r_0^2)) \\
    + O_{\delta_0, r_0} (\epsilon^{3/2} |\hat{\eta}| \ln \sqrt{\epsilon} + 1 + |\overline{d}_s| (\epsilon + \epsilon^{3/2} |\hat{\eta}| + |\overline{d}_s|)) \\
  d_3 + O_{\delta_0, r_0} (\epsilon^2 + \epsilon^3 |\hat{\eta}|^2 + |\overline{d}_s|^2) \\
  O_{\delta_0, r_0} (\epsilon |d| + |d|^2 + \epsilon^2 + \epsilon^3 |\hat{\eta}|^2) - \overline{d}_s
\end{pmatrix}.
\]

Note that \( F(0; 0) = 0 \) and that \( D F(0; 0) \) is invertible for all sufficiently small \( \delta_0 \) and sufficiently large \( r_0 \) since \( a_0 \neq 0 \): hence, we can apply the implicit function theorem near the origin and, afterwards, match orders in \( \epsilon \) to find the expansions

\[
d_1 = \epsilon (a_0 + O(\delta_0^2 + 1/r_0^2) + O_{\delta_0, r_0} (\epsilon^{3/2} \ln \sqrt{\epsilon})), \quad d_3 = O_{\delta_0, r_0} (\epsilon^2), \quad \overline{d}_s = O_{\delta_0, r_0} (\epsilon^2).
\]

Using these expansions, we see that the remaining equation for the \( V_2^0 \) coordinate becomes \( \epsilon^{3/2} \mathcal{G}(\hat{\eta}; \epsilon) = 0 \) where

\[
\mathcal{G}(\hat{\eta}; \epsilon) = \hat{\eta} (a_0 + O(\delta_0^2 + 1/r_0^2)) + O_{\delta_0, r_0} (\sqrt{\epsilon} \ln \sqrt{\epsilon} + \sqrt{\epsilon} |\hat{\eta}| + |\hat{\eta}|^2 \sqrt{\epsilon} \ln \sqrt{\epsilon}).
\]
We again apply the implicit function theorem to find zeros of $G(\cdot; \epsilon)$ and, matching orders in $\epsilon$, we arrive at $\hat{\eta} = O_{\delta_0, r_0}(\sqrt{\epsilon} \ln \sqrt{\epsilon})$.

4.4 Dynamics on the center manifold

We complete the proof of Theorem 4.2 by proving Proposition 4.3.8.

4.4.1 Rescaling and transition charts

We first define $z := B/A = A_r/A$. Next, we augment (3.7) by the evolution equation for $\epsilon$, given by $\epsilon_r = 0$. We then blow-up the vector field and all four variables in two different directions. First, we blow-up the vector field (3.7) in the $\kappa$ direction using the coordinates

\[ A_1 := \frac{A}{\kappa^2}, \quad z_1 := \frac{z}{\kappa}, \quad \kappa_1 := \kappa, \quad \epsilon_1 := \frac{\epsilon}{\kappa^2}, \quad \tau := \ln r, \quad (4.1) \]

called the “transition chart” coordinates. Using the rescaled time $e^\tau := r$, we obtain

\[
\begin{align*}
\partial_\tau A_1 &= A_1(z_1 + 2) \\
\partial_\tau z_1 &= \epsilon_1 + c_2^0 A_1 - z_1^2 + \kappa_1^2 R_1(A_1, z_1, \epsilon_1) \\
\partial_\tau \kappa_1 &= -\kappa_1 \\
\partial_\tau \epsilon_1 &= 2\epsilon_1 
\end{align*}
\] (4.2)
with $R_1(A_1, \epsilon_1) = O(\epsilon_1 + A_1 + z_1)$. The vector field (4.2) has the fixed point $P_1 = (0, 0, 0, 0)$. The linearization of (4.2) about $P_1$ is

$$
\begin{pmatrix}
\tilde{A}_1 \\
\tilde{z}_1 \\
\tilde{\kappa}_1 \\
\tilde{\epsilon}_1
\end{pmatrix}
= \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & c_2^0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_1 \\
\tilde{z}_1 \\
\tilde{\kappa}_1 \\
\tilde{\epsilon}_1
\end{pmatrix}
$$

with eigenvalues $\{2, 0, -1, 2\}$ with associated eigenvectors

$$v_1^1 = (2, c_2^0, 0, 0)^T, \quad v_2^2 = (0, 1, 0, 0)^T, \quad v_3^3 = (0, 0, 1, 0)^T, \quad v_4^4 = (-1, 0, 0, c_2^0)^T.$$

Therefore, the strong stable manifold $W^{ss}_{\tau_{-\epsilon}}$ of $P_1$ is two-dimensional and given to leading order near $P_1$ by $\text{span}\{v_1^1, v_4^4\}$.

Next, we blow-up the vector field (3.7) in the $\epsilon$-direction using the coordinates

$$A_2 := \frac{A}{\epsilon}, \quad z_2 := \frac{z}{\sqrt{\epsilon}}, \quad \kappa_2 := \frac{\kappa}{\sqrt{\epsilon}}, \quad \epsilon_2 := \epsilon, \quad s := \sqrt{\epsilon r} = \frac{1}{\kappa_2},$$

(4.4)
called the “rescaling chart” coordinates. We remark that solutions in the coordinate systems (4.1) and (4.4) are related through the relationships

$$A_1 = s^2 A_2, \quad z_1 = s z_2, \quad \kappa_1^2 \epsilon_1 = \epsilon_2, \quad \epsilon_1 = c_2 \epsilon_2 = \frac{1}{\kappa_2^2} = s^2.$$

(4.5)
Using (4.4) and the rescaled time $s := \epsilon r$, we obtain

\[
\begin{align*}
\partial_s A_2 &= z_2 A_2 \\
\partial_s z_2 &= -\kappa_2 z_2 + c_2^0 A_2 - z_2^2 + 1 + \epsilon_2 R_2(A_2, z_2, \kappa_2) \\
\partial_s \kappa_2 &= -\kappa_2^2 \\
\partial_s \epsilon_2 &= 0
\end{align*}
\]  
(4.6)

with $R_2(A_2, z_2, \kappa_2) = O(1 + \kappa_2^2 + A_2 + \kappa_2 z_2)$. We first set $\epsilon = \epsilon_2 = 0$. Then the vector field (4.6)$_{\epsilon, \epsilon_2 = 0}$ has the two fixed points $P_{2}^{\pm} = (0, \pm 1, 0, 0)$. We will show at the end of this section that $P_{2}^{-}$ is associated with exponentially decaying solutions to (3.7) in the original coordinates. The linearization of (4.6) about $P_{2}^{-}$ is

\[
\begin{pmatrix}
\tilde{A}_2 \\
\tilde{z}_2 \\
\tilde{\kappa}_2 \\
\tilde{\epsilon}_2
\end{pmatrix}_s =
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
c_2^0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
A_2 \\
z_2 \\
\kappa_2 \\
\epsilon_2
\end{pmatrix}.
\]  
(4.7)

The linearization (4.7) has eigenvalues \{-1, 2, 0, 0\} with associated eigenvectors

\[
v_2^1 = (-3, c_2^0, 0, 0)^T, \quad v_2^2 = (0, 1, 0, 0)^T, \quad v_2^3 = (0, -1, 2, 0)^T, \quad v_2^4 = (0, 0, 0, 1)^T.
\]

For $\epsilon = 0$, the two-dimensional center-stable manifold $W_{s+}^{cs}$ of $P_{2}^{-}$ is given to leading order by span \{\$v_2^1, v_2^3\$. We remark that $\epsilon_2(s) = \epsilon_2 = \epsilon$ is constant for all $s$; therefore, the $v_4$ direction is neutral and not contained in $W_{s+}^{cs}$, to leading order.

For all $\epsilon$ small enough, the fixed points $P_{2}^{\pm}(\epsilon)$ and all invariant manifolds $W_{r-}^{u}(\epsilon)$, $W_{s+}^{cs}(\epsilon)$ persist and depend smoothly on $\epsilon$. For simplicity of notation, all fixed points and invariant manifolds are evaluated at $\epsilon = 0$ unless we explicitly indicate their
dependence on $\epsilon$.

### 4.4.2 Singular connecting orbit between transition and rescaling charts

We first set $\epsilon = 0$ so that $\epsilon_2 = 0$. Inspecting (4.5) shows that $\kappa_1 = 0$ as well. The rescaling-chart vector field (4.6) then reduces to

$$
\begin{align*}
\partial_s A_2 &= z_2 A_2 \\
\partial_s z_2 &= -\kappa_2 z_2 + c_2^0 A_2 - z_2^2 + 1 \\
\partial_s \kappa_2 &= -\kappa_2^2
\end{align*}
$$

(4.8)

and the transition chart vector field (4.2) reduces to

$$
\begin{align*}
\partial_\tau A_1 &= A_1(z_1 + 2) \\
\partial_\tau z_1 &= \epsilon_1 + c_2^0 A_1 - z_1^2 \\
\partial_\tau \epsilon_1 &= \epsilon_1.
\end{align*}
$$

(4.9)

Note that the equations for $\kappa_2$ and $\epsilon_1$ decouple from the rest of the system. Using $\kappa_2(s) = 1/s$ we can rewrite (4.8) as a second-order equation with $a := A_2$

$$
\begin{align*}
a_{ss} + \frac{a_s}{s} &= a + c_2^0 a^2, \quad s > 0.
\end{align*}
$$

(4.10)

We have the following result.

**Lemma 4.4.1.** For $c_2^0 \neq 0$, (4.10) has a unique monotonically decreasing, nontrivial, bounded solution $q_0(s)$. Furthermore, there exist constants $a_0, a_2, c_1 > 0$ such that
The solution $q_0(s)$ satisfies

$$q_0(s) = a_0 - a_2 s^2 + O(s^4 \ln s) \quad \text{as } s \to 0,$$

and

$$q_0(s) = K_0(s) \left( c_1 + O(e^{-2s}) \right) \quad \text{as } s \to \infty.$$

Lastly, the linearization of (4.10) about $q_0(s)$ does not have a nontrivial solution that is bounded uniformly on $\mathbb{R}^+$. For $c_3^0 > 0$, the only bounded solution to (4.10) is $a(s) \equiv 0$.

**Proof.** First consider $c_2^0 < 0$. Without loss of generality, let $c_2^0 = -1$; otherwise rescale $a$. Then equation (4.10) reduces to

$$a_{ss} + a_s/s = a(1 - a). \quad (4.11)$$

Equation (4.11) arises in many applications and has been well studied; it is related, for example, to the nonlinear Schrödinger equation. The existence of a positive, radial solution $q_0(s) \in H^1(\mathbb{R}^2)$ to $\Delta a - a = -a^2$ in two dimensions which exponentially decays as $|s| \to \infty$ follows from an application of [51, Thm. 2, 3]. In fact, since $\partial_s q_0(s) \in L^2(\mathbb{R})$, we actually have by equation (4.11) that $\partial_s q_0(s) \in L^2(\mathbb{R}^2)$ so that $q_0(s) \in H^2(\mathbb{R}^2)$. Repeating the argument twice we have $q_0(s) \in H^4(\mathbb{R}^2)$; and by the Sobolev embedding theorem, $q_0(s) \in C^2_0(\mathbb{R}^2)$ (the space of continuous bounded functions which tend to 0 as $s \to \infty$) since $H^4(\mathbb{R}^2) \subset C^2_0(\mathbb{R}^2)$ [52, Theorem 6.7]. Thus the hypotheses of [24, Thm. 1] are satisfied and $q_0(s)$ is monotone; it is unique by [34]. The nonexistence of a nontrivial bounded solution of the linearization is shown in [10, Lemma 2.1]. The asymptotics of $q_0(s)$ follow from the variation-of-constants formula and a standard fixed-point argument in each limit; the sign of $a_2$ then follows from monotonicity. The derivation of the asymptotic expansion of $q_0$ is completely analogous to the standard oscillon case. See Appendices C and D for
details on standard oscillons in the limits \( s \to 0 \) and \( s \to \infty \), respectively.

For \( c_0^2 > 0 \), replace \( a \) by \(-a\). \(\blacksquare\)

We now show that \( q_0(s) \) gives a connecting orbit between the fixed points \( P_1 \) and \( P_2^- \).

**Lemma 4.4.2.** Assume \( c_2^0 \neq 0 \) so that Lemma 4.4.1 is true and set \( \epsilon = \epsilon_2 = \kappa_1 = 0 \). Then \( Q_0 \), given in the transition and rescaling chart coordinates by

\[
\begin{align*}
Q_0^1(\tau) &= \left\{ (A_1, z_1, \epsilon_1)(\tau) = \left( e^{2\tau} q_0(e^\tau), e^\tau \frac{q_0'(e^\tau)}{q_0(e^\tau)}, e^{2\tau} \right) \right\} \text{ and } \\
Q_0^2(s) &= \left\{ (A_2, z_2, \kappa_2)(s) = \left( q_0(s), \frac{q_0(s)}{q_0'(s)}, \frac{1}{s} \right) \right\}
\end{align*}
\]

(4.12)

respectively, forms a connecting orbit between \( P_1 \) of (4.9) and \( P_2^- \) of (4.8), which lies in the intersection \( W^{u}_{\tau_-} \cap W^{cs}_{s^+} \). Moreover, the intersection \( W^{u}_{\tau_-} \cap W^{cs}_{s^+} \) is transverse.

**Proof.** The proof is essentially the same as in [38, Lemma 2.4]. Using the asymptotic expansions for \( q_0(s) \) given in Lemma 4.4.1, it is easy to check that \( Q_0^1(\tau) \) converges to \( P_1 \) exponentially as \( \tau \to -\infty \) and that \( Q_0^2(s) \) converges to \( P_2^- \) as \( s \to \infty \). The argument is completely analogous to the standard oscillon case, the details of which are explained in Appendix sections C.3 and D.3, respectively. Since \( Q_0^1(\tau) \) and \( Q_0^2(s) \) satisfy (4.9) and (4.8), respectively, we have that \( Q_0 \in W^{u}_{\tau_-} \cap W^{cs}_{s^+} \).

To show that the intersection of \( W^{u}_{\tau_-} \) and \( W^{cs}_{s^+} \) along \( Q_0 \) is transverse, we invoke the nondegeneracy condition in Lemma 4.4.1 and argue by contradiction. Assume that the intersection \( W^{u}_{\tau_-} \cap W^{cs}_{s^+} \) is not transverse, then there exists a nonzero solution \( \hat{Q} \in T_{Q_0^1(\tau)} W^{u}_{\tau_-} \cap T_{Q_0^2(s)} W^{cs}_{s^+} \). Let \( \hat{Q}_1(\tau) \) and \( \hat{Q}_2(s) \) denote the solution \( \hat{Q} \) written in
the transition and rescaling chart coordinates, respectively; we write

\[ \tilde{Q}_2(s) =: (\tilde{A}_2, \tilde{z}_2, \tilde{\kappa}_2)(s). \]

Linearizing (4.6) about \( Q_2^0(s) \), we find that \( \tilde{\kappa}_2 = 0 \) and that \( \tilde{A}_2 \) and \( \tilde{z}_2 \) satisfy

\[
\begin{align*}
\partial_s \tilde{A}_2 &= \frac{q_0'}{q_0} \tilde{A}_2 + q_0 \tilde{z}_2 \\
\partial_s \tilde{z}_2 &= -\frac{1}{s} \tilde{z}_2 + c_2^0 \tilde{A}_2 - 2\frac{q_0'}{q_0} \tilde{z}_2. \quad (4.13)
\end{align*}
\]

Letting \( \tilde{a} := \tilde{A}_2 \), a straightforward computation shows that (4.13) is equivalent to

\[
\tilde{a}_{ss} + \frac{\tilde{a}_s}{s} = \tilde{a}(1 + 2c_2^0 q_0), \quad (4.14)
\]

the linearization of (4.10) about \( q_0(s) \). In particular, \( \tilde{A}_2(s) \) is a nonzero bounded solution of the linearization of (4.11) about \( q_0(s) \), in contradiction to Lemma 4.4.1.

\[ \blacksquare \]

### 4.4.3 Formal scaling argument

We begin with a formal argument to build intuition about the expected solution scaling. We first transform \( z_1 \mapsto \tilde{z}_1 \) so that \( \tilde{z}_1 \equiv 0 \) along \( Q_1^0(\tau) \). To leading order,
vector field (4.2) is given by

\[ \begin{align*}
\partial_\tau A_1 &= 2A_1 \\
\partial_\tau \tilde{z}_1 &= -\tilde{z}_1^2 \\
\partial_\tau \kappa_1 &= -\kappa_1 \\
\partial_\tau \epsilon_1 &= 2\epsilon_1.
\end{align*} \] (4.15)

We explicitly solve \( \epsilon_1 = \delta_0 e^{2\tau}, \kappa_1 = \sqrt{\epsilon/\delta_0} e^{-\tau}, \) with \( \tau = 0 \) at the Poincare section \( \epsilon_1 = \delta_0 \). As discussed in Lemma 4.4.2, the intersection \( \mathcal{W}^u_\tau \cap \mathcal{W}^s_\kappa \) is transverse. Since the transformation into the \( \tilde{z}_1 \) coordinates puts \( \mathcal{W}^u_\tau \) on the \( A_1 \) axis for every fixed \( \epsilon_1 = \delta_0 \), this means that the intersection \( \mathcal{W}^{cs}_\kappa \cap \{ \tilde{z}_1 \} \) is transverse. Thus, we can parametrize \( \mathcal{W}^{cs}_\kappa \) near \( Q^0_1(0) \) in the \( \tilde{z}_1 \) direction; this idea was visualized for standard oscillons in Figure 3.7b. We let \( \eta \ll 1 \) be the resulting parametrization so that the initial data is \( (A_1, \tilde{z}_1)|_{\tau=0} = (a, -\eta) \), where \( a = O(1) \) is the \( A_1 \) component of \( Q^0_1(0) \). Then (4.15) is solved by

\[ \begin{align*}
A_1(\tau) &= a e^{2\tau}, \quad \tilde{z}_1(\tau) = \frac{\eta}{1 + \eta \tau}.
\end{align*} \]

Letting \( \delta_0 = r_0 = a = 1 \), we find that \( \kappa_1 = 1/r_0 \) at time \( \tau_* = \ln \sqrt{\epsilon} \) so that

\[ A_1(\tau_*) = \epsilon, \quad \tilde{z}_1(\tau_*) = \eta. \]

In the original \( A, B \) coordinates we therefore have

\[ Pc\tilde{\mathcal{W}}^{cs}_+(\epsilon)|_{\tau=r_0} = \epsilon V^0_1 + \eta \epsilon V^0_2. \]
We match with \( \tilde{W}^{-cu}(\epsilon)|_{\tau=r_0} \) within the center manifold, where, to leading order,

\[
P^{c}\tilde{W}^{-cu}(\epsilon)|_{\tau=r_0} = \{d_1V_1^0 + O(d_1(\epsilon + d_1))V_2^0 : d_1 \in \mathbb{R}, \text{ small}\}.
\]

(4.16)

Then intersection \( P^{c}\tilde{W}^{-cu}(\epsilon)|_{\tau=r_0} \cap P^{c}\tilde{W}^{s}(\epsilon)|_{\tau=r_0} \) satisfies

\[
V_1^0 : \quad \epsilon = d_1
\]

\[
V_2^0 : \quad \eta\epsilon = O(d_1(\epsilon + d_1)).
\]

We find that \( d_1 = O(\epsilon) \) and \( \eta = O(\epsilon) \).

### 4.4.4 The dynamics near \( P_1 \) in the transition chart coordinates

Our goal is to use Lemma 4.4.2 to trace \( W^{\epsilon,c}_{s+}(\epsilon) \) backwards in time to the equilibrium \( P_1 \) and beyond for all \( \epsilon < \epsilon_0 \). We will find it convenient to first transform the vector field (4.2)

\[
\begin{align*}
\partial_{\tau}A_1 &= A_1(z_1 + 2) \\
\partial_{\tau}z_1 &= -z_1^2 + \epsilon_1 + \epsilon_2^0A_1 + \kappa_1^2 O(\epsilon_1 + A_1 + z_1) \\
\partial_{\tau}\kappa_1 &= -\kappa_1 \\
\partial_{\tau}\epsilon_1 &= 2\epsilon_1
\end{align*}
\]

(4.17)

in the transition chart coordinates into a more convenient form by straightening the center, stable, and unstable manifolds near \( P_1 = (0,0,0,0) \) as well as the strong stable and strong unstable fibers.
Lemma 4.4.3. There is a smooth change of coordinates of the form

\[ \tilde{z}_1 = z_1 + O(z_1\kappa_1^2 + A_1 + \epsilon_1) \]  

(4.18)

that transforms vector field (4.22) near \( P_1 \) into

\[ \partial_\tau A_1 = A_1 \left( 2 + O(|\tilde{z}_1|(1 + \kappa_1^2) + A_1 + \epsilon_1) \right) \]

\[ \partial_\tau \tilde{z}_1 = -\tilde{z}_1^2 + \kappa_1^2 O(A_1 + \epsilon_1) \]

\[ \partial_\tau \kappa_1 = -\kappa_1 \]

\[ \partial_\tau \epsilon_1 = 2\epsilon_1. \]  

(4.19)

The inverse transformation is given by

\[ z_1 = \tilde{z}_1 + O(\tilde{z}_1\kappa_1^2 + A_1 + \epsilon_1). \]  

(4.20)

Proof. We first remove the lower order terms in the evolution equation for \( z_1 \) in (4.17) through the transformation

\[ z_1 = z_1 - \frac{e_0^0}{2} A_1 - \frac{1}{2} \epsilon_1. \]  

(4.21)

Transformation (4.21) puts (4.17) in the form

\[ \partial_\tau A_1 = A_1 \left( 2 + O(|z_1| + A_1 + \epsilon_1) \right) \]

\[ \partial_\tau z_1 = -z_1^2 + z_1(3e_0^0 A_1/2 + \epsilon_1) + O \left( \kappa_1^2(\epsilon_1 + A_1 + z_1) + A_1^2 + \epsilon_1^2 \right) \]

\[ \partial_\tau \kappa_1 = -\kappa_1 \]

\[ \partial_\tau \epsilon_1 = 2\epsilon_1 \]  

(4.22)
and has the effect of rotating the linearized eigenspaces so that now

\[ \begin{align*}
\overline{v}_1 & = (1, 0, 0, 0)^T, \\
\overline{v}_2 & = (0, 1, 0, 0)^T, \\
\overline{v}_3 & = (0, 0, 1, 0)^T, \\
\overline{v}_4 & = (0, 0, 0, 1)^T.
\end{align*} \]

We note that the center-stable and center-unstable manifolds of \( P_1 \) are given by

\[ \mathcal{W}^{cs}_{\tau = 0} = \{ A_1 = \epsilon_1 = 0 \} \] and \( \mathcal{W}^{cu}_{\tau = 0}(\epsilon) = \{ \kappa_1 = 0 \} \), respectively. Furthermore,

\[ \begin{align*}
\mathcal{W}^s_{\tau = 0} & \subset \mathcal{W}^{cs}_{\tau = 0}(\epsilon) = \{ A_1 = z_1 = \epsilon_1 = 0 \} \\
\mathcal{W}^c_{\tau = 0} & := \mathcal{W}^{cs}_{\tau = 0}(\epsilon) \cap \mathcal{W}^{cu}_{\tau = 0}(\epsilon) = \{ A_1 = \kappa_1 = \epsilon_1 = 0 \}.
\end{align*} \]

We also know that solutions on the center manifold are given by

\[ \bar{z}_1 = \overline{z}_1(\tau) := \frac{1}{c_1 + \tau} \quad \text{with} \quad c_1 = \frac{1}{\overline{z}_1(0)}. \]

We first show that there exists a smooth change of coordinates of the form

\[ \tilde{z}_1 = z_1 + O((A_1 + \epsilon_1)(A_1 + \epsilon_1 + \tilde{z}_1)) \quad (4.23) \]

which transforms the evolution of \( z_1 \) in equation (4.22) near \( P_1 \) into

\[ \partial_\tau \tilde{z}_1 = -\tilde{z}_1^2 + \kappa_1^2 O(\epsilon_1 + A_1 + \tilde{z}_1). \quad (4.24) \]

To achieve this, we straighten out the strong unstable fibers within the center-unstable manifold: let \((A_1, z_1, \epsilon_1) = (0, \tilde{z}_1, 0)\) be a point on the center manifold within the center-unstable manifold. Then, for every \( A_1, \tilde{z}_1, \epsilon_1 \ll 1 \), the strong
unstable fiber associated with base point \( z_1 = \tilde{z}_1 \) can be written as a graph

\[
\tilde{z}_1 = z_1(\tilde{z}_1; A_1, \epsilon_1) = \tilde{z}_1 + O((|A_1| + |\epsilon_1|)(|A_1| + |\epsilon_1| + |\tilde{z}_1|)).
\]  

(4.25)

Now the evolution of \( \tilde{z}_1 \) within the center-unstable manifold is independent of \( A_1 \) and \( \epsilon_1 \) and must therefore be of the form (4.24).

