Amy McCarthy MA341 Number Theory at BU

CONTINUED FRACTIONS

Introduction

There are many ways to write a number, for example, as a fraction, decimal, logarithm, powers, percentages, and even in a word format, with each representation being more convenient for one purpose or another. Until reaching sophomore year in college, *continued fractions,* which seems to be one of the most revealing representations of numbers, has seem to be omitted from my personal mathematics education. Numbers whose decimal expansions look unremarkable and featureless are revealed to have extraordinary symmetries and patterns embedded which seem to unfold when represented in a continued fraction. Continued fractions also provide us with a way of constructing rational approximations to irrational numbers and discovering the most irrational numbers, which can become very useful to a mathematician or math student.

Continued fractions first appeared in the works of the Indian mathematician Aryabhata in the 6th