Next we straighten the strong stable fibers within the center-stable manifold. We let \((\tilde{z}_1, \kappa_1) = (\tilde{z}_1, 0)\) be a point on the center manifold within the center-stable manifold after transformation (4.23). Completely analogously, we straighten the stable fibers within the center-stable manifold via the transformation

\[
\tilde{z}_1 = \tilde{z}_1(\tilde{z}_1; \kappa_1) = \tilde{z}_1(1 + O(\kappa_1)).
\]  

(4.26)

We claim that the graph (4.26) is actually of the form

\[
\tilde{z}_1 = \tilde{z}_1(\tilde{z}_1; \kappa_1) = \tilde{z}_1(1 + O(\kappa_1^2)).
\]  

(4.27)

Letting \( \hat{z}_1 \) be the linearized \( z_1 \) coordinate, we have that \( |\hat{z}_1(\tau) - \tilde{z}_1^*(\tau)| = O(e^{\nu \tau}) \) for some \( \nu > 0 \) in the linearized strong stable fiber with \( \tau \geq 0 \). Solving the linearized vector field in the center-stable manifold

\[
\partial_\tau \hat{z}_1 = -2\tilde{z}_1^*(\tau)\hat{z}_1
\]

for \( \hat{z}_1 \), we have \( \hat{z}_1 = c_2/(c_1 + \tau)^2 \) so that \( c_2 = 0 \). The expansion for the strong unstable fiber which has tangent space \( \hat{z}_1 = 0 \) is given by (4.27). The evolution of \( \tilde{z}_1 \) is independent of \( \kappa_1 \) within the center-stable manifold. It is also still independent of \( A_1 \) and \( \epsilon_1 \) within the center-unstable manifold since (4.27) has no effect when \( \kappa_1 = 0 \). Therefore, the evolution of \( \tilde{z}_1 \) must be of the form given in (4.19).
By combining (4.25) and (4.27) we obtain the inverse transformation (4.20). We apply the implicit function theorem near \((A_1, \tilde{z}_1, \epsilon_1, \kappa_1; \tilde{z}_1) = (0, 0, 0, 0; 0)\) to invert (4.20); the result is the transformation (4.18).

4.4.5 Passage through transition chart coordinates

We use the transversality of \(q_0(s)\) stated in Lemma 4.4.2 to parametrize \(\mathcal{W}_{s+}^{cs}(\epsilon)\) near \(P_1(\epsilon)\) in the coordinates of (4.18). Throughout this section we use a tilde to denote any object which has been transformed into the coordinates (4.18).

**Lemma 4.4.4.** Assume \(c_2^0 \neq 0\) so that Lemma 4.4.2 is true. For each sufficiently small \(\delta_0 > 0\), there exist constants \(\eta_0, \epsilon_0 > 0\) such that, for all \(0 \leq \epsilon \leq \epsilon_0\), the following is true. Define the Poincare section \(\Sigma_{\delta_0} := \{\epsilon_1 = \delta_0^2\}\) and let \(\mathcal{W}_{s+}^{cs}(\epsilon)\) denote the center-stable manifold transformed into the coordinates of (4.18). Then

\[
\mathcal{W}_{s+}^{cs}(\epsilon) \cap \Sigma_{\delta_0} = \{(A_1, \tilde{z}_1, \kappa_1, \epsilon_1) = (\delta_0^2 q_0(\delta_0) + O(|\eta| + \epsilon), -\eta, \sqrt{\epsilon/\delta_0}, \delta_0^2) : \eta \in (-\eta_0, \eta_0)\} \quad (4.28)
\]

near \(\tilde{Q}_2^0(s)\).

**Proof.** First consider \(\epsilon = 0\). Lemma 4.4.2 shows that the connecting orbit \(Q_1^0(\tau)\) is contained in the unstable manifold of \(P_1\) so that, after transformation into the coordinates of (4.4.3), the \(\tilde{z}_1\) component of \(Q_1^0(\tau)\) is zero for all \(\tau\). Therefore, the connecting orbit in the transformed coordinates is given by

\[
\tilde{Q}_1^0(\tau) \cap \Sigma_{\delta_0} = \{(A_1, \tilde{z}_1, \kappa_1, \epsilon_1) = (\delta_0^2 q_0(\delta_0), 0, 0, \delta_0)\}.
\]

In the transformed coordinates, the strong unstable manifold \(\tilde{W}_{s-}^u\) of \(P_1\) is the \(A_1\)-
axis. Using the transversality of the intersection $\tilde{W}_{s+}^{cs} \cap \tilde{W}_{\tau}^{u}$ we can parametrize $\tilde{Q}_1^0 \cap \Sigma_{\delta_0}$ by $\eta$, a small offset in the $\tilde{z}_1$ direction, so that

$$\tilde{W}_{s+}^{cs}(0) \cap \Sigma_{\delta_0} = \{(A_1, \tilde{z}_1, \kappa_1, \epsilon_1) = (\delta_0 q_0(\delta_0) + O(|\eta|), -\eta, 0, \delta_0)\}.$$  

The fixed points, invariant manifolds, and connecting orbit are all smooth in $\epsilon$ so that for $0 \leq \epsilon \leq \epsilon_0$ the parametrization is given by (4.28).

Lastly, we propagate the initial data (4.28) under vector field (4.19) backwards until the matching point $\kappa_1 = 1/r_0$.

**Lemma 4.4.5.** Assume $c_2^0 \neq 0$ so that Lemma 4.4.4 is true. Fix $\hat{\eta}_0 > 0$ and $r_0 > 1/\rho_1$, where $\rho_1$ was determined in the proof of Proposition 4.3.2. Then, for each fixed $\delta_0 > 0$ small enough there is an $\epsilon_0 > 0$ so that for all $\eta$ of the form $\eta = \sqrt{\epsilon} \hat{\eta}$ with $\hat{\eta} \in [0, \hat{\eta}_0]$ and all $0 < \epsilon \leq \epsilon_0$, we can solve (4.19) with initial data given by (4.28) at time $\tau = 0$ back to $\tau_* = \ln(r_0 \sqrt{\epsilon}/\delta_0)$. The associated solution at $\tau = \tau_*$ is given by

$$A_1(\tau_*) = \epsilon r_0^2 \left(a_0 + O(\delta_0^2 + 1/r_0^2)\right) + O_{\delta_0, r_0} \left(\epsilon^{3/2}(|\hat{\eta}| \ln \sqrt{\epsilon} + 1)\right)$$

$$\tilde{z}_1(\tau_*) = -\sqrt{\epsilon} \hat{\eta} + O_{\delta_0, r_0}(\epsilon \ln \sqrt{\epsilon})$$

$$\epsilon_1(\tau_*) = \epsilon r_0^2$$

$$\kappa_1(\tau_*) = 1/r_0.$$  (4.29)

In the original coordinates the solution becomes

$$A(\tau_*) = \epsilon \left(a_0 + O(\delta_0^2 + 1/r_0^2)\right) + O_{\delta_0, r_0} \left(\epsilon^{3/2}(|\hat{\eta}| \ln \sqrt{\epsilon} + 1)\right)$$

$$B(\tau_*) = -\epsilon^{3/2} \hat{\eta} \left(a_0 + O(\delta_0^2 + 1/r_0^2)\right)$$

$$+ O_{\delta_0, r_0} \left(\epsilon^{3/2}(|\hat{\eta}|^2 \sqrt{\epsilon} \ln \sqrt{\epsilon} + \sqrt{\epsilon}|\hat{\eta}| + \sqrt{\epsilon} \ln \sqrt{\epsilon})\right).$$  (4.30)
where the Landau symbol $O_{\delta_0, r_0}$ means that the bounding constant and the region of validity may depend on $\delta_0$ and $r_0$.

**Proof.** We set $\tau = 0$ at $\epsilon_1 = \delta_0$, which is possible because the vector field (4.19) is autonomous. A formal analysis shows that, as a first approximation,

$$A_1(\tau) \sim e^{2\tau}, \quad \eta = O(\epsilon), \quad \text{and} \quad z_1(\tau_*) = O(\eta).$$

We therefore define $\eta := \sqrt{\epsilon} \hat{\eta}, \tilde{z}_1 := \sqrt{\epsilon} \hat{z}_1$, and $A_1 := e^{2\tau} \hat{A}_1$ and consider the fixed point system

$$e^{2\tau} \hat{A}_1(\tau) = \left[ \delta_0^2 q_0(\delta_0) + \sqrt{\epsilon} O(\hat{\eta}) \right] e^{2\tau} e^{\int_0^\tau O(\sqrt{\epsilon} \hat{z}_1(1+\kappa_1^2) + e^{2\tau} \hat{A}_1 + \epsilon_1) d\sigma}$$

$$\sqrt{\epsilon} \hat{z}_1(\tau) = -\sqrt{\epsilon} \hat{\eta} + \int_0^\tau \left[ -\epsilon \hat{z}_1^2 + O(\kappa_1^2 (\hat{A}_1 e^{2\sigma} + \epsilon_1)) \right] d\sigma$$

(4.31)

in the $(\hat{A}_1, \hat{z}_1)$ coordinates for $\tau \in [\tau_*, 0]$. Due to the variation of parameters formula, discussed in Appendix A.2, a pair of smooth functions

$$(A_1, z_1) = (e^{2\tau} \hat{A}_1, \epsilon \hat{z}_1)$$

satisfies (4.19) with initial data given by (4.28) if, and only if, $(\hat{A}_1, \hat{z}_1)$ is a fixed point of (4.31). We show that (4.31) has a unique fixed point.

We observe that $\kappa_1$ and $\epsilon_1$ decouple from the rest of the system with $\kappa_1 = \frac{\sqrt{\epsilon} e^{-\tau}}{\delta_0}$ and $\epsilon_1 = \delta_0^2 e^{2\tau}$. We substitute these expressions into (4.31) and explicitly integrate the $O(\kappa_1^2 + \epsilon_1^2)$ terms in the equation for $\hat{A}_1$. Then (4.31) is equivalent to the system

$$\hat{A}_1(\tau) = \left[ \delta_0^2 q_0(\delta_0) + \sqrt{\epsilon} O(\hat{\eta}) \right] e^{O(\delta_0^2) + \int_0^\tau O_{\epsilon_0}(\sqrt{\epsilon} \hat{z}_1 + e^{2\sigma} \hat{A}_1) d\sigma}$$

$$\hat{z}_1(\tau) = -\hat{\eta} + \sqrt{\epsilon} \int_0^\tau \left[ -\hat{z}_1^2 + O_{\delta_0} \left( \hat{A}_1 + 1 \right) \right] d\sigma.$$  

(4.32)
Using a standard contraction mapping principal argument on the space of continuous functions with \( \tau \in [\tau_*, 0] \), one can show that (4.32) has a unique fixed point. The argument is completely analogous to the standard oscillon case, the details of which are explained in Appendix E. By uniqueness, the fixed point can be written

\[
\hat{A}_1(\tau) = [\delta_0^2 q_0(\delta_0) + \sqrt{\epsilon} O(|\hat{\eta}| + 1)] e^{O_{r_0}(\delta_0^2 + \sqrt{\epsilon} \tau + \delta_0^2 e^{2\tau})}
\]

\[
= [\delta_0^2 q_0(\delta_0) + \sqrt{\epsilon} O(|\hat{\eta}| + 1)] [1 + O_{r_0}(\delta_0^2 + \sqrt{\epsilon} \tau + \delta_0^2 e^{2\tau})]
\]

\[
\hat{z}_1(\tau) = -\hat{\eta} + \sqrt{\epsilon} \tau O_{\delta_0}(|\hat{\eta}|^2 + 1).
\]

We transform back into \((A_1, \hat{z}_1)\) and evaluate at \(\tau = \tau^*\) to get

\[
A_1(\tau^*) = \frac{\epsilon r_0^2}{\delta_0^2} [\delta_0^2 q_0(\delta_0) + O_{r_0, \delta_0} (\sqrt{\epsilon} |\hat{\eta}| + \sqrt{\epsilon} + |\hat{\eta}| \sqrt{\epsilon \ln \sqrt{\epsilon}})]
\]

\[
= \epsilon r_0^2 \left( a_0 + O(\delta_0^2 + 1/r_0^2) \right) + O_{\delta_0, r_0} \left( \epsilon^{3/2} (|\hat{\eta}| \ln \sqrt{\epsilon} + 1) \right)
\]

\[
\tilde{z}_1(\tau^*) = -\sqrt{\epsilon} \hat{\eta} + \epsilon \ln \frac{\sqrt{\epsilon} r_0}{\delta_0} O(|\hat{\eta}|^2)
\]

\[
= -\sqrt{\epsilon} \hat{\eta} + O_{\delta_0, r_0} (\epsilon \ln \sqrt{\epsilon}),
\]

which proves (4.29). We invert the transformation \(\tilde{z}_1\) from Lemma 4.4.3 to recover

\[
z_1 = \tilde{z}_1(\tau^*) + O \left( \tilde{z}_1(\tau^*) \kappa_1^2(\tau^*) + A_1(\tau^*) + \epsilon_1(\tau^*) \right)
\]

\[
= -\sqrt{\epsilon} \hat{\eta} \left( 1 + O(1/r_0^2) \right) + O_{\delta_0, r_0} (\epsilon \ln \sqrt{\epsilon}).
\]

Finally, we write \((A_1, z_1)\) in the original \((A, B)\) coordinates by inverting the transition chart transformation (4.1). Then \(A(\tau^*) = A_1(\tau^*)/r_0\) and \(B(\tau^*) = A_1(\tau^*)z_1(\tau^*)/r_0^2\), which are given by (4.30).

This concludes the proof of Proposition 4.3.8.
4.4.6 **Singular connecting orbit: \( A \equiv 0 \)**

We briefly discuss a second connecting orbit between the fixed points \( P_1 \) and \( P_2^- \). The schematic for standard oscillons is shown in Figure 3.8. We will see that this second orbit cannot be used to construct localized solutions. Let \( A = A_1 = A_2 = 0 \) so that the transition chart vector field (4.2) becomes

\[
\begin{align*}
\partial_\tau A_1 &= 0 \\
\partial_\tau z_1 &= \epsilon_1 - z_1^2 + \kappa_1^2 R_1(A_1, z_1, \epsilon_1) \\
\partial_\tau \kappa_1 &= -\kappa_1 \\
\partial_\tau \epsilon_1 &= 2\epsilon_1 
\end{align*}
\]

(4.33)

and the rescaling chart vector field (4.6) becomes

\[
\begin{align*}
\partial_s A_2 &= 0 \\
\partial_s z_2 &= -\kappa_2 z_2 - z_2^2 + 1 + \epsilon_2 R_2(A_2, z_2, \kappa_2) \\
\partial_s \kappa_2 &= -\kappa_2^2 \\
\partial_s \epsilon_2 &= 0 
\end{align*}
\]

(4.34)

The variables \( \kappa_1(\tau) = \kappa_0 e^{-\tau}, \epsilon_1(\tau) = \epsilon_0 e^{2\tau}, \kappa_2(s) = 1/s, \) and \( \epsilon_2(s) = \epsilon \) decouple from the rest of the system and can be explicitly solved. Letting \( \tau = 0 \) at \( \epsilon_1 = \delta_0 \), we have \( \epsilon_1 = \delta_0 e^{2\tau} \) and \( \kappa_1 = \sqrt{\epsilon / \delta_0} e^{-\tau} \).

First set \( \kappa_1 = \epsilon_2 = \epsilon = 0 \) and define \( a(s) \) so that \( z_2 =: a s / a \). Then equation (4.36) is equivalent to

\[
a_{ss} + \frac{as}{s} - a = 0 
\]

(4.35)
the zeroth order modified Bessel equation, solved by a linear combination of the zeroth order modified Bessel functions \( a(s) = c_1 K_0(s) + c_2 I_0(s) \). Exactly as with standard oscillons, in Section 3.4.6, it can be shown that the orbit \( \tilde{Q}_0 \) given in the transition and rescaling coordinates by

\[
\tilde{Q}_0^0(\tau) := \left\{ (A_1, z_1, \kappa_1, \epsilon_1) = \left( 0, -e^{\tau} \frac{K_1(e^{\tau})}{K_0(e^{\tau})}, 0, e^{\tau} \right) \right\} \quad \text{and} \\
\tilde{Q}_0^0(s) := \left\{ (A_2, z_2, \kappa_2, \epsilon_2) = \left( 0, -\frac{K_1(s)}{K_0(s)}, \frac{1}{s}, 0 \right) \right\},
\]

respectively, forms a connecting orbit between \( (A_1, z_1) = (0, 0) \) and \( (A_2, z_2) = (0, -1) \), and that \( z_1(\tau) \) actually approaches zero algebraically from \( z_1 < 0 \), for \( \ln \tau \ll 1 \). Moreover, \( \mathcal{W}_{s+}^{cs} \cap \{ A_1 \equiv 0 \} \) transversely near \( \tilde{Q}_0^0(\tau) \) since the tangent space of the manifold \( \mathcal{W}_{s+}^{cs} \) is given by span \{ \( v_1, v_3 \) \}, where

\[
v_1 = (-3, c_0^0, 0, 0)^T \quad \text{and} \quad v_2^3 = (0, -1, 2, 0)^T.
\]

Thus we can parametrize \( \mathcal{W}_{s+}^{cs} \) in the \( A_1 \) direction near \( \tilde{Q}_0^0(0) \); this parametrization was visualized in Figure 3.8b for standard oscillons. We let \( \eta \ll 1 \) be the resulting parametrization so that the initial data is \( (A_1, z_1)|_{\tau=0} = (\eta, -a) \), where \( a = O(1) \) is the \( z_1 \) component of \( \tilde{Q}_0^0(0) \). We also have \( a > 0 \) provided we make \( \delta_0 \) small enough.

The remainder of this argument is formal, as in Section 4.4.3. To leading order, vector field (4.2) is given by

\[
\partial_\tau A_1 = A_1(2 + z_1) \\
\partial_\tau z_1 = -z_1^2,
\]

(4.36)
solved by

\[ A_1(\tau) = \eta(1 - a\tau)e^{2\tau}, \quad z_1(\tau) = \frac{-a}{1 - a\tau}. \]

Letting \( \delta_0 = r_0 = a = 1 \) we find that \( \kappa_1 = 1/r_0 \) at time \( \tau_* = \ln \sqrt{\epsilon} \) so that

\[ A_1(\tau_*) = \eta\epsilon(1 - \ln \sqrt{\epsilon}), \quad z_1(\tau_*) = \frac{-1}{1 - \ln \sqrt{\epsilon}}. \]

In the original \( A, B \) coordinates we therefore have

\[ P^c\tilde{W}^s_{\pm}(\epsilon)|_{r=r_0} =\eta\epsilon(1 - \ln \sqrt{\epsilon})V^0_1 - \eta\epsilon V^0_2. \]

We match with \( \tilde{W}^{cu}(\epsilon)|_{r=r_0} \) within the center manifold, where, to leading order,

\[ P^c\tilde{W}^{cu}(\epsilon)|_{r=r_0} = \{ d_1V^0_1 + O(d_1(\epsilon + d_1))V^0_2 : d_1 \in \mathbb{R}, \text{ small} \}. \]  

(4.37)

The intersection \( P^c\tilde{W}^{cu}(\epsilon)|_{r=r_0} \cap P^c\tilde{W}^s_{\pm}(\epsilon)|_{r=r_0} \) satisfies

\[ V^0_1 : \quad \eta\epsilon(1 - \ln \sqrt{\epsilon}) = d_1 \]

\[ V^0_2 : \quad \eta\epsilon = O(d_1(\epsilon + d_1)). \]

We find that

\[ O(\epsilon + d_1) = 1/(1 - \ln \sqrt{\epsilon}) \]

and

\[ 1 = O \left( \epsilon(1 - \ln \sqrt{\epsilon})(1 + \eta(1 - \ln \sqrt{\epsilon})) \right) \]

so that \( d_1 = O \left( (1 - \ln \epsilon)^{-1} \right) \) and \( \eta = O \left( \frac{1}{\epsilon}(1 - \ln \epsilon)^{-2} \right) \). We observe that \( \eta \to \infty \) as \( \epsilon \to 0 \), which means that a non-trivial intersection \( P^c\tilde{W}^{cu}(\epsilon)|_{r=r_0} \cap P^c\tilde{W}^s_{\pm}(\epsilon)|_{r=r_0} \) cannot be parametrized near \( \tilde{Q}^0_1 \) uniformly in \( \epsilon \). The formal analysis motivates the
following conjecture.

**Conjecture 4.4.6.** For $\mathcal{W}^{cs}_{s+}(\epsilon)$ parametrized near $\tilde{Q}_{1}^{0}(\tau)$ in the transition chart coordinates, the only uniformly bounded intersection with $\tilde{\mathcal{W}}^{cs}_{s+}(\epsilon)$ as $\epsilon \rightarrow 0$ is $d_1 = \eta = 0$.

### 4.5 Conclusion

In this chapter, we have rigorously shown that reciprocal oscillons with monotone tails exist as solutions of the form

$$u(r) = u^+_\text{unit} + \tilde{u}(r), \quad \tilde{u}(r) \rightarrow 0 \text{ as } r \rightarrow \infty$$

to the steady state planar radial forced complex Ginzburg–Landau equation

$$0 = (1 + i\alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma \bar{u}$$

near the bifurcation curve

$$\Gamma_b := \{(\mu, \gamma) : (1 + \beta^2)\gamma^2 = (\omega + \beta\mu)^2\}.$$
CHAPTER FIVE

Numerics
In this chapter, we summarize our numerical results on the existence of localized steady-state solutions to the planar forced complex Ginzburg–Landau equation (CGL)

\[ u_t = (1 + i\alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma \bar{u}, \quad u \in \mathbb{C}. \quad (0.1) \]

These results are taken from our work [40]. We study all four distinct steady-state localized solution types shown in one space-dimension in Figure 5.1. This is not an exhaustive investigation and further study is required to systematically characterize all of the possible behaviors of oscillons far from onset. We performed our numerical investigation by using the continuation package AUTO07p [16] applied to the steady-state problem

\[ 0 = (1 + i\alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma \bar{u} \quad (0.2) \]

on the interval \([0, L]\) with Neumann boundary conditions at \(r = 0\) and Dirichlet boundary conditions at \(r = L\). We choose \(L = 50\) for most computations; however, we occasionally find that solutions broaden so that \(L\) must be larger. We used Matlab’s sparse eigenvalue solver EIGS to compute the linear PDE stability of solutions.

The following symmetries of (0.2) simplify our computations. First, we can re-
strict our computations to the half plane $\gamma > 0$, because (0.2) respects the gauge symmetry

$$(\alpha, \beta, \gamma, \omega, \mu, u) \mapsto (\alpha, \beta, \gamma e^{i\phi}, \omega, \mu, u e^{i\phi/2}).$$

We similarly restrict our computations to $\beta > 0$, because (0.2) respects the symmetry

$$(\alpha, \beta, \gamma, \omega, \mu, u) \mapsto (-\alpha, -\beta, \gamma, -\omega, \mu, u).$$

Finally, note that (0.2) is equivariant under $u \mapsto -u$ so that all nontrivial solutions come in pairs; whenever we discuss solution counts, we mean the number of group orbits of solutions.

Below, we will define several bifurcation curves $\Gamma_j$. These curves were originally derived in [8] for the one-dimensional CGL and are explained in more detail in that work; we summarized those results in Section 2.2. For each bifurcation curve $\Gamma_j$, we also define $\gamma = \gamma_j = \gamma_j(\omega)$ such that

$$(\omega, \gamma_j) \in \Gamma_j \quad \text{for all other parameters fixed.} \quad (0.3)$$

For example, $\gamma = \gamma_0$ means that $(\omega, \gamma_0) \in \Gamma_0$.

**Standard oscillons:** The bifurcation curves for standard oscillons come from a spatial eigenvalue analysis of the linearization of (0.2) about $u = 0$ at $r = \infty$. These spatial eigenvalues as a function of the parameter values are shown in detail in Figure 2.7. Here we state only the relevant results. Oscillons with monotone tails may bifurcate from

$$\Gamma_0 := \{ (\mu, \gamma) : \gamma^2 = \mu^2 + \omega^2 \},$$
into the region $\gamma < \gamma_0$ provided that also $\alpha \omega < \mu$. Oscillons with oscillatory tails may bifurcate from

$$\Gamma_a := \{ (\omega, \gamma) : (1 + \alpha^2)\gamma^2 = (\omega + \alpha \mu)^2 \},$$

into the region $\gamma < \gamma_a$ provided that $\mu > \alpha \omega$. Along both $\Gamma_0$ and $\Gamma_a$, we only observe bifurcations provided that $\mu < \beta \omega$, a condition which also arises in the proof of Theorem 3.1 in Chapter 3.

**Reciprocal oscillons:** Reciprocal oscillons have a nonzero background state, denoted $u = u_{\text{unif}}^+$; thus, we use the spatial eigenvalues for the linearization of (0.2) about $u = u_{\text{unif}}^+$ at $r = \infty$ to predict the bifurcation curve for reciprocal oscillons. The nontrivial uniform solution exists near

$$\Gamma_b := \{ (\omega, \gamma) : (1 + \beta^2)\gamma^2 = (\omega + \beta \mu)^2 \},$$

in the region $\gamma > \gamma_b$ provided that $\omega > \omega_\beta$ with $\omega_\beta := \mu/\beta$. The spatial eigenvalues for the linearization about $u_{\text{unif}}^+$ are shown in more detail in Figure 2.8. Reciprocal oscillons with monotone tails may bifurcate from $\Gamma_b$ into the region $\gamma > \gamma_b$ provided that $z(\omega - \omega_z) < 0$, where \footnote{We remark that [8] defines $-z$ instead of $z$. Hence, all inequalities involving $z$ are reversed in [8] from the inequalities discussed in this section.}

$$z := \alpha(1 - \beta^2) - 2\beta \quad \text{and} \quad \omega_z := \mu(1 - \beta^2 + 2\alpha \beta)/z.$$  

Reciprocal oscillons with oscillatory tails bifurcate from $\Gamma_d$ into the region $\gamma > \gamma_d$ provided that $z(\omega - \omega_z) > 0$, where $\Gamma_d$ can be computed numerically using `fsolve` in Matlab as was discussed in Section 2.2.2.
Figure 5.2: Color and line style legend for the various bifurcation curves shown in Figure 5.3 in the \((\omega, \gamma)\) plane. The insets for \(\Gamma_0, \Gamma_a, \Gamma_b,\) and \(\Gamma_d\) represent the spatial eigenvalues of the linearization of (0.2) at \(r = \infty\) about either \(u = 0\) or \(u = u_{\text{unif}}^\pm\). The spatial eigenvalues are shown for the parameter regions indicated; for parameter values in the opposite parameter region, the spatial spectra should be rotated by 90 degrees. The inset for \(\Gamma_*\) depicts a representative solution profile along the curve \(\Gamma_*\).

The curves \(\Gamma_j\) are represented in the figures below by different line styles and colors, the legend for which is shown in Figure 5.2.

Our numerical results for standard planar oscillons as solutions to (0.2) are summarized schematically in Figure 5.3 for four different parameter sets for \((\beta, \alpha, \mu)\), which are listed in Table 5.1. The schematic diagrams in Figure 5.3 are not meant to represent all possible behaviors of oscillons far from onset; a more detailed study is needed. As expected from the one-dimensional analysis, standard oscillons with monotone tails bifurcate from \(\Gamma_0\), while standard oscillons with oscillatory tails bifurcate from \(\Gamma_a\). We observe that solutions terminate at \(\Gamma_*\); we discuss this curve in Section 5.1 below. For reference, we also plot \(\Gamma_b\) and \(\Gamma_d\), the bifurcation curves for
reciprocal oscillons. Each bifurcation curve $\Gamma_j$ is highlighted whenever the spatial eigenvalues are as shown in Figure 5.2; otherwise (i.e., when the spatial eigenvalues are rotated by 90 degrees from the orientation in Figure 5.2), we represent $\Gamma_j$ with the same line style as in the legend but thin and black. Our numerical results are plotted either as graphs of $||u||_{L^2}$ versus $\gamma$ or in the $(\omega, \gamma)$ plane. In either case, it is understood that the other parameters are held constant.

5.1 Existence and stability of standard oscillons bifurcating from $\Gamma_0$

To simplify notation, we define the critical parameter values $\omega_\beta := \mu/\beta$ and $\omega_\alpha := \mu/\alpha$. The hypotheses in Theorem 3.2 from Chapter 3 on standard oscillons are then equivalent to

(i) $\omega > \omega_\beta$: equivalent to the hypothesis $c_3^0 < 0$, and

(ii) $\alpha \omega_\alpha > \alpha \omega$: necessary so that all four spatial eigenvalues immediately below $\Gamma_0$ are real (see Figure 5.2, and also Figure 2.7 which contains additional information about spatial eigenvalues in the $(\mu, \gamma)$ plane).

Alternatively, the existence conditions (i) and (ii) are equivalent to $\omega_\alpha > \omega > \omega_\beta$ for $\alpha > 0$, and $\omega > \omega_\alpha$, $\omega_\beta$ for $\alpha < 0$. Since the existence region for $\omega$ rescales with the magnitude of the parameters $\alpha$, $\beta$, and $\mu$, we define parameter regions based solely on the signs of $\alpha$ and $\mu$, which we label counterclockwise in the $(\alpha, \mu)$ plane starting
Figure 5.3: Expected existence regions of standard oscillons in each of the parameter quadrants. We refer to Figure 5.2 for the interpretation of the curves shown here: the segments of the curves $\Gamma_j$ that coincide with bifurcations are highlighted; otherwise, we use the same line style as in Figure 5.2, but the curve is thin and black. Small amplitude localized solutions bifurcate into the parameter region below $\Gamma_0$ and $\Gamma_a$ provided that also $\omega > \omega_{\beta} := \mu/\beta$. The dark salmon shaded regions indicate the numerically observed existence region for localized solutions with monotone tails; the light green shaded regions indicate the existence region for localized solutions with oscillatory tails. Solutions bifurcating from $\Gamma_0$ are observed to develop oscillatory tails for $\gamma < \gamma_a$ (with the notation of (0.3)). In parameter regions $Q_1$, $Q_2$, and $Q_4$, all localized solutions terminate in a stationary one-dimensional front at $\Gamma_s$. In $Q_3$, $\Gamma_s$ ends at the curve $\Gamma_d$; the termination of solutions for parameter values to the left of $\Gamma_d$ in $Q_3$ is not well understood. Stable oscillons were found in between two saddle nodes, as also shown in Figure 5.4, in a small subset of $Q_2$. In Figures 5.4-5.7, we show the results of numerical continuation along the vertical blue lines indicated above.
with $\alpha > 0$, $\mu > 0$

\[ Q_1 := \alpha > 0, \mu > 0, \quad Q_2 := \alpha < 0, \mu > 0, \]

\[ Q_3 := \alpha < 0, \mu < 0, \quad Q_4 := \alpha > 0, \mu < 0. \]

We also set $\alpha < \beta$ in region $Q_1$ and $\alpha > \beta$ in region $Q_4$ so that both conditions (i) and (ii) are satisfied simultaneously. We performed a few select example computations in each parameter region and further study is required to determine whether the solution behavior indicated in Figure 5.3 is representative, as our study is not exhaustive. In particular, large amplitude solutions may interact with $\Gamma_b$ and $\Gamma_d$ in complicated ways, based on the sign of $z$. Below, we report on one computation for a single parameter set, listed in Table 5.1, in each of the quadrants.

An example bifurcation diagram for localized solutions emerging from $\Gamma_0$ is shown in Figure 5.4 along with some representative solution profiles. Near $\Gamma_0$, localized solutions have small amplitude. As $\gamma$ is decreased away from $\gamma_0$, the profile grows and narrows; as a result, the solution $L^2$ norm increases. For $\gamma < \gamma_a$, solutions develop oscillatory tails. We observe that the solutions resemble one-dimensional stationary fronts for $\gamma$ far enough below $\gamma_0$. At this point, a small decrease in the value of $\gamma$ causes the front interface to shift to the right, which, in turn, increases the solution $L^2$ norm; in fact, we observe that the continuation curve asymptotes in the $(\gamma, ||u||_{L^2})$-plane at some value of $\gamma$, which we call $\gamma_* := \gamma_*(\omega)$. In two-parameter $(\omega, \gamma)$-space we denote the boundary of the oscillon existence region corresponding to $(\omega, \gamma_*(\omega))$ by $\Gamma_*$. We observe that solutions along $\Gamma_*$ resemble one-dimensional stationary fronts. To verify this observation, we computed the curve in $(\omega, \gamma)$ space along which one-dimensional stationary fronts connect $u = u_{\text{unif}}^+$ and $u = 0$ and compared it with $\Gamma_*$: they coincide, and the resulting solution profiles look identical.
Table 5.1: Values of the fixed parameters for our numerical computations in each region.

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$\mu$</th>
<th>$z := \alpha(1 - \beta^2) - 2\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1$</td>
<td>10.0</td>
<td>0.5</td>
<td>10.0</td>
<td>-69.5</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>2.5</td>
<td>-2.0</td>
<td>0.5</td>
<td>5.5</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>2.5</td>
<td>-2.0</td>
<td>-0.5</td>
<td>5.5</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>0.5</td>
<td>10.0</td>
<td>-5.0</td>
<td>6.5</td>
</tr>
</tbody>
</table>

Figure 5.4: Standard oscillons with monotone tails in region $Q_2$ (bifurcation diagram [left] and sample solution profiles [right]) with $\beta = 2.5$, $\alpha = -2.0$, and $\mu = 0.5$. We choose $\omega = 3.65$ so that the double saddle node exists along the continuation curve in $\gamma$. Solutions are stable in the section of the bifurcation curve in between the two saddle nodes, as indicated by the solid line style. Note that solution profiles are plotted as a function of $s = r/L \in [0, 1]$ with $L = 26.1$. The location of the continuation curve in $(\omega, \gamma)$-space is indicated by the blue vertical line in Figure 5.3.

Since $u^\pm_{\text{unit,bifurcate from } \Gamma_b}$ for $\omega > \omega_\beta$, this observation provides insight into why $\Gamma_s$ bifurcates from the point $\omega = \omega_\beta$. Similarly, since standard oscillons terminate at $\Gamma_s$, this provides another rationale for the standard oscillon existence condition $\omega > \omega_\beta$, which was necessary in the proof of Theorem 3.2 in Chapter 3 and which we observed numerically, as indicated in Figure 5.3. Finally, these observations explain why we observe $\gamma^* > \gamma_b$, provided $\Gamma_s$ exists, as is also indicated in Figure 5.3.

We also investigated the PDE stability of the solutions. All solutions were unstable near onset. In parameter regions $Q_3$ and $Q_4$ the essential spectrum was in the right half-plane due to the instability of the background state $u = 0$, as can
Figure 5.5: Stability of standard oscillons with monotone tails in region $Q_2$ (bifurcation diagram [left] and numerical spectrum associated with sample solution profiles along the bifurcation curve [right]) with $\beta = 2.5$, $\alpha = -2.0$, and $\mu = 0.5$. We choose $\omega = 3.65$ so that the double saddle node exists along the continuation curve in $\gamma$. The eigenvalue with largest real part is indicated by the large star. The location of the continuation curve in $(\omega, \gamma)$-space is indicated by the blue vertical line in Figure 5.3.

be seen from the following computation: We decompose $u = v + iw$ into real and imaginary parts with $v, w \in \mathbb{R}$, and define $\tilde{U} := (v, w)$ so that the linearization of the time-dependent CGL (0.1)

$$u_t = (1 + i\alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma\bar{u}, \quad u \in \mathbb{C}. \quad (1.1)$$

about $u = 0$ with $r = \infty$ becomes the real system $\tilde{U}_t = \mathcal{L}\tilde{U}$ where

$$\mathcal{L} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \partial_{rr} + \begin{pmatrix} -\mu + \gamma & -\omega \\ \omega & -\mu - \gamma \end{pmatrix}.$$
Figure 5.6: Standard oscillons with oscillatory tails in region $Q_1$ (bifurcation diagram [left] and sample solution profiles [right]) with $\beta = 10.0$, $\alpha = 0.5$, and $\mu = 10.0$. We choose $\omega = 59.0$ so that $\omega > \mu/\alpha = 20.0$. Note that solution profiles are plotted as a function of $s = r/L \in [0, 1]$ with $L = 16.0$. The location of the continuation curve in $(\omega, \gamma)$-space is indicated by the blue vertical line in Figure 5.3.

Thus, the essential spectrum of standard oscillons can be computed from

$$0 = \det \begin{pmatrix}
-\mu + \gamma - k^2 - \lambda & \alpha k^2 - \omega \\
-\alpha k^2 + \omega & -\mu - \gamma - k^2 - \lambda
\end{pmatrix}
= (-\mu - k^2 - \lambda)^2 - \gamma^2 + (\alpha k^2 - \omega)^2
= \lambda^2 + 2\lambda(k^2 + \mu) + \mu^2 + \omega^2 - \gamma^2 + (1 + \alpha^2)k^4 - 2k^2(-\mu + \alpha \omega)
$$

so that

$$\lambda = (-\mu - k^2) \pm \sqrt{\gamma^2 - (\alpha k^2 - \omega)^2}. \quad (1.2)$$

Hence, whenever $\mu < 0$, some part of the essential spectrum is always located in the right half plane; thus, localized solutions in the parameter regions $Q_3$ and $Q_4$ can never stabilize. We do, however, observe saddle nodes in these regions related to the oscillons with oscillatory tails; these are discussed in Section 5.2.
Figure 5.7: Standard oscillons with oscillatory tails in region $Q_3$ (bifurcation diagram [left] and sample solution profiles [right]) with $\beta = 2.5$, $\alpha = -2.0$, and $\mu = -0.5$. We choose $\omega = 0.13$ so that $\omega$ is in the parameter region to the left of $\Gamma_d$. Solutions do not stabilize at the saddle nodes due to the instability of the background state. Note that solution profiles are plotted as a function of $s = r/L \in [0, 1]$ with $L = 100.0$. The location of the continuation curve in $(\omega, \gamma)$-space is indicated by the blue vertical line in Figure 5.3.

In region $Q_2$, solutions were found to be stable in between two saddle nodes, whenever such saddle nodes exist; see Figure 5.5 for a plot of the bifurcation diagram and the numerical spectrum for various solutions along the bifurcation curve. The numerical spectrum for a solution $u_\ast(r; \alpha, \beta, \gamma, \mu, \omega)$ from the AUTO07p numerical continuation study (with parameters $\alpha$, $\beta$, $\gamma$, $\mu$, and $\omega$ fixed) was computed as follows: we write the radial time dependent CGL

$$u_t = (1 + i\alpha) \left( u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} \right) + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma \bar{u}, \quad u \in \mathbb{C}.$$ 

as a first order real system, with the right hand side linearized about the solution $u_\ast(r)$ (with $u_\ast(r) = v_\ast(r) + iw_\ast(r)$):

$$\tilde{U}_t = \mathcal{L}(u_\ast; \alpha, \beta, \gamma, \mu, \omega)\tilde{U}$$
Figure 5.8: Bifurcation diagram in \((ω, γ)\)-space in \(Q_3\) with \(β = 2.5\), \(α = -2.0\), \(μ = -0.5\) and interval length \(L = 75.0\). The thin solid lines represent the continuation curves of the saddle nodes. The dynamics appear to be organized by \(Γ_d\) in the region where \(Γ_∗\) ceases to exist. In the region shown, we have \(zω > zω_∗\); hence, spatial eigenvalues exist in pairs on the imaginary axis along \(Γ_d\), whilst they are rotated 90 degrees from how they are shown in Figure 5.2 along \(Γ_b\).

where

\[
L(u_∗; α, β, γ, μ, ω)\tilde{U} = \begin{pmatrix} 1 & -α \\ α & -1 \end{pmatrix} \begin{pmatrix} \tilde{U}_{rr} + \frac{\tilde{U}_r}{r} - \frac{k^2\tilde{U}}{r^2} \end{pmatrix} + \begin{pmatrix} -μ + γ & -ω \\ ω & -μ - γ \end{pmatrix} \tilde{U} - |u_∗|^2 \begin{pmatrix} 1 & -β \\ β & 1 \end{pmatrix} \begin{pmatrix} v_*^2 & v_*w_* \\ v_*w_* & w_*^2 \end{pmatrix} \tilde{U}
\]

and where we have assumed that the solution \(\tilde{U}(r, \theta, t) = R(r, t)e^{ik\theta}\) can be separated into radial and angular components with \(k \in \mathbb{Z}\). We turn \(L\) into a matrix \(M\) by discretizing \(\tilde{U}\) (a typical grid size is 0.01, or \(L/5000\) with \(L = 50.0\)) and using the discrete radial Laplacian for \(\tilde{U}_{rr} + \frac{\tilde{U}_r}{r} - \frac{k^2\tilde{U}}{r^2}\) with Neumann boundary conditions at \(r = 0\). Then the numerical spectrum for the solution \(u_∗\) is given by the eigenvalues of the matrix \(M\). The numerical spectrum shown in Figure 5.5 is for \(k = 0\). We also computed the numerical spectrum for \(k = 1, 2,\) and \(6\); the spectrum for the stable solution shown in Figure 5.5(b) remained in the left half plane except with \(k = 1\), in which case there was a single eigenvalue at the origin of the complex plane due to
translation invariance of equation (1.1). Our numerical stability investigation in $Q_2$ was performed with $z > 0$.

Saddle nodes were also observed for the one-dimensional CGL in $Q_2$ with $z < 0$ [8]; however, the region in [8] seems to be orders of magnitude smaller than in the case considered here. A two-parameter continuation of the saddle nodes in region $Q_2$ indicates that they collide in a cusp near $\Gamma_*$ in the parameter region where both $z\omega > z\omega_\ast$ and $\gamma_\ast > \gamma_0$. The numerically computed existence region of stable standard oscillons is indicated by the spotted area in Figure 5.3. We do not observe saddle nodes in region $Q_1$.

Our numerical results on standard oscillons in the planar CGL are consistent with the one-dimensional results in [8], with the following exceptions: In one space dimension, a second pair of stabilizing saddle nodes was observed along the bifurcation curve in $\gamma$ in parameter region $Q_3$. Second, it was observed in one space dimension that the saddle nodes extended into the region below $\gamma_\ast$; in the planar case, we found these curve only for $\gamma_\ast < \gamma < \gamma_0$. A more detailed study is necessary to determine whether there are parameter regions in which the saddle node behavior in two spatial dimensions is the same as in one spatial dimension.

5.2 Standard oscillons near a Turing bifurcation and collapsed snaking

As indicated in Figure 5.3, we find that standard oscillons with oscillatory tails bifurcate from $\Gamma_a$ provided $\omega > \omega_{\alpha_\ast}$ and $\omega > \omega_{\beta_\ast}$, as expected. An example bifurcation diagram and representative solution profiles are shown in Figure 5.6. In $Q_1$ and
$Q_2$ the behavior of these solutions is similar to that of the monotone tail solutions: solutions have small amplitude near the bifurcation curve $\Gamma_a$; the amplitude becomes larger and the envelope narrows as $\gamma$ decreases; the continuation curve again terminates at $\Gamma_*$—the curve of one-dimensional stationary front solutions. None of these solutions were found to be stable. Our numerical results on standard oscillons with oscillatory tails in the planar CGL in $Q_1$ and $Q_2$ are consistent with the one-dimensional results in [8].

In $Q_3$, $\Gamma_*$ cannot bifurcate from $\Gamma_0$; see Figure 5.3: instead, we observe that $\Gamma_*$ terminates at $\Gamma_d$, where $u_{\text{unif}}^+$ undergoes a bifurcation to spatially periodic waves; hence, one-dimensional stationary fronts in this regime have oscillatory core. In parameter regions to the left of $\Gamma_d$, localized solutions do not terminate in one-dimensional stationary fronts. An example bifurcation diagram and representative solution profiles in the parameter region to the left of $\Gamma_d$ are shown in Figure 5.7: Near $\Gamma_a$, oscillons have small amplitude with oscillatory tails. As $\gamma$ is decreased, oscillations are added to the core; as a result, the bifurcation curve demonstrates collapsed snaking. We also performed a two-parameter continuation of the saddle nodes in the collapsed-snaking region: the result is shown in Figure 5.8. We believe that these saddle-node curves, and therefore collapsed snaking, exist all the way to the left of $\Gamma_d$ until $\omega > -\alpha \mu$, which is the intersection point of $\Gamma_a$ with the $\gamma = 0$ axis: First, we observe that solution profiles exhibit similar behavior to that shown in Figure 5.7 along the bifurcation curves everywhere to the left of $\Gamma_d$. In particular, as the parameter $\gamma$ is decreased, we find that the profiles contain more and more core oscillations. We believe that each new oscillation is introduced at a saddle node, but that the difference between successive saddle nodes becomes too small to resolve numerically. As is shown in Figure 5.8, the dynamics of the saddle nodes in parameter space near $\Gamma_d$ is complicated and not well understood. We expect the
same behavior of solutions in $Q_3$ regardless of the sign of $z$, with the difference that $\Gamma_*$ may bifurcate from $\Gamma_b$ rather than $\Gamma_d$; in this case, we expect that the front solution will have a monotone rather than oscillatory core. However, future study is needed to verify this hypothesis.

In $Q_4$, the curve $\Gamma_*$ similarly terminates at $\Gamma_b$, the curve along which $u^\pm_{\text{unif}}$ bifurcates. Hence, for $\omega$ to the right of this intersection point, solution profiles do not approach one-dimensional fronts, as indicated in Figure 5.3. Preliminary computations indicate that in $Q_4$ to the right of $\Gamma_b$ the one-parameter bifurcation diagram and solution profiles resemble those for $Q_3$ to the left of $\Gamma_d$: we observe saddle nodes as oscillations are added to the core. More detailed numerical computations are needed in order to characterize the similarities and differences between solutions in $Q_3$ and in $Q_4$.

Lastly, we discuss a major difference between our results on the planar CGL and the one-dimensional case. In [8, Appendix B], it was shown that oscillons with oscillatory tails cannot bifurcate from $\Gamma_a$ when $\beta < \alpha$: the sign of $(\beta - \alpha)$ is related to a criticality condition on the leading nonlinear term in the normal form for the one-dimensional CGL near $\Gamma_a$; it is analogous to the condition $c^0_3 < 0$ in Theorem 3.2, which is necessary for oscillons to bifurcate from $\Gamma_0$ in one and two dimensions. However, our numerical computations indicate that oscillons do bifurcate from $\Gamma_a$ in $Q_4$ with $\beta < \alpha$ in the planar case, and an example bifurcation curve and representative solution profiles are shown in Figure 5.9. The existence of oscillons near $\Gamma_a$ for $\beta < \alpha$ is reminiscent of an analytical result on spots in the planar Swift–Hohenberg equation: it was analytically shown that spots bifurcate regardless of the sign of $c^0_3$, where now $c^0_3$ represents the coefficient of the leading nonlinear term in the Swift–Hohenberg equation [36].
Figure 5.9: Standard oscillons with oscillatory tails in $Q_4$ (bifurcation diagram [left] and sample solution profiles [right]) with $\beta = 0.5$, $\alpha = 10.0$, and $\mu = -5.0$. We choose $\omega = 0.93$ so that $\omega > \omega_\alpha$ but to the left of the bifurcation curve $\Gamma_b$. The one-dimensional theory predicts that oscillons should not bifurcate from $\Gamma_a$ in this region since $\beta < \alpha$; we nevertheless observe oscillons with oscillatory tails. Note that solution profiles are plotted as a function of $s = r/L \in [0, 1]$ with $L = 150.0$. The location of the continuation curve in $(\omega, \gamma)$-space is indicated by the blue vertical line in Figure 5.3.

Figure 5.10: Reciprocal oscillons with monotone tails in $Q_1$ (bifurcation diagram [left] and sample solution profiles [right]) with $\beta = 0.5$, $\alpha = 0.33$, and $\mu = 1.0$. We choose $\omega = 13.86$ so that a double saddle node exists. Solutions are stable in the section of the bifurcation curve in between the two saddle nodes, as indicated by the solid line style. Note that solution profiles are plotted as a function of $s = r/L \in [0, 1]$ with $L = 33.7$. $\Gamma^*$ represents the curve of one-dimensional stationary reciprocal fronts.
5.3 Existence and stability of reciprocal oscillons

As with standard oscillons, we expect that the behavior of reciprocal oscillons far from onset will depend on the values of the fixed parameters $\alpha$, $\beta$ and $\mu$. We report here on our preliminary findings and note that a more exhaustive study is needed. An example bifurcation diagram and representative solution profiles are shown in Figure 5.10. As expected from the one-dimensional spatial eigenvalue analysis, we found that planar reciprocal oscillons bifurcate into the region above $\Gamma_b$. These solutions terminate in a one-dimensional stationary reciprocal front, analogous to the terminating curve for standard oscillons. As with standard oscillons, we observe two saddle nodes in some parameter regions provided $\omega$ is large enough; the minimum value of $\omega$ is not well understood. All localized solutions are unstable at onset and, in the case shown in Figure 5.10, stabilize in the region between the two saddle nodes.
Chapter Six

Conclusion
6.1 Summary of main results

In this thesis, we carried out numerical computations and a theoretical analysis of localized solutions in the steady state planar radial 2:1 forced complex Ginzburg–Landau equation (CGL)

\[
0 = (1 + i\alpha) \left( u_{rr} + \frac{u_r}{r} \right) + (-\mu + i\omega)u - (1 + i\beta)|u|^2u + \gamma \bar{u}.
\]  

(1.1)

The work in this thesis was motivated by prior results on localized solutions to the one-dimensional CGL [8, 66].

6.1.1 Theory

Review of one-dimensional results from [8]

We used the bifurcation curves for the one-dimensional CGL from [8] (reviewed in Section 2.2), obtained via a spatial eigenvalue analysis. Four types of localized solutions were found in the one-dimensional CGL, as is shown in Figure 6.1. Standard oscillon (Figures 6.1a and 6.1b) bifurcation curves are predicted from the linearization of (1.1) about \( u \equiv 0 \); reciprocal oscillons (Figures 6.1c and 6.1d) are predicted using the linearization of (1.1) about a nontrivial uniform solution \( u = u_{\text{unit}}^+ \). We expect that monotone solutions (Figures 6.1a and 6.1c) bifurcate from the curves along which two spatial eigenvalues collide at the origin, provided that the other two spatial eigenvalues are real and bounded away from the origin; see Figure 6.2a. We expect that solutions with oscillatory tails (Figures 6.1b and 6.1d) bifurcate from the curves along which pairs of eigenvalues collide on the imaginary axis away from
Figure 6.1: Localized solutions to equation (2.2). The dotted line represents $u \equiv 0$.

Figure 6.2: Possible spatial eigenvalue arrangements for the bifurcation of localized solutions. Spatial eigenvalues collide on the imaginary axis and then move away.

the origin; see Figure 6.2b. Sketches of the various bifurcation curves and associated spatial eigenvalues were shown in Figures 2.7 and 2.8.

Our results in two spatial dimensions

In Chapter 3 we proved rigorously that standard oscillons bifurcate from

$$\Gamma_0 := \{ (\mu, \gamma) : \gamma = \sqrt{\mu^2 + \omega^2} \}$$

into the region $\gamma < \gamma_0$, provided $\alpha \omega < \mu$ so that the spatial eigenvalues are purely real and $\mu < \beta \omega$ so that the bifurcation is subcritical. In Chapter 4 we proved
rigorously that reciprocal oscillons bifurcate from

\[ \Gamma_b := \{ (\mu, \gamma) : (1 + \beta^2)\gamma^2 = (\omega + \beta\mu)^2 \} \]

into the region \( \gamma > \gamma_b \), provided \( \mu < \beta\omega \) so that the nontrivial uniform solution \( u^{+}_{\text{unif}} \) exists and \( z(\omega - \omega_z) > 0 \) so that the spatial eigenvalues are purely real, where

\[ z := \alpha\beta^2 - \alpha + 2\beta \quad \text{and} \quad \omega_z := \frac{\mu(1 - \beta^2 + 2\alpha\beta)}{z}. \]

(1.2)

Our method consisted of matching solutions that stay bounded at the core with those that decay in the far field. To find the latter, we utilized the geometric blow-up strategy of [38]. A major difference from previous work is that the matching analysis required a center-manifold reduction and expansions of invariant manifolds and foliations.

### 6.1.2 Numerics

Using the numerical continuation software package AUTO07, we studied the existence of large amplitude localized solutions to equation (1.1) far from onset. We observed the existence of all four localized radial steady-state solutions to the planar CGL, as well as standard and reciprocal front-like solutions. We observed that the localized solutions bifurcate from the same curves as in the one-dimensional case, as reported on in [8]. We also observed that the curve of stationary one-dimensional fronts bifurcates from the codimension-two point at which the normal form along the bifurcation curve \( \Gamma_0 \) changes from sub- to super-critical.

Our numerical study of standard oscillons was fairly comprehensive. Away from
onset, the behavior of standard localized solutions is complicated and depends on the interaction of the standard oscillon bifurcation curves with the reciprocal oscillon bifurcation curves $\Gamma_b$ and $\Gamma_d$. In most parameter regimes, standard oscillons terminate at the curve of one-dimensional stationary fronts. Standard oscillons with monotone tails were found to be stable in a small region of parameter space. Our numerical results for four different parameter regimes were summarized in Figure 5.3. Preliminary computations indicate that the analogous statements hold for the reciprocal version of each solution type. We emphasize that a more detailed numerical study is needed to completely characterize the behavior of solutions far from onset.

A major difference between the 1D and 2D CGL is that we found numerical evidence for standard oscillons of type (ii) in the region $\beta < \alpha$, whereas no oscillons were found in this region in the 1D case. This observation is in line with results for Turing bifurcations of the 1D and planar Swift–Hohenberg equation: in [36], it was shown that planar spots exist in regions where 1D spots do not exist.

6.2 Open questions

Analysis of other solution types

In future work, we will apply the blow-up coordinates used in both this paper and [38] to prove the existence of localized solutions with oscillatory tails (see Figures 6.1b and 6.1d). We expect that the general framework and strategy will be the same, with the following modifications: the dimension of the center manifold near onset will be four, and the dynamics will be different from the one we encountered here. The analysis on the center manifold should be similar to the one carried out in
[36], though preliminary computations indicate that the amplitude scaling might be different. Since oscillons of type (ii) were observed in parameter space regions where the one-dimensional theory predicts they should not bifurcate, the existence proof for these solutions will provide insight into the differences between the planar and one-dimensional cases.

We also plan to prove the existence of the one-dimensional stationary fronts using the same blow-up analysis as for the localized solutions. Since these fronts bifurcate from a codimension-two point, the analysis will be more complicated than for the localized solutions. We also expect that the leading nonlinear term on the center manifold will be quintic rather than cubic or quadratic (as for standard and reciprocal oscillons, respectively), so that the dynamics on the center manifold will be different than in the cases already encountered.

**Rigorous justification of the complex Ginzburg–Landau equation**

We also plan to justify the CGL as a normal form for periodically forced reaction-diffusion equations near a supercritical Hopf bifurcation. As discussed in Section 2.1 (particularly in Section 2.1.3), while a formal multiple scales analysis indicates clearly that the CGL describes, to leading-order, the long-scale modulation of the amplitude of solutions to such systems, we believe it would be useful to make this connection rigorous using the following two complementary approaches:

First, we plan to extend the spatial dynamics for time-periodic patterns that were developed in [45, Chapter 4] for autonomous reaction-diffusion systems near Hopf bifurcations to nonautonomous systems with periodic forcing. We expect to be able to prove that the stationary forced CGL is the normal-form equation in this setting.
on an appropriate spatial center manifold in a space of time-periodic functions.

Second, we plan to show that the time-dependent forced CGL governs the dynamics of the envelope \( A(X, T) \) of solutions \( u(x, t) \) of the form

\[
  u(x, t) = \epsilon A(\epsilon x, \epsilon^2 t)e^{i\omega_0 t} + \epsilon^2 R(x, t) + c.c.
\]

of the forced reaction-diffusion system over time scales of order \( 1/\epsilon^2 \) by showing that the remainder \( R(x, t) \) remains bounded over that time scale; this is the approach taken to justify the complex Ginzburg–Landau equation for unforced reaction-diffusion equations in [46]. One possible strategy for proving boundedness of the remainder is to use the space-time normal forms described recently in [20–23].

Questions raised by the numerical results

We would like to flesh out our numerical study in the following two ways. First, we would like to continue the numerical study for standard oscillons to verify that the behavior displayed in Figure 5.3 is exhaustive. Second, we would like to study reciprocal oscillons in the same level of detail as for standard oscillons, so as to have results analogous to those for standard oscillons displayed in Figure 5.3, including results on the stability region.

Our numerical results indicate several other interesting directions for study. First, we would like to understand the collapsed snaking region, as is indicated in Figure 5.7.

Finally, we would like to understand the stability for both standard and reciprocal oscillons. As is indicated in Figure 5.3, standard oscillons with monotone tails were found to be stable in a small region of parameter space. The stability region
was observed far from the bifurcation curve $\Gamma_0$, where we do not currently have an existence result. Hence, such an analytical stability proof would be challenging.
Appendix A

A collection of theorems
In Appendices B-E we will show that there exists a unique solution with certain
properties to some vector field

\[ \dot{u} = A(t)u + f(u; \mu) \quad (0.1) \]

\[ u(t_0) = u_0. \quad (0.2) \]

The argument in each case will follow the same outline. First, we construct solu-
tions to (0.2) using the variation of parameters formula and the fundamental matrix
solution. The resulting equation will be of the form

\[ u = F(u). \quad (0.3) \]

We will use Banach’s Fixed Point Theorem, sometimes in conjunction with the Im-
plicit Function Theorem, to show that there exists a unique solution to (0.3).

In this Appendix we collect the relevant formulae and theorems.

A.1 Fundamental matrix solution

Taken from [11, §2.1.3]. Consider the \( n \)-dimensional homogeneous linear initial value
problem

\[ \dot{u} = A(t)u \quad u \in \mathbb{R}^n, \ A(t) \in \mathbb{R}^{n \times n} \]

\[ u(t_0) = u_0. \quad (1.1) \]

**Definition 1.1.1.** A set of \( n \) solutions of the homogenous linear differential equation
(1.1), all defined on the same open interval \( J \), is called a fundamental set of
solutions on \( J \) if the solutions are linearly independent functions on \( J \).

**Proposition 1.1.2.** If \( \mathcal{F} \) is a fundamental set of solutions of the linear system (1.1) on the interval \((a, b)\), then every solution defined on \((a, b)\) can be expressed as a linear combination of the elements of \( \mathcal{F} \).

**Definition 1.1.3.** An \( n \times n \) matrix function \( t \mapsto \Psi(t) \), defined on an open interval \( J \), is called a **matrix solution** of the homogeneous linear system (1.1) if each of its columns is a (vector) solution. A matrix solution is called a **fundamental matrix solution** if its columns form a fundamental set of functions. In addition, a fundamental matrix solution \( t \mapsto \Psi(t) \) is called the **principal fundamental matrix solution** at \( t_0 \in J \) if \( \Psi(t_0) = I \).

By Proposition 1.1.2, there exists a fundamental matrix solution \( \Psi(t) \) to (1.1) on the interval \( J \). By definition, \( \dot{\Psi}(t) = A(t)\Psi(t) \). Moreover, there exists a principal fundamental matrix solution at every \( t_0 \in J \), which is given by \( \chi(t) := \Psi(t)^{-1}\Psi(t_0) \).

We will use the fundamental matrix solution to compute a solution \( u(t) \) at some time \( t \) from the solution \( u(\tau) \) at time \( \tau \).

**Definition 1.1.4.** The **state transition matrix** for the homogeneous linear system (1.1) on \( J \) is the family of fundamental matrix solutions \( t \mapsto \Phi(t, s) \) parametrized by \( s \in J \) such that \( \Phi(s, s) = I \), where \( I \) denotes the \( n \times n \) identity matrix.

**Proposition 1.1.5.** If \( t \mapsto \Psi(t) \) is a fundamental matrix solution for the system (1.1) on \( J \), then \( \Phi(t, s) := \Psi(t)\Psi^{-1}(s) \) is the state transition matrix and, by definition,

\[
\frac{d}{dt} \Phi(t, s) = A(t)\Phi(t, s).
\]

The state transition matrix \( \Phi(t, s) \) satisfies the Chapman–Kolmogorov identities

\[
\Phi(s, s) = I, \quad \Phi(t, \tau)\Phi(\tau, s) = \Phi(t, s)
\]
and the identities

$$
\Phi(t, s)^{-1} = \Phi(s, t), \quad \frac{\partial}{\partial s} \Phi(t, s) = -\Phi(t, s)A(s).
$$

**Remark 1.1.6.** Due to Proposition 1.1.5, we frequently call the transition matrix the fundamental matrix solution.

This ends a review of results from [11]. The state transition matrix can be expressed in terms of the fundamental set of solutions to the adjoint problem, as we show in the two following propositions.

**Proposition 1.1.7.** Let $\{Vj(t)\}_{j=1}^n$ be a fundamental set of solutions to the $n$-dimensional primary problem (1.1) and $\{Wj(t)\}_{j=1}^n$ be a fundamental set of solutions to the adjoint problem

$$
\dot{w} = -A^T(t)w, \quad w \in \mathbb{R}^n, \quad A^T(t) \in \mathbb{R}^{n \times n} \quad (1.2)
$$
on open interval $J \subset \mathbb{R}$. In the above, $A^T(t)$ denotes the transpose of matrix $A(t)$. Furthermore, let $\Psi(t)$ and $\Phi(t, s)$ be fundamental matrix solution and state transition matrix, respectively, for (1.1) created from $\{Vj\}_{j=1}^n$. Then

(i) $\Phi^T(s, t)$ is the state transition matrix for (1.2);

(ii) $(\Psi^T(t))^{-1} = (\Psi^{-1}(t))^T =: \Psi^{-T}(t)$ is a fundamental matrix solution for (1.2);

(iii) and $\langle V_i(t), W_j(t) \rangle = \langle V_i(s), W_j(s) \rangle = \delta_{ij}$ is a constant for all $s, t \in J$.

**Proof.** (i) We first show $\frac{d}{dt} \Phi^T(s, t) = -A^T(t)\Phi^T(s, t)$. Using the properties in
Proposition 1.1.5, we have \( \Phi(t, s) \Phi(s, t) = I \) so that

\[
0 = \frac{d}{dt} [\Phi(t, s) \Phi(s, t)] = \frac{d}{dt} \Phi(t, s) \Phi(s, t) + \Phi(t, s) \frac{d}{dt} \Phi(s, t)
\]

\[
\Rightarrow \frac{d}{dt} \Phi(s, t) = -\Phi(s, t) \frac{d}{dt} \Phi(t, s) \Phi(s, t)
\]

\[
= -\Phi(s, t) A(t) \Phi(t, s) \Phi(s, t)
\]

\[
= -\Phi(s, t) A(t)
\]

\[
\Rightarrow \frac{d}{dt} \Phi^T(s, t) = -A^T(t) \Phi^T(s, t)
\]

as desired. Then (ii) immediately follows since

\[
(\Phi(s, t))^T = (\Psi(s) \Psi^{-1}(t))^T = \Psi^{-T}(t) \Psi^T(s) = \Psi^{-T}(t) (\Psi^{-T}(s))^{-1}.
\]

(iii) Let \( \chi(t, s) \) denote the fundamental matrix solution for (1.2). Then \( \langle V_i(s), W_j(s) \rangle \) is the \( i \)th row and \( j \)th column of \( \Psi^T(s) \chi(s) \). But by property (ii), \( \Psi^T(s) \chi(s) = I \). ■

**Proposition 1.1.8.** The action of the state transition matrix \( \Phi(t, s) \) on a vector \( v(s) \) can be computed from \( \Phi(t, s)v(s) = \sum_{j=1}^{n} V_j(t) \langle W_j(s), v(s) \rangle \).

**Proof.** By Proposition 1.1.2, any solution to (1.1) \( v(s) \) can be written

\[
v(s) = \sum_{j=1}^{n} c_j V_j(s)
\]

where \( c_j \in \mathbb{R} \) are constant for all time. By Proposition 1.1.7(iii), \( c_j = \langle W_j, v(s) \rangle \) so that

\[
\Phi(t, s)v(s) = \sum_{j=1}^{n} \Phi(t, s)V_j(s) \langle W_j, v(s) \rangle = \sum_{j=1}^{n} V_j(t) \langle W_j(s), v(s) \rangle.
\]
A.2 Variation of parameters formula

The variation of parameters formula is also known as variation of constants formula. It is a formula for calculating solutions to a non-homogeneous ordinary differential equation from the fundamental matrix solution.

**Proposition 1.2.1** (variation of parameters formula). [11, Proposition 2.67] Consider the initial value problem

\[ \dot{u} = A(t)u + g(x,t), \quad u(t_0) = u_0, \quad (2.1) \]

defined on some interval \( J \) containing \( t_0 \). Let \( t \mapsto \Psi(t) \) be a fundamental matrix solution for the homogeneous system

\[ \dot{u} = A(t)u \]

and \( \Phi(t,s) := \Psi(t)\Psi^{-1}(s) \) be the associated state transition matrix, with \( 0 < \det(\Phi(t)) < \infty \) for all \( t \in J \). If \( t \mapsto \phi(t) \) is the solution of the initial value problem defined on some subinterval of \( J \), then

\[ \phi(t) = \Phi(t,t_0)u_0 + \int_{t_0}^{t} \Phi(t,\tau)g(\phi(\tau),\tau)d\tau. \quad (2.2) \]

The solution \( \phi(t) \) shows up on both the lefthand and righthand sides of equation (2.2); thus, the variation of parameters formula defines a fixed point equation. This equation usually cannot be solved explicitly. However, we may be able to apply Banach’s Fixed Point Theorem, discussed in Section A.3, to show the existence of a unique solution. One usually then uses the smoothness of \( g \) to Taylor expand \( \phi(t) \).

**Remark 1.2.2.** By Propositions 1.1.7 and 1.1.8, whenever there exists a fundamen-
tal matrix solution $\Psi(t)$ for the homogeneous problem, with $0 < \det(\Psi(t)) < \infty$ for all $t \in J$, formula (2.2) can be written

$$\phi(t) = \sum_{j=1}^{n} V_j(t) \langle W_j(s), u_0 \rangle + \sum_{j=1}^{n} \int_{t_0}^{t} V_j(t) \langle W_j(s), g(\phi(\tau), \tau) \rangle d\tau, \quad (2.3)$$

where $V_j(t)$ are the columns of $\Psi(t)$ and $W_j(t)$ are the columns of $\Psi^{-T}(t)$ (equivalently: $W_j(t)$ are the columns of the fundamental matrix solution for the adjoint problem $\dot{w} = -A^T(t)w$).

**Remark 1.2.3.** Proposition 1.2.1 can be extended to include parameter dependent equations

$$\dot{u} = A(t)u + g(x,t;\mu), \quad u(t_0;\mu) = u_0 \quad (2.4)$$

by appending the equation $\dot{\mu} = 0$ to equation (2.4) and applying formula (2.2) to the resulting $n+1$-dimensional ordinary differential equation.

### A.3 Banach’s Fixed Point Theorem

Taken from [11, §1.11.2]. Suppose that $(X,d)$ is a complete metric space. We are interested in fixed points of a map $T : X \to X$.

**Definition 1.3.1.** A point $x_0 \in X$ is called a fixed point of $T : X \to X$ if $T(x_0) = x_0$.

One can show the existence of a unique fixed point for a map $T : X \to X$ provided that $T$ is a contraction.
Figure A.1: The mapping $T$ maps $X$ into a subset of $X$. Then $\lim_{n \to \infty} T^n(X) = x_0$, a single “fixed” point.

Definition 1.3.2. Suppose that $T : X \to X$, and $\lambda$ is a real number such that $0 \leq \lambda < 1$. The function $T$ is called a contraction (with contraction constant $\lambda$) if

$$d(T(x), T(y)) \leq \lambda d(x, y)$$

whenever $x, y \in X$.

Theorem 1.1 (Contraction Mapping Theorem). If the function $T$ is a contraction on the complete metric space $(X, d)$ with contraction constant $\lambda$, then $T$ has a unique fixed point $x_0 \in X$. Moreover, if $x \in X$, then the sequence $\{T^n(x)\}_{n=0}^\infty$ converges to $x_0$ as $n \to \infty$ and

$$d(T^n(x), x_0) \leq \frac{\lambda^n}{1-\lambda} d(T(x), x).$$

Theorem 1.1 is represented pictorially in Figure A.1. If the mapping $T$ depends on some parameter $\mu$, then we require the contraction to be uniform in $\mu$.

Definition 1.3.3. Suppose that $A$ is a set, $T : X \times A \to X$, $d$ is a metric on $X$, and $\lambda \in \mathbb{R}$ is such that $0 \leq \lambda < 1$. The function $T$ is a uniform contraction if

$$d(T(x,a), T(y,a)) \leq \lambda d(x, y)$$

whenever $x, y \in X$ and $a \in A$. 

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Figure A.2: Near the point \((x, y) = (1, 0)\) the circle can be described as a graph \(x = \beta(y)\), as indicated by the thick blue curve. However, there is no graph \(\tilde{\beta}\) so that \(y = \tilde{\beta}(x)\).

The resulting fixed “point” will be as smooth as \(T\) is, as described by the Uniform Contraction Theorem.

**Theorem 1.2** (Uniform Contraction Theorem). Suppose that \(X\) and \(Y\) are Banach spaces and define metric \(d\) in terms of the Banach space norm \(d(x, y) := ||x - y||\).

Let \(U \subseteq X\) and \(V \subseteq Y\) be open subsets and \(\overline{U}\) denote the closure of \(U\). Furthermore, assume that the function \(T : \overline{U} \times V \to \overline{U}\) is a uniform contraction with contraction constant \(\lambda\). Finally, for each \(y \in V\), let \(g(y)\) denote the unique fixed point of the contraction \(x \mapsto T(x, y)\) in \(\overline{U}\) given by Theorem 1.1. Then, if \(k\) is a non-negative integer and \(T \in C^k(\overline{U} \times V, X)\), then \(g : V \to X\) is in \(C^k(V, X)\). Also, if \(T\) is real analytic, then so is \(g\).

### A.4 Implicit Function Theorem

The implicit function theorem gives conditions under which a function \(F\) can be described, locally, as a graph.

**Theorem 1.3** (Implicit Function Theorem). \([11, \text{Theorem 1.259}]\) Suppose that \(X\), \(Y\), and \(Z\) are Banach spaces, \(U \subseteq X\), \(V \subseteq Y\) are open sets, \(F : U \times V \to Z\) is a
$C^1$ function, and $(x_0, y_0) \in U \times V$ with $F(x_0, y_0) = 0$. If $F_x(x_0, y_0) : X \to Z$ has a bounded inverse, then there is a product neighborhood $U_0 \times V_0 \subseteq U \times V$ with $(x_0, y_0) \in U_0 \times V_0$ and a $C^1$ function $\beta : V_0 \to U_0$ such that $\beta(y_0) = x_0$ and $F(\beta(y), y) \equiv 0$. Moreover, if $F(x, y) = 0$ for $(x, y) \in U_0 \times V_0$, then $x = \beta(y)$.

**Example 1.4.1** [Circle] It is useful to keep in mind the example of the unit circle: \{(x, y) : x^2 + y^2 = 1\}. As is shown in Figure A.2, the circle can be described near the point $(x, y) = (1, 0)$ as the graph $x = \beta(y)$ where $\beta(y) := \sqrt{1 - y^2}$. However, no graph of the form $y = \tilde{\beta}(x)$ exists since $y = \pm \sqrt{1 - x^2}$ has both the positive and negative solutions. We formulate this example in terms of the hypotheses of the Implicit Function Theorem.

We let $X = Y = Z = \mathbb{R}$, $x \in U := [-1, 1]$, $y \in V = [-1, 1]$, $x_0 = 1$, $y_0 = 0$, and $F(x, y) := x^2 + y^2 - 1$. Then $F(x_0, y_0) = 0$ and $F_x(x_0, y_0) = 2x_0 = 2 \neq 0$. Then, by the Implicit Function Theorem, there exists a graph $\beta : U \times V \to Z$ so that $\beta(y_0) = x_0$ and $F(\beta(y), y) \equiv 0$. Using $\beta(y) = \sqrt{1 - y^2}$ we check that $F(\beta(y), y) = (1 - y^2) + y^2 - 1 = 0$ as claimed. However, since $F_y(x_0, y_0) = 2y_0 = 0$, there does not exist a graph $\tilde{\beta}$ so that $y = \tilde{\beta}(x)$ and $F(x, \tilde{\beta}(x)) = 0$. \triangle
Appendix B

Construction of the core manifold
In this section we prove Lemma 3.2.3, Chapter 3. In particular, we construct the set of all solutions to

$$\partial_r U = \mathcal{A}(1/r)U + \mathcal{F}(U, \epsilon^2)$$  \hspace{1cm} (0.1)$$

which remain bounded as $r \to 0$. In equation (0.1), $U = (\tilde{U}, \tilde{V})^T \in \mathbb{R}^4$,

$$\mathcal{A}(\kappa) := \begin{pmatrix} 0 & I \\ C_1 & -\kappa I \end{pmatrix},$$

$$\mathcal{F}
(\begin{pmatrix} \tilde{U}, \tilde{V} \end{pmatrix}^T; \nu) = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix}
(\begin{pmatrix} \tilde{U}, \tilde{V} \end{pmatrix}^T; \nu) := \begin{pmatrix} 0 \\ \nu C_2 \tilde{U} + |\tilde{U}|^2 C_3 \tilde{U} \end{pmatrix},$$

$I \in \mathbb{R}^2$ is the identity matrix and $C_1, C_2, C_3 \in \mathbb{R}^2$. Equation (0.1) comes up in the study of standard oscillons. For reciprocal oscillons, the leading nonlinear term is quadratic rather than cubic; the resulting fixed point argument is completely analogous and we omit the details.

The proof will follow a standard fixed point theorem argument on solutions constructed using variation of parameters.
B.1 Construction of the fundamental matrix solution

Equation (0.1) can be solved via the variation of parameters formula from Appendix A.2

\[ U(r) = \Phi(r, r_0)U(r_0) + \int_{r_0}^{r} \Phi(r, s)F\left(\tilde{U}(s), \tilde{V}(s); \epsilon^2\right) ds \]  \hspace{1cm} (1.1)

where \( \Phi(r, s) \), described in Appendix A.1, satisfies \( \Phi' = A(1/r)\Phi \) with \( \Phi(r, r) = I \) and is known as the fundamental matrix solution. In particular,

\[ \Phi = V_1(\tilde{V}_2, \cdot) + V_2(\tilde{V}_1, \cdot) + V_3(\tilde{V}_4, \cdot) + V_4(\tilde{V}_3, \cdot) \]  \hspace{1cm} (1.2)

where \( V_j \in \mathbb{R}^4 \) are the linearly independent solutions to \( \partial_r V_j = A(1/r)V_j \) and \( \tilde{V}_j \in \mathbb{R}^4 \) are the linearly independent solutions of the adjoint problem \( \partial_r \tilde{V}_j = \tilde{A}(1/r)\tilde{V}_j \) with

\[ \tilde{A}(1/r) = -A(1/r)^T = \begin{pmatrix} 0 & -C_1^T \\ -I & 1/r \end{pmatrix}. \]

The solutions \( V_j \) and \( \tilde{V}_j \) will be related to the eigenvectors of the matrices \( C_1 \) and \( C_1^T \) respectively. It was shown in Section 3.2.1 that on the bifurcation curve for standard oscillons, the matrix \( C_1 \) has eigenvalues \( \lambda_0 = 0, \lambda_1 = m^2, m \in \mathbb{R} \), with associated eigenvectors \( \tilde{U}_0, \tilde{U}_1 \in \mathbb{R}^2 \). Then \( C_1^T \) has the same eigenvalues with associated eigenvectors \( \tilde{U}_0, \tilde{U}_1 \in \mathbb{R}^2 \). We normalize these eigenvectors so that \( \langle \tilde{U}_i, \tilde{U}_j \rangle = \delta_{ij} \). We remark that the value of these eigenvectors does not affect the analysis below.
We make the ansatz

\[
V_j(r) = \begin{cases} 
\begin{pmatrix} \tilde{a}_j(r)\tilde{U}_0, \tilde{b}_j(r)\tilde{U}_0 \end{pmatrix}^T & : j = 1, 2 \\
\begin{pmatrix} \tilde{a}_j(r)\tilde{U}_1, \tilde{b}_j(r)\tilde{U}_1 \end{pmatrix}^T & : j = 3, 4
\end{cases}
\] (1.3)

and

\[
\tilde{V}_j(r) = \begin{cases} 
\begin{pmatrix} \tilde{a}_j(r)\tilde{U}_0, \tilde{b}_j(r)\tilde{U}_0 \end{pmatrix}^T & : j = 1, 2 \\
\begin{pmatrix} \tilde{a}_j(r)\tilde{U}_1, \tilde{b}_j(r)\tilde{U}_1 \end{pmatrix}^T & : j = 3, 4
\end{cases}
\] . (1.4)

It remains to find the solutions \((\tilde{a}_j(r), \tilde{b}_j(r))\) and \((\tilde{a}_j(r), \tilde{b}_j(r))\). We first observe that \(\tilde{b}_j = \partial_r \tilde{a}_j\) and \(\tilde{b}_j = -\partial_r \tilde{a}_j/\lambda_k\). Then

\[
\partial_{rr} \tilde{a}_j + \frac{\partial_r \tilde{a}_j}{r} - \lambda_k \tilde{a}_j = 0, \quad k = \lfloor j/2 \rfloor
\] (1.5)

and

\[
\partial_{rr} \tilde{a}_j - \frac{\partial_r \tilde{a}_j}{r} - \lambda_k \tilde{a}_j = 0, \quad k = \lfloor j/2 \rfloor,
\] (1.6)

respectively. We also observe that \(\tilde{a}_j = -r \partial_r \tilde{a}_j\) since

\[
\partial_r(-r \partial_r \tilde{a}_j) = -\partial_r \tilde{a}_j - r \partial_{rr} \tilde{a}_j = r(-\lambda_k \tilde{a}_j) \quad \text{and}
\]

\[
\partial_{rr}(-r \partial_r \tilde{a}_j) = -\lambda_k \tilde{a}_j - r \lambda_k \partial_r \tilde{a}_j
\]
Table B.1: The asymptotic behavior of the modified Bessel functions for small argument $z \ll 1$ quoted from [1, (9.6.10)-(9.6.13)] where $\gamma = \lim_{n \to \infty} \left( \sum_{j=1}^{n} \frac{1}{n} - \ln n \right)$ is the Euler-Mascheroni constant.

\[
\begin{array}{cccc}
\text{First Kind} & \text{Second Kind} \\
I_0(z) & 1 + O(z^2) & K_0(z) & - \ln \left( \frac{z}{2} \right) - \gamma + O(z^2 \ln z) \\
I_1(z) & \frac{z}{2} + O(z^3) & K_1(z) & \frac{1}{z} + O(z \ln z) \\
\end{array}
\]

so that the left hand side of (1.6) becomes

\[
(-\lambda_k \tilde{a}_j - r \lambda_k \partial_r \tilde{a}_j) - (-\lambda_k \tilde{a}_j) - (\lambda_k (-r \partial_r \tilde{a}_j)) = 0
\]

as claimed. It is then straightforward to compute $\tilde{b}_j = -\partial_r \tilde{a}_j / \lambda_k = r \tilde{a}_j$.

So it remains to find $\tilde{a}_j(r)$, since all other functions can be computed from it.

There are two cases

(i) $\lambda_k = \lambda_0 = 0$. Then (1.5) is solved by $\tilde{a}_1 = 1$ and $\tilde{a}_2 = \ln r$.

(ii) $\lambda_k = \lambda_1 = m^2$. Then (1.5) is the modified Bessel equation solved by the modified Bessel functions $\tilde{a}_3 = I_0(mr)$ and $\tilde{a}_4 = K_0(mr)$. We use $\partial_z I_0(z) = I_1(z)$ and $\partial_z K_0(z) = -K_1(z)$ [1, (9.6.27)] to find $\tilde{b}_j$. We will also find it useful that $\partial_z (z I_1(z)) = z I_0(z)$ and $\partial_z (-z K_1(z)) = z K_0(z)$ [1, (9.6.28)]. The asymptotic expansion of these functions as $z \to 0$ is given in Table B.1.

Let $\tilde{\nu}_j := (\tilde{a}_j, \tilde{b}_j)$ and $\bar{\nu}_j := (\bar{a}_j, \bar{b}_j)$. Then

\[
\begin{align*}
\tilde{\nu}_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \tilde{\nu}_2 &= \begin{pmatrix} \ln r \\ 1/r \end{pmatrix}, & \tilde{\nu}_3 &= \begin{pmatrix} I_0(mr) \\ m I_1(mr) \end{pmatrix}, & \tilde{\nu}_4 &= \begin{pmatrix} K_0(mr) \\ -mK_1(mr) \end{pmatrix}
\end{align*}
\]
and

\[ q v_1 = \begin{pmatrix} 1 \\ -r \ln r \end{pmatrix}, q v_2 = \begin{pmatrix} m r I_1(m r) \\ -r I_0(m r) \end{pmatrix}, q v_3 = \begin{pmatrix} m r K_1(m r) \\ r K_0(m r) \end{pmatrix}. \]

We observe \( \langle \tilde{v}_j, \tilde{v}_j \rangle = 0 \) by definition of \( \tilde{b}_j, \tilde{a}_j, \) and \( \tilde{b}_j \) since in all cases the inner product is given by

\[ \tilde{a}_j \tilde{a}_j + \tilde{b}_j \tilde{b}_j = (\tilde{a}_j)(-r \partial_r \tilde{a}_j) + (\partial_r \tilde{a}_j)(r \tilde{a}_j) = 0. \]

Moreover, using the Wronskian \( I_0(z) K_1(z) + I_1(z) K_0(z) = 1/z [1, (9.6.15)] \), one can easily see that

\[ \langle \tilde{v}_1, \tilde{v}_2 \rangle = \langle \tilde{v}_2, \tilde{v}_1 \rangle = \langle \tilde{v}_3, \tilde{v}_4 \rangle = \langle \tilde{v}_4, \tilde{v}_3 \rangle = 1. \]

Combining these facts with \( \langle \tilde{U}_i, \tilde{U}_j \rangle = \delta_{ij} \), we see that the fundamental matrix solution \( \Phi \) is indeed given by (1.2) with the vectors \( V_j \) and \( \hat{V}_j \) as defined by our ansatz (1.3) and (1.4), respectively.

We substitute these vectors into the variation of constants formula (1.1) to get
the fixed point equation

\[
\begin{pmatrix}
\tilde{U}(r) \\
\tilde{V}(r)
\end{pmatrix} =
d_1 \begin{pmatrix} \tilde{U}_0 \\ 0 \end{pmatrix} + d_2 \begin{pmatrix} \ln r \tilde{U}_0 \\ 1/r \tilde{U}_0 \end{pmatrix} + d_3 \begin{pmatrix} I_0(mr) \tilde{U}_1 \\ mI_1(mr) \tilde{U}_1 \end{pmatrix} + d_4 \begin{pmatrix} K_0(mr) \tilde{U}_1 \\ -mK_1(mr) \tilde{U}_1 \end{pmatrix}
\]

\[
+ \begin{pmatrix} \tilde{U}_0 \\ 0 \end{pmatrix} \int_{r_0}^{r} (-s \ln s) (\tilde{U}_0, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) \, ds
\]

\[
+ \begin{pmatrix} \ln r \tilde{U}_0 \\ 1/r \tilde{U}_0 \end{pmatrix} \int_{r_0}^{r} s (\tilde{U}_0, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) \, ds
\]

\[
+ \begin{pmatrix} I_0(mr) \tilde{U}_1 \\ mI_1(mr) \tilde{U}_1 \end{pmatrix} \int_{r_0}^{r} s K_0(ms) (\tilde{U}_1, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) \, ds
\]

\[
+ \begin{pmatrix} K_0(mr) \tilde{U}_1 \\ -mK_1(mr) \tilde{U}_1 \end{pmatrix} \int_{r_0}^{r} (-s I_0(ms)) (\tilde{U}_1, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) \, ds
\]

(1.7)

**B.2 Fixed point theorem**

With the preliminaries in place, we apply Banach’s Fixed Point Theorem, from Appendix A.3, to show that there exists a unique fixed point \( U \in \mathbb{R}^4 \) to equation (1.7), in appropriately defined function spaces. We consider only the first two rows of the variation of parameters formula (1.7) since the higher order terms are independent.
of $\tilde{V}$ and since $\tilde{V}$ can be computed from $\tilde{V} = \partial_y \tilde{U}$. Thus, we seek fixed points of

$$
\tilde{U}(r) = d_1 \tilde{U}_0 + d_2 \ln r \tilde{U}_0 + d_3 I_0(mr) \tilde{U}_1 + d_4 K_0(mr) \tilde{U}_1 \\
+ \tilde{U}_0 \int_{r_0}^r (-s \ln s) (\tilde{U}_0, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) ds \\
+ \ln r \tilde{U}_0 \int_{r_0}^r s (\tilde{U}_0, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) ds \\
+ I_0(mr) \tilde{U}_1 \int_{r_0}^r s K_0(ms) (\tilde{U}_1, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) ds \\
+ K_0(mr) \tilde{U}_1 \int_{r_0}^r (-s I_0(ms)) (\tilde{U}_1, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) ds 
$$

(2.1)

**B.2.1 Definition of fixed point equation and spaces**

We will use the space $X = \{ \phi \in C^0([0, r_0]) : ||\phi||_X < \infty \}$ equipped with norm $||\phi||_X := \sup_{r \leq r_0} |\phi|$. Then define $D \subset X \times X$, $D := \{ \phi \in X \times X : ||\phi||_D \leq \rho_1 \}$, equipped with norm $|| (\phi_1, \phi_2) ||_D = \sup ( ||\phi_1||_X, ||\phi_2||_X )$.

We seek only solutions which remain bounded as $r \to 0$. Hence, $d_2$ and $d_4$ must be chosen so that

$$
d_2 = - \int_{r_0}^0 s (\tilde{U}_0, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) ds
$$

and

$$
d_4 = - \int_{r_0}^0 (-s I_0(ms)) (\tilde{U}_1, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) ds,
$$

respectively. Furthermore, for all $\tilde{U} \in D$,

$$
\int_{r_0}^0 (-s \ln s) (\tilde{U}_0, e^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)) ds < \infty
$$
and
\[
\int_{r_0}^0 sK_0(ms)\langle \tilde{U}_1, \epsilon^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s) \rangle ds < \infty,
\]
which means that \(d_1\) and \(d_3\) can be redefined via
\[
d_1 \to d_1 + \int_{r_0}^0 (-s \ln s)\langle \tilde{U}_0, \epsilon^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s) \rangle ds
\]
and
\[
d_3 \to d_3 + \int_{r_0}^0 sK_0(ms)\langle \tilde{U}_1, \epsilon^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s) \rangle ds,
\]
respectively. Altogether, equation (2.1) becomes
\[
\tilde{U}(r) = d_1 \tilde{U}_0 + d_3 I_0(mp)\tilde{U}_1 + \ln r \tilde{U}_0 \int_{r_0}^r s\langle \tilde{U}_0, \epsilon^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s) \rangle ds
+ I_0(mp)\tilde{U}_1 \int_{r_0}^r sK_0(ms)\langle \tilde{U}_1, \epsilon^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s) \rangle ds
+ K_0(mp)\tilde{U}_1 \int_{r_0}^r (-sI_0(ms))\langle \tilde{U}_1, \epsilon^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s) \rangle ds
= : G(\tilde{U}; d_1, d_3).
\]

Using the asymptotic expansions in Table B.1, it is easy to check that the right hand side of (2.2) is indeed bounded as \(r \to 0\) for all \(\tilde{U} \in C([0, r_0]; \mathbb{R}^2)\). Since neither \(d_2\) nor \(d_4\) appear in equation (2.2), we define the parameter space \(\mathbb{R}^2 \supset B := \{d_1, d_3 \in \mathbb{R} : |d_1|, |d_3| \leq \rho_2\}\).

We will find it useful to define the following constants: \(c_0 := ||\tilde{U}_0||_D, c_1 := ||\tilde{U}_1||_D,\)
\(\tilde{c}_0 := ||\tilde{U}_0||_D, \tilde{c}_1 := ||\tilde{U}_1||_D, c_2 := \sup_{||\phi||=1} ||C_2 \phi||_D, c_3 := \sup_{||\phi||=1} ||C_3 \phi||_D\). Furthermore, it is clear from the asymptotics expansions in Table B.1 that \(I_0(z), I_1(z),\)

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$zK_0(z)$ and $zK_1(z)$ are bounded over any bounded interval $z \in [0, z_0], z_0 < \infty$; we can therefore define $c_{I_0, r_0} := ||I_0(mr)||_X$, $c_{I_1, r_0} := ||I_1(mr)||_X$, $c_{r K_0, r_0} := ||rK_0(mr)||_X$, and $c_{r K_1, r_0} := ||rK_1(mr)||_X$. Similarly, $c_{r^2, r_0} := ||r^2||_X$, and $c_{r^2 \ln r, r_0} := ||r^2 \ln r||_X$. The notation $c_{\phi, r_0}$ reinforces the fact that the bounds depend on $r_0$. All of the above constants are finite.

In the following we will show that equation (2.2) satisfies the hypotheses of Banach’s Fixed Point Theorem, from Appendix A.3:

(i) $\mathcal{G} : \mathcal{D} \times \mathcal{B} \rightarrow \mathcal{D}$

(ii) $\exists l < 1 : ||\mathcal{G}(\phi; d) - \mathcal{G}(\psi; d)||_D \leq l||\phi - \psi||_D \quad \forall \phi, \psi \in \mathcal{D}$ and $d := (d_1, d_3) \in \mathcal{B}$.

and conclude that there exists a unique fixed point.
B.2.2 Hypothesis (i): Mapping into itself

Let \( \tilde{U}(r) \in \mathcal{D} \). It is straightforward to observe that \( \mathcal{G}(\tilde{U}) \in \mathcal{C}^0([s_0, \infty)) \times \mathcal{C}^0([s_0, \infty)) \).

It remains to show \( ||\mathcal{G}(\tilde{U})||_\mathcal{D} \leq \rho_1 \).

\[
||\mathcal{G}(\tilde{U}; d_1, d_3)||_\mathcal{D} \\
\leq \rho_2 c_0 + \rho_2 c_1 ||I_0(mr)||_x \\
+ (\epsilon^2 c_2 \rho_1 + c_3 \rho_1^3) \left[ c_0 \hat{c}_0 \left(||r^2 - s \ln s ds||_x + ||\ln r \int_0^{r} s ds||_x\right) \\
+ c_1 \hat{c}_1 \left(||I_0(mr) \int_0^{r} sK_0(ms)ds||_x + ||K_0(mr) \int_0^{r} -sI_0(ms)ds||_x\right)\right] \\
\leq \rho_2 (c_0 + c_1 ||I_0(mr)||_x) \\
+ (\epsilon^2 c_2 \rho_1 + c_3 \rho_1^3) \left[ c_0 \hat{c}_0 \left(||r^2 + 0.5r^2 \ln r||_x + ||0.5r^2 \ln r||_x\right) \\
+ c_1 \hat{c}_1 \left(||rI_0(mr)(mK_1(mr) + 1)/m^2||_x + ||rK_0(mr)I_1(mr)/m||_x\right)\right] \\
\leq \rho_2 (c_0 + c_1 c_{I_0,r_0}) + (\epsilon^2 c_2 \rho_1 + c_3 \rho_1^3) \ldots \\
* \left[ c_0 \hat{c}_0 (c_{r^2,r_0} + 2c_{r^2 \ln r,r_0}) + c_1 \hat{c}_1 (c_{I_0,r_0} (c_{rK_1,r_0} + 1/m^2) + c_{rK_0,r_0} c_{I_1,r_0}) \right].
\]

Set

\[
C_{r_0} := c_0 \hat{c}_0 (c_{r^2,r_0} + 2c_{r^2 \ln r,r_0}) + c_1 \hat{c}_1 (c_{I_0,r_0} (c_{rK_1,r_0} + 1/m^2) + c_{rK_0,r_0} c_{I_1,r_0})
\]

Thus, Hypothesis (i) reduces to showing

\[
\rho_2 (c_0 + c_1 c_{I_0,r_0}) + (\epsilon^2 \rho_1 + c_3 \rho_1^3) C_{r_0} \leq \rho_1. \quad (2.3)
\]
This is true by first fixing $\rho_1 < \rho_1^*$ where $\rho_1^*$ satisfies

$$\epsilon^2 + c_3(\rho_1^*)^2 < \frac{1}{2C_{r_0}}$$

and then fixing $\rho_2 < \rho_2^*$ where

$$\rho_2^* < \frac{\rho_1}{2(c_0 + c_1 c_{I_0, r_0})}.$$ 

In the above $\epsilon$ is small enough that $\rho_1^* > 0$.

### B.2.3 Hypothesis (ii): Contraction

We will find it convenient to denote the first and second rows of $\mathcal{G}$ and $\tilde{U}$ by

$$\mathcal{G} =: \begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \end{pmatrix} \quad \text{and} \quad \tilde{U} =: \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix},$$

respectively. The space $\mathcal{D}$ is convex so the Lipschitz constant is equal to the norm of the Jacobian

$$\|D\mathcal{G}\|_{\mathcal{D}} = \sup_{\phi \in \mathcal{G}} \frac{\|D\mathcal{G}\phi\|_{\mathcal{D}}}{\|\phi\|_{\mathcal{D}}},$$

where

$$DF = \begin{pmatrix} \partial_{u_1} \mathcal{G}_1 & \partial_{u_2} \mathcal{G}_1 \\ \partial_{\tilde{u}_1} \mathcal{G}_2 & \partial_{\tilde{u}_2} \mathcal{G}_2 \end{pmatrix}.$$
We use the Gâteaux derivative

\[(\partial_x G_y)\bar{x} = \lim_{\delta \to 0} \frac{G_y(\bar{x} + \delta \bar{x}) - G_y(\bar{x})}{\delta} .\]

We note first that

\[||DG(\phi_1, \phi_2)^T||_D = ||\partial_{\bar{u}_1} G_1 \phi_1 + \partial_{\bar{u}_2} G_1 \phi_2||_x + ||\partial_{\bar{u}_1} G_2 \phi_1 + \partial_{\bar{u}_2} G_2 \phi_2||_x \]
\[\leq ||\partial_{\bar{u}_1} G_1 \phi_1||_x + ||\partial_{\bar{u}_2} G_1 \phi_2||_x + ||\partial_{\bar{u}_1} G_2 \phi_1||_x + ||\partial_{\bar{u}_2} G_2 \phi_2||_x .\]

Therefore,

\[||DG||_D \leq \sup_{(\phi_1, \phi_2) \in D} \frac{(||\partial_{\bar{u}_1} G_1 \phi_1||_x + ||\partial_{\bar{u}_2} G_1 \phi_2||_x) + (||\partial_{\bar{u}_1} G_2 \phi_1||_x + ||\partial_{\bar{u}_2} G_2 \phi_2||_x)}{||\phi_1||_x + ||\phi_2||_x} .\]
The derivatives in the numerator differ only by a constant and can, in fact, be uniformly bounded.

\[ \| \partial_{\tilde{u}} G_i \phi_j \|_X \]

\begin{align*}
\leq & \ c_0 \tilde{c}_0 \left\| \int_0^r -s \ln (\epsilon^2 c_2 \tilde{u}_j + 3c_3 \rho_1^2 \tilde{u}_j) \, ds \right\|_X \\
& + \ c_0 \tilde{c}_0 \left\| \ln r \int_0^r (\epsilon^2 c_2 \tilde{u}_j + 3c_3 \rho_1^2 \tilde{u}_j) \, ds \right\|_X \\
& + \ c_1 \tilde{c}_1 \left\| I_0 (mr) \int_0^r s K_0 (ms) (\epsilon^2 c_2 \tilde{u}_j + 3c_3 \rho_1^2 \tilde{u}_j) \right\|_X \right| \, ds \\
& + \ c_1 \tilde{c}_1 \left\| K_0 (mr) \int_0^r -s I_0 (ms) (\epsilon^2 c_2 \tilde{u}_j + 3c_3 \rho_1^2 \tilde{u}_j) \right\|_X \right| \, ds \\
\leq & \ (\epsilon^2 c_2 + 3c_3 \rho_1^2) \left[ c_0 \tilde{c}_0 \left( \left\| \int_0^r -s \ln s ds \right\|_X + \right\| \ln r \int_0^r s ds \right\|_X \right) \\
& + \ c_1 \tilde{c}_1 \left( \left\| I_0 (mr) \int_0^r s K_0 (ms) \right\|_X \right| \, ds + \left\| K_0 (mr) \int_0^r -s I_0 (ms) \right\|_X \right| \right] \| \tilde{u}_j \|_X \\
\leq & \ (\epsilon^2 c_2 + 3c_3 \rho_1^2) \| \tilde{u}_j \|_X \]

\begin{align*}
\ast & \left[ c_0 \tilde{c}_0 \left( c_{r^2, r_0} + 2c_{r^2 \ln r, r_0} \right) + c_1 \tilde{c}_1 \left( c_{l_0, r_0} (c_{r K_1, r_0} + 1/m^2) + c_{r K_0, r_0} c_{l_1, r_0} \right) \right] \\
= & \ C(\epsilon^2 c_2 + 3c_3 \rho_1^2) \| \tilde{u}_j \|_X
\end{align*}

Substituting into \( \| D\mathcal{G} \|_D \), we have that Hypothesis (ii) is satisfied if

\[ \sup_{(\pi_1, \pi_2) \in \mathcal{D}} \frac{2C(\epsilon^2 c_2 + 3c_3 \rho_1^2) (\| \tilde{u}_1 \|_X + \| \tilde{u}_2 \|_X)}{\| \tilde{u}_1 \|_X + \| \tilde{u}_2 \|_X} < 1/2. \]

Hence, we need to show that \( \epsilon^2 c_2 + 3c_3 \rho_1^2 < 1/(4C) \). We first fix \( \epsilon < \epsilon_0 \) small enough so that \( 1/(4c) - \epsilon^2 c_2 > 0 \). Then the inequality is satisfied by choosing \( \rho_1 < \rho_1^* \) where

\[ 3c_3(\rho_1^*)^2 < 1/(4C) - \epsilon^2 c_2. \]
Thus \( G \) is a contraction and Hypothesis (ii) is satisfied. Hypotheses (i) and (ii) are simultaneously satisfied by taking \( \rho_1 = \rho_1^\dagger := \min(\rho_1^*, \rho_1^\dagger) \) and then \( \rho_2 < \rho_2^\dagger \) where

\[
\rho_2^\dagger < \frac{\rho_1}{2(c_0 + c_1c_{I_0, r_0})}.
\]

### B.2.4 Conclusion and refinement of higher order error terms

By combining the results of sections B.2.2 and B.2.3 we can conclude that, for every fixed \( r_0 < \infty \), there exists a unique fixed point \( \tilde{U} = G(\tilde{U}) \) for \( \rho_1, \rho_2 \), and \( \epsilon_0 \) small enough. Hence, there exists a unique solution \( U \in \mathbb{R}^4 \) to the variation of parameters formula

\[
\begin{pmatrix}
\tilde{U}(r) \\
\tilde{V}(r)
\end{pmatrix} = d_1 \begin{pmatrix}
\tilde{U}_0 \\
0
\end{pmatrix} + d_3 \begin{pmatrix}
I_0(mr)\tilde{U}_1 \\
mI_1(mr)\tilde{U}_1
\end{pmatrix}
+ \begin{pmatrix}
\tilde{U}_0 \\
0
\end{pmatrix} \int_{r_0}^r (-s \ln s)(\tilde{U}_0, \epsilon_0^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s))ds
+ \begin{pmatrix}
\ln r\tilde{U}_0 \\
1/r\tilde{U}_0
\end{pmatrix} \int_{0}^r s(\tilde{U}_0, \epsilon_0^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s))ds
+ \begin{pmatrix}
I_0(mr)\tilde{U}_1 \\
mI_1(mr)\tilde{U}_1
\end{pmatrix} \int_{r_0}^r sK_0(ms)\langle\tilde{U}_1, \epsilon_0^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)\rangle ds
+ \begin{pmatrix}
K_0(mr)\tilde{U}_1 \\
-mK_1(mr)\tilde{U}_1
\end{pmatrix} \int_{0}^r (-sI_0(ms))\langle\tilde{U}_1, \epsilon_0^2 C_2 \tilde{U}(s) + |\tilde{U}(s)|^2 C_3 \tilde{U}(s)\rangle ds
\]

(2.4)
after inverting the redefinition of \(d_1\) and \(d_2\) from Section B.2.1. Moreover, we evaluate (2.4) at \(r = r_0\) to obtain

\[ U(r_0) = d_1V_1(r_0) + d_3V_3(r_0) + g_2(d_1, d_3; \epsilon)V_2(r_0) + g_4(d_1, d_3; \epsilon)V_4(r_0), \]  

(2.5)

where \(g_j(d_1, d_3; \epsilon) = O_{r_0}(\epsilon^2|d| + |d|^3)\) are smooth functions.

It remains to argue that \(\tilde{U} \in C^0([0, r_0]; \mathbb{R}^4)\) is contained in \(\tilde{W}_{cs}^{\epsilon}\), the set of all bounded solutions to (0.1) as \(r \to 0\), if, and only if, \(\tilde{U}\) satisfies (2.4), and hence (2.5). Let \(\tilde{U} \in C^0([0, r_0]; \mathbb{R}^4)\) be a bounded solution to (0.1). Then \(\tilde{U}\) satisfies the variation of constants formula (1.7), which differs from (2.4) by the additional \(d_2V_2(r)\) and \(d_4V_4(r)\) terms, and the limits on the corresponding integrals. But we have already shown in Section B.2.1 that the right hand side of (1.7) will be unbounded unless \(d_2\) and \(d_4\) are chosen so that (1.7) becomes (2.4). Conversely, every solution \(\tilde{U} \in C^0([0, r_0]; \mathbb{R}^4)\) of (2.4) gives a solution to (0.1); since the right hand side is bounded on \(r \in [0, r_0]\), so is \(\tilde{U}\).
Appendix C

Connecting orbit $q_0$: Small argument asymptotics
In this section we consider the behavior of \( q_0(s) \) as \( s \to 0 \), where \( q_0(s) \) is a solution to

\[
a_{ss} + \frac{a_s}{s} = a - a^3. \tag{0.1}
\]

Equation (0.1) comes up in the analysis of the far-field center manifold associated with standard oscillons. For reciprocal oscillons, the leading nonlinear term is quadratic rather than cubic; the resulting fixed point argument is completely analogous and we omit the details.

The proof will follow a standard fixed point theorem argument on solutions constructed using variation of parameters.

\section*{C.1 Construction of the fundamental matrix solution}

It is known that \( q_0(s) \) is the unique nonnegative monotone solution to (0.1) [24, 34, 39]; hence, \( q_0(s) \) approaches a constant as \( s \to 0 \), say \( a_0 < \infty \). We write \( a(s) = a_0 + \tilde{a}(s) \) where \( \tilde{a}(0) = 0 \) and \( \tilde{a}(s) = o(1) \). Then (0.1) becomes

\[
\tilde{a}_{ss} + \frac{\tilde{a}_s}{s} + (3a_0^2 - 1)\tilde{a} = a_0(1 - a_0^2) - \tilde{a}^2(3a_0 + \tilde{a}) \tag{1.1}
\]

Equation (1.1) can be written as a first order system

\[
\tilde{u}_s = \mathcal{A}(s)\tilde{u} + \mathcal{F}(\tilde{a}, \tilde{b}) \tag{1.2}
\]
where

\[ \tilde{u} := \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}, \quad \mathcal{A}(s) := \begin{pmatrix} 0 & 1 \\ 1 - 3a_0^2 & -\frac{1}{s} \end{pmatrix}, \quad \mathcal{F}(\tilde{a}, \tilde{b}) := \begin{pmatrix} 0 \\ a_0(1 - a_0^2) - \tilde{a}^2(3a_0 + \tilde{a}) \end{pmatrix}. \] (1.3)

Equation (1.2) can be solved via the variation of parameters formula from Appendix ??

\[ \tilde{u}(s) = \Phi(s, s_0)\tilde{u}(s_0) + \int_{s_0}^{s} \Phi(s, \tau)\mathcal{F}(\tilde{u}(\tau), \tilde{b}(\tau)) \, d\tau \] (1.4)

where \( \Phi(s, \tau) \), described in Appendix A.1, satisfies \( \Phi' = \mathcal{A}(s)\Phi \) with \( \Phi(s, s) = I \) and is known as the fundamental matrix solution. In particular,

\[ \Phi_\cdot = \tilde{v}_1(\tilde{v}_2, \cdot) + \tilde{v}_2(\tilde{v}_1, \cdot) \] (1.5)

where \( \tilde{v}_j \) are the two linearly independent solutions of \( \tilde{v}_j' = \mathcal{A}(s)\tilde{v}_j \) and \( \hat{v}_j \) are the two linearly independent solutions of the adjoint problem

\[ \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} 0 & 3a_0^2 - 1 \\ -1 & \frac{1}{s} \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}. \] (1.6)

In fact, let \( \tilde{a}_j \) be a solution of

\[ \partial_s \tilde{a}_j + \frac{\partial \tilde{a}_j}{s} + (3a_0^2 - 1)\tilde{a}_j = 0. \] (1.7)
Then $\tilde{v}_j = (\tilde{a}_j, \partial_s \tilde{a}_j)^T$ solves the linear problem and $\hat{v}_j = (-s \partial_s \tilde{a}_j, s \tilde{a}_j)^T$ (or $\hat{v}_j = (s \partial_s \tilde{a}_j, -s \tilde{a}_j)^T$) solves the adjoint linear problem since

$$
\partial_s (-s \partial_s \tilde{a}_j) = -\partial_s \tilde{a}_j - s \partial_{ss} \tilde{a}_j = (3a_0^2 - 1)(s \tilde{a}_j) \quad \text{and}
\partial_s (s \tilde{a}_j) = \tilde{a}_j + s \partial_s \tilde{a}_j = -(-s \partial_s \tilde{a}_j) + \frac{1}{s} (s \tilde{a}_j).
$$

It remains to find the solutions $\tilde{a}_j$ to (1.7). There are 3 cases to consider based on the sign of $3a_0^2 - 1$. We will argue that, in fact, all three options give the same asymptotic behavior for $\tilde{v}_i$, $\hat{v}_i$ and hence for the fundamental matrix solution as $s \to 0$. Letting $3a_0^2 - 1 = c$ and

$$
\bar{c} = \begin{cases} 
|c| : & c \neq 0 \\
1 : & c = 0
\end{cases}
$$

we have

(i) $3a_0^2 - 1 = 0$. Then equation (1.7) is solved by $\tilde{a}_j(s) = 1$ or $\ln s$. Then the fundamental matrix solution becomes

$$
\Phi(s, \tau) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left\langle \begin{pmatrix} \bar{c} \\ -\tau \ln \bar{c} \tau \end{pmatrix}, \cdot \right\rangle + \begin{pmatrix} \ln \bar{c} s \\ \frac{1}{s} \end{pmatrix} \left\langle \begin{pmatrix} 0 \\ \tau \end{pmatrix}, \cdot \right\rangle.
$$

(ii) $3a_0^2 - 1 < 0$. Then equation (1.7) is the modified Bessel equation, solved by the modified Bessel functions $I_0(cs)$ and $K_0(cs)$. We note that $\partial_z I_0(z) = I_1(z)$ and $\partial_z K_0(z) = -K_1(z)$ [1, (9.6.27)] so that the fundamental matrix solution
Table C.1: The asymptotic behavior of the Bessel functions and modified Bessel functions for small argument \( z \ll 1 \) quoted from [1, (9.1.10)-(9.1.13), (9.6.10)-(9.6.13)], respectively, where \( \gamma = \lim_{n \to \infty} \left( \sum_{j=1}^{n} \frac{1}{n} - \ln n \right) \) is the Euler-Mascheroni constant.

<table>
<thead>
<tr>
<th></th>
<th>Bessel Functions</th>
<th>Modified Bessel Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J_0(z) )</td>
<td>( 1 + O(z^2) )</td>
<td>( I_0(z) )</td>
</tr>
<tr>
<td>( J_1(z) )</td>
<td>( \frac{z}{2} + O(z^3) )</td>
<td>( I_1(z) )</td>
</tr>
<tr>
<td>( Y_0(z) )</td>
<td>( \ln \left( \frac{z}{2} \right) + \gamma + O(z^2 \ln z) )</td>
<td>( K_0(z) )</td>
</tr>
<tr>
<td>( Y_1(z) )</td>
<td>( -\frac{1}{z} + O(z \ln z) )</td>
<td>( K_1(z) )</td>
</tr>
</tbody>
</table>

(iii) \( 3a_0^2 - 1 > 0 \). Then equation (1.7) is the Bessel equation, solved by the Bessel functions \( J_0(cs) \) and \( Y_0(cs) \). We note that \( \partial_z J_0(z) = -J_1(z) \) and \( \partial_z Y_0(z) = -Y_1(z) \) [1, (9.1.28)] so that the fundamental matrix solution becomes

\[
\Phi(s, \tau) = \frac{J_0(\tau)}{\tau J_1(\tau)} \langle \begin{pmatrix} \tau K_1(\tau) \\ -\tau K_0(\tau) \end{pmatrix}, \begin{pmatrix} \tau I_1(\tau) \\ -\tau I_0(\tau) \end{pmatrix} \rangle + \frac{1}{z} \langle \begin{pmatrix} K_0(\tau) \\ -\tau K_1(\tau) \end{pmatrix}, \begin{pmatrix} \tau I_1(\tau) \\ -\tau I_0(\tau) \end{pmatrix} \rangle
\]

In the above we have appropriately scaled the Bessel functions so that the Wronskians [1, (9.1.16), (9.6.15)] become

\[
J_1(z)Y_0(z) - J_0(z)Y_1(z) = \frac{1}{z}
\]

\[
I_0(z)K_1(z) + I_1(z)K_0(z) = \frac{1}{z}.
\]

Using the Wronskian, one can easily observe that, for all three cases, \( \langle \hat{v}_i, \hat{v}_j \rangle = (1 - \delta_{ij}) \) as desired so that the fundamental matrix solution is given by (1.5). The asymptotic expansion of these functions as \( z \to 0 \) is given in Table C.1.
Using the expansions in Table C.1 we see that for \( c \geq 0 \) the fundamental matrix solution \( \Phi \) can be expanded

\[
\Phi(s, \tau) = \begin{pmatrix}
1 + \{O(s^2)\} \\
0 + \{O(s)\}
\end{pmatrix} \begin{pmatrix}
1 + \{O(\tau^2)\} \\
-\tau (\ln c \tau + \{-\gamma + \ln 2 + O(\tau^2 \ln \tau)\})
\end{pmatrix},
\]

\[
+ \begin{pmatrix}
\ln c s + \{\gamma - \ln 2 + O(s^2 \ln s)\} \\
\frac{1}{s} + \{O(s)\}
\end{pmatrix} \begin{pmatrix}
0 + \{O(\tau^2)\} \\
\tau + \{O(\tau^3)\}
\end{pmatrix},
\]

and for \( c < 0 \) the fundamental matrix solution \( \Phi \) can be expanded

\[
\Phi(s, \tau) = \begin{pmatrix}
1 + O(s^2) \\
0 + O(s)
\end{pmatrix} \begin{pmatrix}
1 + O(\tau^2) \\
-\tau (\ln c \tau + -\gamma + \ln 2 + O(\tau^2 \ln \tau))
\end{pmatrix},
\]

\[
+ \begin{pmatrix}
-(\ln c s + \gamma - \ln 2) + O(s^2 \ln s) \\
-\frac{1}{s} + O(s)
\end{pmatrix} \begin{pmatrix}
0 + O(\tau^2) \\
-\tau + O(\tau^3)
\end{pmatrix}.
\]

The terms in brackets are higher order terms not present for the case \( 3a_0^2 - 1 = 0 \); for simplicity of notation we omit the brackets from now on. We observe that the difference between (1.8) and (1.9) is the sign in front of the leader order terms in the second inner product. Hence, to leading order, (1.8) and (1.9) are equivalent.

### C.2 Fixed point theorem

With the preliminaries in place, we apply Banach’s Fixed Point Theorem, from Appendix A.3, to show that there exists a unique fixed point \((\tilde{a}, \tilde{b})\) to equation (1.4), in appropriately defined function spaces. We consider only the first row of the variation of parameters formula (1.4) since the higher order terms are independent
of $\tilde{b}$ and since $\tilde{b}$ can be computed from $\tilde{b} = \partial_s \tilde{a}$.

C.2.1 Definition of fixed point equation and spaces

We will use the space $\mathcal{X} = \{ \tilde{a}(s) \in C^0([0, s_0]) : ||\tilde{a}(s)|| < \infty \}$ equipped with norm $||\tilde{a}|| := \sup_{s \leq s_0} |\tilde{a}(s)|$. Then define $\mathcal{D} \subset \mathcal{X}$, $\mathcal{D} = \{ \tilde{a}(s) \in \mathcal{X} : ||\tilde{a}(s)|| \leq \rho \}$.

We first observe that the integrands in (1.4) are

\[
\left\langle \begin{pmatrix} 1 + \{O(\tau^2)\} \\ -\tau (\ln \tau + \{ -\gamma + \ln 2 + O(\tau^2 \ln \tau) \}) \end{pmatrix}, \begin{pmatrix} 0 \\ a_0(1 - a_0^2) - \tilde{a}^2(3a_0 + \tilde{a}) \end{pmatrix} \right\rangle = \left[ -\tau (\ln \tau + \gamma - \ln 2 + O(\tau^2 \ln \tau)) \right] \left[ a_0(1 - a_0^2) - \tilde{a}(\tau)^2(3a_0 + \tilde{a}(\tau)) \right]
\]

and

\[
\left\langle \begin{pmatrix} 0 + \{O(\tau^2)\} \\ \tau + \{O(\tau^3)\} \end{pmatrix}, \begin{pmatrix} 0 \\ a_0(1 - a_0^2) - \tilde{a}^2(3a_0 + \tilde{a}) \end{pmatrix} \right\rangle = \left[ \tau (1 + O(\tau^2)) \right] \left[ a_0(1 - a_0^2) - \tilde{a}(\tau)^2(3a_0 + \tilde{a}(\tau)) \right].
\]

Using the above expansions, the first row of (1.4) becomes

\[
\tilde{a}(s) = c_1(1 + O(s^2)) + c_2 \left( \ln \tau s + \gamma - \ln 2 + O(s^2 \ln s) \right) \\
+ \left[ 1 + O(s^2) \right] \int_{s_0}^s -\tau \left[ \ln \tau + \gamma - \ln 2 + O(\tau^2 \ln \tau) \right] \left[ a_0(1 - a_0^2) - \tilde{a}^2(3a_0 + \tilde{a}) \right] d\tau \\
+ \left[ \ln \tau s + \gamma - \ln 2 + O(s^2 \ln s) \right] \int_{s_0}^s \tau \left[ 1 + O(\tau^2) \right] \left[ a_0(1 - a_0^2) - \tilde{a}^2(3a_0 + \tilde{a}) \right] d\tau.
\]

(2.1)

We will restrict the analysis to solutions in $\mathcal{X} := \{ \tilde{a} : ||\tilde{a}|| < \infty \}$ with $||\phi|| := \ldots$
\[ \sup_{s \leq s_0} |\phi(s)|. \] Therefore, \( c_2 \) must be chosen so that

\[ c_2 = -\int_{s_0}^{0} \tau \left[ 1 + O(\tau^2) \right] \left[ a_0(1 - a_0^2) - \tilde{a}^2(3a_0 + \tilde{a}) \right] d\tau. \]

Furthermore, \( \int_{s_0}^{0} O(\tau \ln \tau (1 + \tilde{a}^2)) d\tau < \infty \) so that \( c_1 \) can be redefined

\[ c_1 \rightarrow c_1 + \int_{s_0}^{0} -\tau \left[ \ln \tau + \gamma - \ln 2 + O(\tau^2 \ln \tau) \right] \left[ a_0(1 - a_0^2) - \tilde{a}^2(3a_0 + \tilde{a}) \right] d\tau. \]

Altogether, equation (2.1) becomes

\[ \tilde{a}(s) = c_1(1 + O(s^2)) + \left[ 1 + O(s^2) \right] \int_{0}^{s} -\tau \left[ \ln \tau + \gamma - \ln 2 + O(\tau^2 \ln \tau) \right] \left[ a_0(1 - a_0^2) - \tilde{a}^2(3a_0 + \tilde{a}) \right] d\tau \]

\[ + \left[ \ln \tau s + \gamma - \ln 2 + O(s^2 \ln s) \right] \int_{0}^{s} \tau \left[ 1 + O(\tau^2) \right] \left[ a_0(1 - a_0^2) - \tilde{a}^2(3a_0 + \tilde{a}) \right] d\tau. \]

We observe that \( \tilde{a}(0) = 0 \) implies that \( c_1 = 0 \) and explicitly integrate the terms with no \( \tilde{a} \) dependence to get

\[ \tilde{a}(s) = a_0(1 - a_0^2) \frac{s^2}{4} + O(s^4 \ln s) \]

\[ + \left[ 1 + O(s^2) \right] \int_{0}^{s} \tau \left[ \ln \tau s + \gamma - \ln 2 + O(\tau^2 \ln \tau) \right] \tilde{a}^2(3a_0 + \tilde{a}) d\tau \]

\[ - \left[ \ln \tau s + \gamma - \ln 2 + O(s^2 \ln s) \right] \int_{0}^{s} \tau \left[ 1 + O(\tau^2) \right] \tilde{a}^2(3a_0 + \tilde{a}) d\tau \]

\[ =: \mathcal{F}(\tilde{a}(s)). \] (2.2)

In the following we will show that equation (2.2) satisfies the hypotheses of Banach’s Fixed Point Theorem, from Appendix A.3:

(i) \( \mathcal{F} : \mathcal{D} \rightarrow \mathcal{D} \)
(ii) \( \exists l < 1 : \| F(\phi) - F(\psi) \| \leq l \| \phi - \psi \| \quad \forall \phi, \psi \in D \)

and conclude that there exists a unique fixed point.

### C.2.2 Hypothesis (i): Mapping into itself

Let \( \tilde{a}(s) \in D \). It is straightforward to observe that \( F(\tilde{a}(s)) \in C^0([s_0, \infty)) \). It remains to show \( \| F(\tilde{a}(s)) \| \leq \rho \).

\[
\| F(\tilde{a}(s)) \| \leq \left| a_0 (1 - a_0^2)\frac{s_0^2}{4} + O(s_0^4 \ln s_0) \right| \\
+ \left| \left[ 1 + O(s^2) \right] \int_{s_0}^{s} \tau \left[ \ln \tau + \gamma - \ln 2 + O(\tau^2 \ln \tau) \right] \tilde{a}^2(3a_0 + \tilde{a}) d\tau \right| \\
+ \left| \left[ \ln s_0 + \gamma - \ln 2 + O(s^2 \ln s) \right] \int_{s_0}^{s} \tau \left[ 1 + O(\tau^2) \right] \tilde{a}^2(3a_0 + \tilde{a}) d\tau \right| \\
= \left| a_0 (1 - a_0^2)\frac{s_0^2}{4} + O(s_0^4 \ln s_0) \right| + (I) + (II) \tag{2.3}
\]

We compute each integral individually using the pointwise bound \( |\tilde{a}(\tau)| \leq \rho \) for all \( \tau \leq s_0 \) since \( \tilde{a}(\tau) \in D \).

\[
(I) \sup_{s \leq s_0} \left| \left[ 1 + O(s^2) \right] \int_{0}^{s} \tau \left[ \ln \tau + \gamma - \ln 2 + O(\tau^2 \ln \tau) \right] \tilde{a}^2(3a_0 + \tilde{a}) d\tau \right| \\
\leq \sup_{s \leq s_0} \left| \left[ 1 + O(s^2) \right] \int_{0}^{s} \tau \left[ \ln \tau + \gamma - \ln 2 + O(\tau^2 \ln \tau) \right] \rho^2(3a_0 + \rho) d\tau \right| \\
\leq \sup_{s \leq s_0} \left| \left[ 1 + O(s^2) \right] \rho^2(3a_0 + \rho) \left[ \frac{\tau^2}{4} (2 \ln \tau - 1 + 2 \gamma - 2 \ln 2) + O(\tau^4 \ln \tau) \right] \right| \bigg|_{0}^{s} \\
= \rho^2(3a_0 + \rho) \left( s_0^2 / 2 (|\ln \tau s_0| + \gamma + \ln 2 + 1/2) + O(s_0^4 \ln s_0) \right)
\]
\[
\begin{align*}
(I) & \quad \sup_{s \leq s_0} \left| \ln \bar{v}s + \gamma - \ln 2 + O(s^2 \ln s) \right| \int_0^s \tau \left[ 1 + O(\tau^2) \right] \bar{a}^2 (3a_0 + \bar{a}) d\tau \\
\leq & \sup_{s \leq s_0} \left| \ln \bar{v}s + \gamma - \ln 2 + O(s^2 \ln s) \right| \int_0^s \tau \left[ 1 + O(\tau^2) \right] \rho^2 (3a_0 + \rho) d\tau \\
\leq & \sup_{s \leq s_0} \left| \ln \bar{v}s + \gamma - \ln 2 + O(s^2 \ln s) \right| \rho^2 (3a_0 + \rho) \left[ \frac{\tau^2}{2} + O(\tau^4) \right]_0^s \\
= & \left| \ln \bar{v}s_0 + \gamma + \ln 2 \right| \rho^2 (3a_0 + \rho) \left( s_0^2/2 + O(s_0^4 \ln s_0) \right) .
\end{align*}
\]

We substitute these estimates back into (2.3) and find that Hypothesis (i) reduces to showing

\[
\left| a_0 (1 - a_0^2) \frac{s_0^2}{4} + \rho^2 (3a_0 + \rho) (s_0^2 |\ln \bar{v}s_0| + O(s_0^2)) \right| \leq \rho .
\] (2.4)

Consider

\[
0 = \left| a_0 (1 - a_0^2) \frac{s_0^2}{4} + \rho^2 (3a_0 + \rho) (s_0^2 |\ln \bar{v}s_0| + O(s_0^2)) - \rho/2 \right| =: f(s_0; \rho) .
\] (2.5)

The function \( f \) satisfies the hypotheses of the Implicit Function Theorem, from Appendix A.4: \( f(0; 0) = 0 \) and \( (\partial_\rho f)(0; 0) = -1/2 \neq 0 \). Therefore, for each \( s_0 \leq s_* \) small enough there exists a unique \( \rho_*(s_0) \) such that \( f(s_0; \rho_*(s_0)) = 0 \) and inequality (2.4) is satisfied.
C.2.3 Hypothesis (ii): Contraction

The space $D$ is convex so the Lipschitz constant is equal to the norm of the derivative

$$||\partial \tilde{a} F|| = \sup_{\tilde{a} \in D} \frac{||(\partial \tilde{a} F)\tilde{a}||}{||\tilde{a}||}.$$  

We use the Gâteaux derivative

$$(\partial \tilde{a} F)\tilde{a} = \lim_{\delta \to 0} \frac{F(\tilde{a} + \delta \tilde{a}) - F(\tilde{a})}{\delta}$$

so that

$$||(\partial \tilde{a} F)\tilde{a}|| \leq \left| \left| \left[ 1 + O(s^2) \right] \int_0^s \tau \left[ \ln \tau + \gamma - \ln 2 + O(\tau^2 \ln \tau) \right] \tilde{a}(6a_0 + 3\tilde{a}) \tilde{a} d\tau \right| \right|$$

$$+ \left| \left| \left[ \ln \tau + \gamma - \ln 2 + O(s^2 \ln s) \right] \int_0^s \tau \left[ 1 + O(\tau^2) \right] \tilde{a}(6a_0 + 3\tilde{a}) \tilde{a} d\tau \right| \right|$$

$$\leq \left| \left| \left[ 1 + O(s^2) \right] \int_0^s \tau \left[ \ln \tau + \gamma - \ln 2 + O(\tau^2 \ln \tau) \right] d\tau \right| \right| \rho(6a_0 + 3\rho)||\tilde{a}||$$

$$+ \left| \left| \left[ \ln \tau + \gamma - \ln 2 + O(s^2 \ln s) \right] \int_0^s \tau \left[ 1 + O(\tau^2) \right] d\tau \right| \right| \rho(6a_0 + 3\rho)||\tilde{a}||$$

$$\leq \rho(6a_0 + 3\rho)(s_0^2 |\ln \tau s_0| + O(s_0^2)) ||\tilde{a}||$$

where the last line follows from the same calculations as in Section C.2.2. Hence, Hypothesis (ii) reduces to showing

$$\rho(6a_0 + 3\rho)(s_0^2 |\ln \tau s_0| + O(s_0^2)) < 1/2.$$  \hspace{1cm} (2.6)

We fix $\rho_1 = \sup_{s \leq s_*} \rho_*(s)$ where $\rho_*$ and $s_*$ were defined in Section C.2.2 (and make $s_*$ small enough that $\rho_1 < \infty$, if it is not already). We then choose $s_0 \leq s_*$ small enough so that $\rho_1(6a_0 + 3\rho_1)(s_0^2 |\ln \tau s_0| + O(s_0^2)) < 1/2$. Then $F$ is a contraction and
Hypothesis (ii) is satisfied.

C.2.4 Conclusion and refinement of higher order error terms

By combining the results of sections C.2.2 and C.2.3, we can conclude that there exists a unique fixed point \( \tilde{a} = \mathcal{F}(\tilde{a}) \) for \( \rho \) and \( s_0 \) small enough. Moreover, by matching orders in \( s \) we observe that 
\[
\tilde{a} = a_0 + \frac{a_0(1-a_0^2)}{4} s^2 + O(s^4 \ln s).
\]
Then 
\[
a(s) = a_0 + \frac{a_0(1-a_0^2)}{4} s^2 + O(s^4 \ln s).
\]
We comment that the known property \( a(s) \) is monotonically decreasing \([34]\) means that \( a_{ss}(0) < 0 \) so that \( a_0 > 1 \).

C.3 Convergence to the fixed point \( (A_1, z_1) = (0,0) \) in the transition chart coordinates

We now use the asymptotic expansion 
\[
a(s) = a_0 + \frac{a_0(1-a_0^2)}{4} s^2 + O(s^4 \ln s)
\]
to show that 
\[
z_1(\tau) = e^{\tau} a_{s} (e^{\tau}) \to 0 \text{ as } \tau \to -\infty.
\]
We remark that \( A_1(\tau) = e^{\tau} a(e^{\tau}) \to 0 \text{ as } \tau \to -\infty \) is clear. We first find the derivative 
\[
\partial_s a(s) = \frac{a_0(1-a_0^2)}{2} s + O(s^3 \ln s).
\]
Then 
\[
z_1(\tau) = e^{\tau} a_0 (1-a_0^2) e^{\tau} + O(\tau e^{3\tau})
\]
\[
\frac{2a_0 + O(e^{2\tau})}{2a_0 + O(e^{2\tau})}.
\]
It is then clear that \( z_1(\tau) \to 0 \text{ as } \tau \to -\infty \) as desired.
Connecting orbit $q_0$: Large argument asymptotics
We now consider the behavior of \( q_0(s) \) as \( s \to \infty \), where we recall that \( q_0(s) \) is a solution to

\[
a_{ss} + \frac{a_s}{s} = a - a^3, \tag{0.1}
\]

which is related to the study of standard oscillons. As in Appendix C, the resulting fixed point argument is completely analogous for reciprocal oscillons and we omit the details.

The proof will follow a standard fixed point theorem argument on solutions constructed using variation of parameters.

### D.1 Construction of the fundamental matrix solution

Equation (0.1) can be written as a first order system

\[
u_s = \mathcal{A}(s)u + \mathcal{F}(a,b) \tag{1.1}
\]

where

\[
u := \begin{pmatrix} a \\ b \end{pmatrix}, \quad \mathcal{A}(s) := \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{s} \end{pmatrix}, \quad \mathcal{F}(a,b) := \begin{pmatrix} 0 \\ -a^3 \end{pmatrix}. \tag{1.2}
\]
Equation (1.1) can be solved via the variation of parameters formula from Appendix A.2

\[ u(s) = \Phi(s, s_0)u(s_0) + \int_{s_0}^{s} \Phi(s, \tau)\mathcal{F}(a(\tau), b(\tau)) \, d\tau \]  

(1.3)

where \( \Phi(s, \tau) \), described in Appendix A.1 satisfies \( \Phi' = A(s)\Phi \) with \( \Phi(s, s) = I \) and is known as the fundamental matrix solution. In particular,

\[ \Phi \cdot = v_1(\hat{v}_2, \cdot) + v_2(\hat{v}_1, \cdot) \]

where \( v_j \) are the two linearly independent solutions of \( v'_j = \mathcal{A}(s)v_j \) and \( \hat{v}_j \) are the two linearly independent solutions of the adjoint problem

\begin{equation}
\begin{pmatrix}
\hat{a} \\
\hat{b}
\end{pmatrix}_s =
\begin{pmatrix}
0 & -1 \\
-1 & \frac{1}{s}
\end{pmatrix}
\begin{pmatrix}
\hat{a} \\
\hat{b}
\end{pmatrix}.
\end{equation}

(1.4)

In fact, let \( a_j \) be a solution of

\[ \partial_{ss}a_j + \frac{\partial_s a_j}{s} - a_j = 0, \]  

(1.5)

where \( a_j(s) = K_0(s) \) or \( I_0(s) \) since (1.5) is the modified Bessel function. We then observe that \( v_j = (a_j, \partial_s a_j)^T \) and \( \hat{v}_j = (-s\partial_s a_j, sa_j)^T \) (or \( \hat{v}_j = (s\partial_s a_j, -sa_j)^T \)) since

\[ \partial_s(-s\partial_s a_j) = -\partial_s a_j - s\partial_{ss}a_j = -(sa_j) \] and \[ \partial_s(sa_j) = a_j + s\partial_s a_j = -(-s\partial_s a_j) + \frac{1}{s}(sa_j). \]
Then \(\langle v_i, \hat{v}_j \rangle = (1 - \delta_{ij})\) as desired and one can easily check that the fundamental matrix solution is given by

\[
\Phi(s, \tau) = \left( \begin{array}{c} K_0(s) \\ -K_1(s) \end{array} \right) \langle \begin{array}{c} \tau I_1(\tau) \\ -\tau I_0(\tau) \end{array} \rangle \cdot + \left( \begin{array}{c} I_0(s) \\ I_1(s) \end{array} \right) \langle \begin{array}{c} \tau K_1(\tau) \\ \tau K_0(\tau) \end{array} \rangle \cdot, \tag{1.6} \]

since \(\partial_z I_0(z) = I_1(z)\) and \(\partial_z K_0(z) = -K_1(z)\) [1, (9.6.27)]. The asymptotic expansions for the modified Bessel functions is shown in Table D.1.

### D.1.1 Bounds on modified Bessel and exponential functions

In what follows we will need bounds on the modified Bessel and exponential functions. For the modified Bessel functions we use the expansions in Table D.1. Then, for all \(s \in [s_*, \infty)\) with \(s_*\) large there exists constants \(c_2, \tilde{c}_2, c_3\) so that

\[
\tilde{c}_2 \frac{1}{\sqrt{s}} e^{-s} \leq K_0(s) \leq c_2 \frac{1}{\sqrt{s}} e^{-s} \leq I_0(s) \leq c_3 \frac{1}{\sqrt{s}} e^s. \tag{1.7} \]

We will find that the exponential function also plays a role in the fixed point argument below. For \(s\) large enough the exponential function can be expanded [1, (5.1.51)]

\[
E_1(s) = \int_s^\infty \frac{e^{-\tau}}{\tau} d\tau = \frac{e^{-s}}{s} \left( 1 + O \left( \frac{1}{s} \right) \right) \]
so that for all \( s \in [s_*, \infty) \) with \( s_0 \) large there exists a constant \( c_4 \) so that

\[
E_1(s) \leq c_4 \frac{e^{-s}}{s}
\]  

(1.8)

In what follows we assume \( s_0 \geq s_* \) so that the above bounds hold on the interval \( s \in [s_0, \infty) \).

D.2 Fixed point theorem

With the preliminaries in place, we apply Banach’s Fixed Point Theorem, from Appendix A.3, to show that there exists a unique solution to equation (1.3), in appropriately defined function spaces.

D.2.1 Definition of fixed point equation and spaces

As in Appendix C, we consider only the first row of the fixed point equation (1.3).

\[
a(s) = c_1 K_0(s) + c_2 I_0(s) + I_0(s) \int_{s_0}^{s} -\tau K_0(\tau)a(\tau)^3d\tau + K_0(s) \int_{s_0}^{s} \tau I_0(\tau)a(\tau)^3d\tau
\]

(2.1)

Since \( a(s) \) is bounded as \( s \to \infty \) we must have that

\[
c_2 = \int_{\infty}^{s_0} -\tau K_0(\tau)a(\tau)^3d\tau.
\]
Next, we make the ansatz $a(s) = \frac{K_0(s)}{K_0(s_0)} \tilde{a}(s)$, since we are interested in exponentially decaying solutions. The variation of parameters formula (2.1) becomes

$$
\tilde{a}(s) = c_1 + \frac{1}{K_0(s_0)^2} \left[ \frac{I_0(s)}{K_0(s)} \int_{s_0}^{s} -\tau K_0^4(\tau) \tilde{a}(\tau)^3 d\tau + \int_{s_0}^{s} \tau I_0(\tau) K_0^3(\tau) \tilde{a}(\tau)^3 d\tau \right].
$$

(2.2)

We will use the space $X = \left\{ \tilde{a}(s) \in C^0([s_0, \infty)) : ||\tilde{a}(s)|| < \infty \right\}$ equipped with norm $||a|| := \sup_{s \geq s_0} |a(s)|$. Then define $D \subset X$, $D := \{a(s) \in X : ||a(s)|| \leq \rho_1\}$ and $B := \{c_1 \in \mathbb{R} : |c_1| \leq \rho_2\}$.

Based on the bounds from Section D.1.1 we observe that

$$
\left| \int_{s_0}^{\infty} \tau I_0(\tau) K_0^3(\tau) \tilde{a}(\tau)^3 d\tau \right| < \rho_1^3 c_2^3 \left| \int_{s_0}^{\infty} \frac{e^{-2\tau}}{\tau} d\tau \right| < \infty
$$

so that we can redefine $c_1 \to c_1 + \frac{1}{K_0(s_0)^2} \int_{s_0}^{\infty} -\tau I_0(\tau) K_0^3(\tau) \tilde{a}(\tau)^3 d\tau < \infty$. Then (2.2) becomes

$$
\tilde{a}(s) = c_1 + \frac{1}{K_0(s_0)^2} \left[ \frac{I_0(s)}{K_0(s)} \int_{s_0}^{s} -\tau K_0^4(\tau) \tilde{a}(\tau)^3 d\tau + \int_{s_0}^{s} \tau I_0(\tau) K_0^3(\tau) \tilde{a}(\tau)^3 d\tau \right]
=: \mathcal{F}(\tilde{a}(s); c_1)
$$

(2.3)

In the following we will show that equation (2.2) satisfies the hypotheses of Banach’s Fixed Point Theorem, from Appendix A.3:

(i) $\mathcal{F} : D \times B \to D$

(ii) $\exists \ l < 1 : ||\mathcal{F}(\phi; c_1) - \mathcal{F}(\psi; c_1)|| \leq l ||\phi - \psi|| \ \forall \ \phi, \psi \in D, \ c_1 \in B$
and conclude that there exists a unique fixed point.

D.2.2 Hypothesis (i): Mapping into itself

Let $\tilde{a}(s) \in \mathcal{D}$ and $c_1 \in \mathcal{B}$. It is straightforward to observe that $\mathcal{F}(\tilde{a}(s); c_1) \in \mathcal{C}^0([s_0, \infty))$. It remains to show $\|\mathcal{F}(\tilde{a}(s); c_1)\| \leq \rho_1$.

$$\|\mathcal{F}(\tilde{a}(s); c_1)(s)\| = \sup_{s \geq s_0} |\mathcal{F}(\tilde{a}(s), c_1)|$$

$$= \sup_{s \geq s_0} \left| c_1 + \frac{1}{K_0(s_0)^2} \frac{I_0(s)}{K_0(s)} \int_{\infty}^{s} -\tau K_0(\tau)^4 \tilde{a}(\tau)^3 d\tau \right|$$

$$+ \frac{1}{K_0(s_0)^2} \int_{\infty}^{s} -\tau I_0(\tau) K_0(\tau)^3 \tilde{a}(\tau)^3 d\tau$$

$$\leq \sup_{s \geq s_0} |c_1| + \sup_{s \geq s_0} \frac{1}{K_0(s_0)^2} \left| \frac{I_0(s)}{K_0(s)} \int_{\infty}^{s} -\tau K_0(\tau)^4 \tilde{a}(\tau)^3 d\tau \right|$$

$$+ \sup_{s \geq s_0} \frac{1}{K_0(s_0)^2} \left| \int_{\infty}^{s} \tau I_0(\tau) K_0(\tau)^3 \tilde{a}(\tau)^3 d\tau \right|$$

$$= : (I) + (II) + (III)$$

Term (I) is trivially less than $\rho_2$. We compute integrals (II) and (III) individually, using the bounds (1.7), (1.8) and the pointwise bound $|\tilde{a}(\tau)| \leq \rho_1$ for all $\tau \geq s_0$ since
\( \tilde{a}(\tau) \in D. \)

\[
\begin{align*}
(II) \quad & \sup_{s \geq s_0} \frac{1}{K_0(s_0)^2} \int_0^\infty \tau K_0(\tau)^4 \tilde{a}(\tau)^3 d\tau \\
& \leq \sup_{s \geq s_0} \frac{c_3}{c_2} e^{2s} \left| \int_s^\infty \frac{c_4 e^{-4\tau}}{\tau} \rho_1^3 d\tau \right| \\
& = \sup_{s \geq s_0} \frac{4c^2 c_3 \rho_1^3}{K_0(s_0)^2} e^{2s} \left| \int_s^\infty e^{-4\tau} d\tau \right| \\
& = \sup_{s \geq s_0} \frac{4c^2 c_3 \rho_1^3}{K_0(s_0)^2} e^{2s} \left| E_1(4\tau) \right|_s^{\infty} \\
& = \frac{c^4 c_3 c_4 \rho_1^3}{c_2^2}
\end{align*}
\]

\[
\begin{align*}
(III) \quad & \sup_{s \geq s_0} \frac{1}{K_0(s_0)^2} \left| \int_\infty^s \tau I_0(\tau) K_0(\tau)^3 \tilde{a}(\tau)^3 d\tau \right| \\
& \leq \sup_{s \geq s_0} \left| \int_\infty^s c_3 c_2 \rho_1^3 e^{-2\tau} \rho_1^3 d\tau \right| \\
& = \sup_{s \geq s_0} \frac{2c^2 c_3 \rho_1^3}{K_0(s_0)^2} \left| \int_\infty^s e^{-2\tau} d\tau \right| \\
& = \sup_{s \geq s_0} \frac{2c^2 c_3 \rho_1^3}{K_0(s_0)^2} E_1(2\tau) \bigg|_s^{\infty} \\
& \leq \frac{c^2 c_3 c_4 \rho_1^3}{c_2^2}
\end{align*}
\]

We substitute these estimates back into (2.5) and find that Hypothesis (i) reduces to showing

\[
\rho_2 + \frac{c^4 c_3 c_4 \rho_1^3}{c_2^2} + \frac{c^2 c_3 c_4 \rho_1^3}{c_2^2} \leq \rho_1. \tag{2.6}
\]
This is possible by fixing $\rho_1 \leq \rho_1^*$ where

$$(\rho_1^*)^2 \frac{c_2^3 c_3 c_4 (c_2 + \tilde{c}_2)}{c_2^3} = \frac{1}{2}$$

and then $\rho_2 \leq \rho_2^*$ with $\rho_2^* = \rho_1/2$.

**D.2.3 Hypothesis (ii): Contraction**

The space $\mathcal{D}$ is convex so the Lipschitz constant is equal to the norm of the derivative

$$||\partial_{\theta} \mathcal{F}|| = \sup_{\|\theta\|} \frac{||\partial_{\theta} \mathcal{F}||}\|\theta\|.$$  

We use the Gâteaux derivative

$$(\partial_{\theta} \mathcal{F})\theta = \lim_{\delta \to 0} \frac{\mathcal{F}(\tilde{\theta} + \delta \theta) - \mathcal{F}(\tilde{\theta})}{\delta}$$

so that

$$||\partial_{\theta} \mathcal{F}|| = \frac{1}{K_0(s_0)^2} \left| \frac{I_0(s)}{K_0(s)} \int_\infty^s -3\tau K_0(\tau) \tilde{\theta}^2 \tilde{a} d\tau \right|$$

$$+ \left( \int_\infty^s -3\tau I_0(\tau) K_0(\tau) \tilde{\theta}^2 \tilde{a} d\tau \right) ||\tilde{a}||$$

$$\leq \frac{1}{K_0(s_0)^2} \left| \frac{I_0(s)}{K_0(s)} \int_\infty^s -3\tau K_0(\tau) \tilde{\theta}^2 \tilde{a} d\tau \right|$$

$$+ \left( \int_\infty^s -3\tau I_0(\tau) K_0(\tau) \tilde{\theta}^2 \tilde{a} d\tau \right) ||\tilde{a}||$$

$$\leq \rho_1^2 \frac{c_2^3 c_3 c_4 (c_2 + \tilde{c}_2)}{c_2^3} ||\tilde{a}||$$
where the last line follows from the same calculations as in Section D.2.2. Hence, Hypothesis (ii) reduces to choosing $\rho_1 \leq \rho_1^*$ with

$$(\rho_1^*)^2 \frac{3c_2^3c_3c_4(c_2 + \tilde{c}_2)}{c_2^3} = 1/2.$$  \hspace{1cm} (2.7)

By choosing $\rho_1 = \rho_1^* := \min(\rho_1^*, \rho_1^*)$ and $\rho_2 = \rho_1^*/2$ both Hypotheses (i) and (ii) are simultaneously satisfied.

### D.2.4 Conclusion and refinement of higher order error terms

By combining the results of sections D.2.2 and D.2.3, we can conclude that there exists a unique fixed point $\tilde{a} = \mathcal{F}(\tilde{a}; c_1)$ for $\rho_1$ and $\rho_2$ small enough. Moreover, we let $\tilde{a}(s) = c_1 + o(1)$ and substitute back into (2.3) to refine the higher order terms:

$$\tilde{a}(s) = c_1 + \frac{1}{K_0(s_0)^2} \left[ \frac{I_0(s)}{K_0(s)} \int_\infty^s -\tau K_0^2(\tau)O(c_1^3) d\tau + \int_\infty^s \tau I_0(\tau)K_0^3(\tau)O(c_1^3) d\tau \right]$$

$$\leq c_1 + \frac{1}{K_0(s_0)^2} \left[ \frac{c_2^4c_3}{c_2} O(c_1^3)e^{2s} \int_s^\infty \frac{e^{-4\tau}}{\tau} d\tau + c_2^3c_3O(c_1^3) \int_s^\infty \frac{e^{-2\tau}}{\tau} d\tau \right]$$

$$\leq c_1 + \frac{1}{K_0(s_0)^2} \left[ \frac{c_2^4c_3c_4}{c_2^2s^2} O(c_1^3)e^{-2s} \int_s^\infty e^{-4\tau} d\tau + \frac{c_3^2c_3}{s} O(c_1^3)e^{-2s} \right]$$

$$= c_1 + \frac{1}{K_0(s_0)^2} c_1^3O(e^{-2s})$$

We transform back into the original coordinates $a(s) = \frac{K_0(s)}{K_0(s_0)} \tilde{a}(s)$ and let $\frac{c_1}{K_0(s_0)} \to c_1$ so that $a(s) = K_0(c_1 + O(e^{-2s}c_1^3)).$
D.3 Convergence of $q_0(s)$ to the fixed point

$$(A_2, z_2) = (0, -1)$$ in the rescaling chart coordinates.

We now use the asymptotic expansion $a(s) = K_0(c_1 + O(e^{-2s}c_1^3))$ to show that $z_2(s) = \frac{a_2}{a}(s) \to -1$ as $s \to \infty$. We remark that $A_2(s) = a(s) \to 0$ as $s \to \infty$ is clear. We first find the derivative $\partial_s a(s) = -K_1(s) (c_1 + O(e^{-2s}c_1^3)) + K_0(s)O(e^{-2s}c_1^3)$. Then

$$z_2(s) = \frac{-K_1(s) (c_1 + O(e^{-2s}c_1^3)) + K_0(s)O(e^{-2s}c_1^3)}{K_0(s) (c_1 + O(e^{-2s}c_1^3))}.$$ 

We use the asymptotic expansions from Table D.1 to find

$$\lim_{s \to \infty} z_2(s) = \lim_{s \to \infty} -\frac{\sqrt{\frac{\pi}{2s}}e^{-s} \left(1 + O\left(\frac{1}{s}\right)\right) (c_1 + O(e^{-2s}c_1^3)) + \sqrt{\frac{\pi}{2s}}e^{-s} \left(1 + O\left(\frac{1}{s}\right)\right) O(e^{-2s}c_1^3)}{\sqrt{\frac{\pi}{2s}}e^{-s} \left(1 + O\left(\frac{1}{s}\right)\right) (c_1 + O(e^{-2s}c_1^3))} \cdot \frac{\sqrt{\frac{\pi}{2s}}e^{-s} \left(1 + O\left(\frac{1}{s}\right)\right) (c_1 + O(e^{-2s}c_1^3))}{\sqrt{\frac{\pi}{2s}}e^{-s} \left(1 + O\left(\frac{1}{s}\right)\right) (c_1 + O(e^{-2s}c_1^3))} \cdot \frac{\sqrt{\frac{\pi}{2s}}e^{-s} \left(1 + O\left(\frac{1}{s}\right)\right) (c_1 + O(e^{-2s}c_1^3))}{\sqrt{\frac{\pi}{2s}}e^{-s} \left(1 + O\left(\frac{1}{s}\right)\right) (c_1 + O(e^{-2s}c_1^3))} = -1.$$
Appendix E

Fixed point argument: Far-field solution at $r = r_0$
In this Section we provide the fixed point argument necessary to complete the proof of Chapter 3, Lemma 3.4.6. We remark that a completely analogous argument holds for reciprocal oscillons and we omit the details. We show that there exists a unique fixed point to

\[ \hat{A}_1(\tau) = [\delta_0 q_0(\delta_0) + \epsilon O(\hat{\eta} + 1)] e^{O(1/\tau^2 + \delta_0^2)} + \int_0^\tau O(\epsilon \hat{z}_1 + e^{2\sigma} \hat{A}_1^2) d\sigma \]

\[ \hat{z}_1(\tau) = -\hat{\eta} + \epsilon \int_0^\tau \left[ -\hat{z}_1^2 + O \left( \hat{A}_1^2 / \delta_0^2 + 1 \right) \right] d\sigma, \tag{0.1} \]

with \( \tau \in [\tau_*, 0] \), \( \tau_* = \ln \frac{\epsilon \rho_0}{\delta_0} \), in appropriately defined spaces. We define the following notation: Write equation (0.1) as

\[ \begin{pmatrix} \hat{A}_1 \\ \hat{z}_1 \end{pmatrix} = \begin{pmatrix} \mathcal{F}_A(\hat{A}_1, \hat{z}_1; \hat{\eta}) \\ \mathcal{F}_z(\hat{A}_1, \hat{z}_1; \hat{\eta}) \end{pmatrix} =: \mathcal{F}(\hat{A}_1, \hat{z}_1; \hat{\eta}) \tag{0.2} \]

with

\[ \mathcal{F}_A(\hat{A}_1, \hat{z}_1; \hat{\eta}) := [\delta_0 q_0(\delta_0) + \epsilon O(\hat{\eta} + 1)] e^{O(1/\tau^2 + \delta_0^2)} + \int_0^\tau O(\epsilon \hat{z}_1 + e^{2\sigma} \hat{A}_1^2) d\sigma \]

\[ \mathcal{F}_z(\hat{A}_1, \hat{z}_1; \hat{\eta}) := -\hat{\eta} + \epsilon \int_0^\tau \left[ -\hat{z}_1^2 + O \left( \hat{A}_1^2 / \delta_0^2 + 1 \right) \right] d\sigma \tag{0.3} \]

### E.1 Definition of the spaces

We will use the spaces \( \mathcal{X} = \{ \phi(\tau) \in C^0([\tau_*, 0]) : \|\phi(\tau)\|_\mathcal{X} < \infty \} \) equipped with norm \( \|\phi(\tau)\|_\mathcal{X} := \sup_{\tau_* \leq \tau \leq 0} |\phi(\tau)| \). Then define \( \mathcal{D} \subset \mathcal{X} \times \mathcal{X} \), \( \mathcal{D} := \{ \phi \in \mathcal{X} \times \mathcal{X} : \|\phi\|_\mathcal{D} \leq \rho_1 \} \), equipped with norm \( \| (\phi_1, \phi_2) \|_\mathcal{D} = \sup(\|\phi_1\|_\mathcal{X}, \|\phi_2\|_\mathcal{X}) \). We also define \( \mathcal{B} := \)}
\{ \hat{\eta} \in \mathbb{R} : |\hat{\eta}| \leq \rho_2 \}. In what follows we will use bound

\[
\left| \sup_{\tau^* \leq \tau \leq 0} \int_{0}^{\tau} e^{\sigma} d\sigma \right| \leq 1
\]

since \( \tau^* < 0 \). We recall from the statement of Chapter 3, Lemma 3.4.6, that \( 1/r_0 < \rho_1 \) is fixed, where \( \rho_1 \) was determined in the proof of Chapter 3, Proposition 3.3.2. Furthermore, our goal is to show that, for each fixed \( \delta_0 > 0 \) and \( \rho_2 > 0 \) small enough, there is an \( \epsilon_0 > 0 \) so that there exists a unique fixed point to equation (0.1) for all \( \hat{\eta} \in [0, \rho_2] \) and all \( \epsilon \leq \epsilon_0 \). This informs the order in which we choose the constants in the fixed point argument below.

In the following we will show that equation (0.1) satisfies the hypotheses of Banach’s Fixed Point Theorem from Appendix A.3:

(i) \( F : \mathcal{D} \times \mathcal{B} \to \mathcal{D} \)

(ii) \( \exists l < 1 : ||F(\phi; \hat{\eta}) - F(\psi; \hat{\eta})|| \leq l ||\phi - \psi|| \quad \forall \phi, \psi \in \mathcal{D}, \hat{\eta} \in \mathcal{B} \)

and conclude that there exists a unique fixed point.

\textbf{E.2 Hypothesis (i): Mapping into itself}

Let \( (\hat{A}_1, \hat{z}_1) \in \mathcal{D} \) and \( \hat{\eta} \in \mathcal{B} \). It is straightforward to observe that \( F(\hat{A}_1, \hat{z}_1; \hat{\eta}) \in \mathcal{C}^0([\tau_*, 0]) \times \mathcal{C}^0([\tau_*, 0]) \). It remains to show \( ||F_A(\hat{A}_1, \hat{z}_1; \hat{\eta})|| \leq \rho_1 \) and \( ||F_z(\hat{A}_1, \hat{z}_1; \hat{\eta})|| \leq \rho_2 \).
\[ ||\mathcal{F}_A(\hat{A}_1, \hat{z}_1, \hat{\eta})|| = |\delta_0 q_0(\delta_0) + \epsilon O(\hat{\eta} + 1)| \sup_{\tau^* \leq \tau \leq 0} e^{O(1/r_0^2 + \delta_0^2)} + \int_{0}^{\tau} O(\epsilon \hat{z}_1 + e^{2\sigma} \hat{A}_1^2) d\sigma \]

\[ \leq |\delta_0 q_0(\delta_0) + \epsilon O(\hat{\eta} + 1)| \sup_{\tau^* \leq \tau \leq 0} \left[ 1 + O(1/r_0^2 + \delta_0^2) + \int_{0}^{\tau} O(\epsilon \hat{z}_1 + e^{2\sigma} \hat{A}_1^2) d\sigma \right] \]

\[ \leq |\delta_0 q_0(\delta_0) + \epsilon O(\hat{\eta} + 1)| \left[ 1 + O(1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon ||\hat{z}_1||_z + ||\hat{A}_1||_A^2) \right] \]

\[ \leq |\delta_0 q_0(\delta_0) + \epsilon O(\rho_2 + 1)| \left[ 1 + O(1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon \rho_1 + \rho_1^2) \right] \]

and

\[ ||\mathcal{F}_z(\hat{A}_1, \hat{z}_1, \hat{\eta})|| \]

\[ \leq |\hat{\eta}| + \sup_{\tau^* \leq \tau \leq 0} \epsilon \int_{0}^{\tau} ||\hat{z}_1||_z^2 + \epsilon \ln \epsilon \cdot \left[ ||\hat{z}_1||_z^2 + O\left(||\hat{A}_1||_A^2 / \delta_0^2 + 1\right) \right] d\sigma \]

\[ \leq |\hat{\eta}| + \epsilon \ln \epsilon \cdot \left[ ||\hat{z}_1||_z^2 + O\left(||\hat{A}_1||_A^2 / \delta_0^2 + 1\right) \right] \]

\[ \leq \rho_2 + \epsilon \ln \epsilon \cdot \left[ \rho_1^2 + O\left(\rho_1^2 / \delta_0^2 + 1\right) \right] \].

Hypothesis (i) thus reduces to showing

\[ |\delta_0 q_0(\delta_0) + \epsilon O(\rho_2 + 1)| \left[ 1 + O(1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon \rho_1 + \rho_1^2) \right] \leq \rho_1 \quad (2.1) \]

and

\[ \rho_2 + \epsilon \ln \epsilon \cdot \left[ \rho_1^2 + O\left(\rho_1^2 / \delta_0^2 + 1\right) \right] \leq \rho_1. \quad (2.2) \]
We begin with $\epsilon = 0$. For the first inequality we use the implicit function theorem. Consider

$$0 = |\delta_0 q_0(\delta_0)| \left[ 1 + O \left( \frac{1}{\delta_0^2} + \delta_0^2 + \rho_1^2 \right) \right] - \rho_1/2$$

$$:= f(\delta_0; \rho_1).$$

Function $f$ satisfies the hypothesis of the Implicit Function Theorem from Appendix A.4: $f(0; 0) = 0$ and $(\partial_{\rho_1} f)(0; 0) = -1/2 \neq 0$. Thus for each $\delta_0 \leq \delta_*$ small enough there exists a unique $\rho_1(\delta_0)$ such that $f(\delta_0; \rho_1(\delta_0)) = 0$ and inequality (2.1) is satisfied. Fix $\rho_1 = \rho_1^* = \sup \{ \rho_1(\delta_0) : f(\delta_0; \rho_1(\delta_0)) = 0 \}$. Then (2.1) holds for all $\delta_0 \leq \delta_*$. The second inequality (2.2) holds by choosing $\rho_2 \leq \rho_1^*/2$. By continuity, the inequalities (2.1) and (2.2) hold for all $\epsilon \leq \epsilon_0$ with $\epsilon_0$ small enough.

### E.3 Hypothesis (ii): Contraction

The space $\mathcal{D}$ is convex so the Lipschitz constant is equal to the norm of the Jacobian

$$\|DF\|_\mathcal{D} = \sup_{\phi \in \mathcal{D}} \frac{\|DF\phi\|_\mathcal{D}}{\|\phi\|_\mathcal{D}},$$

where

$$DF = \begin{pmatrix} \partial_{\hat{A}_1} F_A & \partial_{\hat{z}_1} F_A \\ \partial_{\hat{A}_1} F_z & \partial_{\hat{z}_1} F_z \end{pmatrix}.$$  

We use the Gâteaux derivative

$$(\partial_{\hat{x}} F_y)_{\hat{x}} = \lim_{\delta \to 0} \frac{F_y(\hat{x} + \delta \hat{x}) - F_y(\hat{x})}{\delta}.$$
We note first that

\[ ||D\mathcal{F}(\phi_1, \phi_2)^T||_D = ||\partial_{\hat{A}_1} \mathcal{F}_A \phi_1 + \partial_{\hat{z}_1} \mathcal{F}_A \phi_2||_X + ||\partial_{\hat{A}_1} \mathcal{F}_z \phi_1 + \partial_{\hat{z}_1} \mathcal{F}_z \phi_2||_X \]

\[ \leq ||\partial_{\hat{A}_1} \mathcal{F}_A \phi_1||_X + ||\partial_{\hat{z}_1} \mathcal{F}_A \phi_2||_X + ||\partial_{\hat{A}_1} \mathcal{F}_z \phi_1||_X + ||\partial_{\hat{z}_1} \mathcal{F}_z \phi_2||_X. \]

Therefore,

\[ ||D\mathcal{F}||_D \leq \sup_{(\phi_1, \phi_2) \in \mathcal{D}} \frac{(||\partial_{\hat{A}_1} \mathcal{F}_A \phi_1||_X + ||\partial_{\hat{A}_1} \mathcal{F}_z \phi_1||_X) + (||\partial_{\hat{z}_1} \mathcal{F}_A \phi_2||_X + ||\partial_{\hat{z}_1} \mathcal{F}_z \phi_2||_X)}{||\phi_1||_X + ||\phi_2||_X}. \]

We now compute the norms in the numerator. We first comment that the higher order terms \(O(\ldots)\) in equations (0.3) may depend on \(\hat{A}_1\) and \(\hat{z}_1\).

\[ ||\partial_{\hat{A}_1} \mathcal{F}_A \hat{A}_1||_X \]

\[ = \left| \left| \left[ \delta_0 q_0(\delta_0) + \epsilon \mathcal{O}(\hat{n} + 1) \right] e^{O(1/r_0^2 + \delta_0^2) + \int_0^\tau \mathcal{O}(\epsilon \hat{z}_1 + e^{2\sigma} \hat{A}_1^2) \, d\sigma} \right| \right|_X \]

\[ \times \left[ O(1/r_0^2 + \delta_0^2) \hat{A}_1 + \int_0^\tau O \left( \epsilon \hat{z}_1 + e^{2\sigma} \hat{A}_1 \right) \hat{A}_1 \, d\sigma \right] \]

\[ = |\delta_0 q_0(\delta_0) + \epsilon \mathcal{O}(\hat{n} + 1)| \sup_{\tau' \leq \tau \leq 0} \left[ 1 + O(1/r_0^2 + \delta_0^2) + \int_0^\tau O \left( \epsilon \hat{z}_1 + e^{2\sigma} \hat{A}_1 \right) \, d\sigma \right] \]

\[ \times \left[ O(1/r_0^2 + \delta_0^2) \hat{A}_1 + \int_0^\tau O \left( \epsilon \hat{z}_1 + e^{2\sigma} \hat{A}_1 \right) \hat{A}_1 \, d\sigma \right] \]

\[ \leq |\delta_0 q_0(\delta_0) + \epsilon \mathcal{O}(\hat{n} + 1)| \left[ 1 + O \left( 1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon \cdot ||\hat{z}_1||_X + ||\hat{A}_1||_X^2 \right) \right] \]

\[ \times \left[ O \left( 1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon \cdot ||\hat{z}_1||_X + ||\hat{A}_1||_X \right) \right] \left| \mathcal{A} \right|_X \]

\[ \leq |\delta_0 q_0(\delta_0) + \epsilon \mathcal{O}(\rho_2 + 1)| \left[ 1 + O \left( 1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon \cdot \rho_1 + \rho_1^2 \right) \right] \]

\[ \times O \left( 1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon \cdot \rho_1 + \rho_1 \right) \left| \mathcal{A} \right|_X \]

\[ =: C_{AA} ||\mathcal{A}||_X \tag{3.1} \]
\[ \| \partial_{\bar{z}_1} F_A \bar{z}_1 \| x \]
\[ = \left\| \left[ \delta_0 q_0 (\delta_0) + \epsilon O(\hat{\eta} + 1) \right] e^{O(1/r_0^2 + \delta_0^2)} \! + \! f_0^\epsilon O(\epsilon \hat{z}_1 + e^{2\sigma} \hat{A}_1^2) \right\|_{\chi} \]
\[ \times \left[ O(1/r_0^2 + \delta_0^2) \bar{z}_1 + \int_0^\tau O (\epsilon + e^{2\sigma} A_1^2) \bar{z}_1 d\sigma \right] \]
\[ = |\delta_0 q_0 (\delta_0) + \epsilon O(\hat{\eta} + 1)| \sup_{\tau^* \leq \tau \leq 0} \left[ 1 + O(1/r_0^2 + \delta_0^2) + \int_0^\tau O \left( \epsilon \hat{z}_1 + e^{2\sigma} \hat{A}_1^2 \right) d\sigma \right] \]
\[ \times \left[ O(1/r_0^2 + \delta_0^2) \bar{z}_1 + \int_0^\tau O (\epsilon + e^{2\sigma} A_1^2) \bar{z}_1 d\sigma \right] \]
\[ \leq |\delta_0 q_0 (\delta_0) + \epsilon O(\hat{\eta} + 1)| \left[ 1 + O \left( 1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon \cdot ||\bar{z}_1||_x + ||A_1||_x^2 \right) \right] \]
\[ \times O \left( 1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon + ||A_1||_x^2 \right) ||\bar{z}_1||_x \]
\[ \leq |\delta_0 q_0 (\delta_0) + \epsilon O(\hat{\eta} + 1)| \left[ 1 + O \left( 1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon \cdot \rho_1 + \rho_1^2 \right) \right] \]
\[ \times O \left( 1/r_0^2 + \delta_0^2 + \epsilon \ln \epsilon + \rho_1^2 \right) ||\bar{z}_1||_x \]
\[ =: C_A ||\bar{z}_1||_x \] \hspace{1cm} (3.2)

\[ \| \partial_{\bar{A}_1} F_{\bar{z}} A_1 \| x = \left\| \epsilon \int_0^\tau O \left( \hat{A}_1/\delta_0^2 + 1 \right) A_1 d\sigma \right\|_{\chi} \]
\[ \leq \sup_{\tau^* \leq \tau \leq 0} \epsilon \int_0^\tau \left| O \left( \hat{A}_1/\delta_0^2 + 1 \right) \right| ||A_1||_x d\sigma \]
\[ \leq \epsilon \ln \epsilon \cdot O \left( ||\hat{A}_1||_x/\delta_0^2 + 1 \right) ||A_1||_x \]
\[ \leq \epsilon \ln \epsilon \cdot O \left( \rho_1/\delta_0^2 + 1 \right) ||A_1||_x \]
\[ =: C_{zA} ||A_1||_x \] \hspace{1cm} (3.3)
Substituting these into $||D\mathcal{F}||_D$, we have that Hypothesis (ii) is satisfied if

$$\sup_{(x_1, x_2) \in \mathcal{D}} \frac{(C_A + C_{A^2}) \cdot ||A_1||_x + (C_{A^1} + C_{A^2}) \cdot ||z_1||_x}{||A_1||_x + ||z_1||_x} < 1/2.$$

Hence, we need to show that each $C_{ij} < 1/4$. We begin with $\epsilon = 0$. Then $C_{zA} = C_{zz} = 0$ are trivially less than 1/8. The other two conditions are

$$|\delta_0 q_0(\delta_0) [1 + O(1/\delta_0^2 + 2 \rho_1^2)] O(1/\delta_0^2 + \rho_1^2) + 1/8$$

and

$$|\delta_0 q_0(\delta_0) [1 + O(1/\delta_0^2 + 2 \rho_1^2)] O(1/\delta_0^2 + \rho_1^2) + 1/8.$$

We fix $\rho_{i*}$ as given in the previous section. Then the inequalities are true by taking $\delta_0 < \delta$, smaller, if necessary. By continuity, each coefficient is less than 1/4 for $\epsilon < \epsilon_0$, small enough.
E.4 Conclusion and estimates on the solution

The contraction mapping principle applies and there exists a unique fixed point to (0.1). By uniqueness, the fixed point can be written

$$\hat{A}_1(\tau) = \left[\delta_0 q_0(\delta_0) + \epsilon O(|\hat{\eta}| + 1)\right] e^{O(1/r_0^2 + \delta_0^2 + \epsilon \hat{\eta} \tau + \epsilon^2 \tau^2)}$$

$$= \delta_0 q_0(\delta_0) + O[\epsilon(|\hat{\eta}| + 1) + \delta_0(1/r_0^2 + \delta_0^2 + \epsilon |\hat{\eta}| \tau + \epsilon^2 \tau^2)]$$

$$\hat{z}_1(\tau) = -\hat{\eta} + \epsilon \tau O(|\hat{\eta}|^2 + 1).$$
Bibliography


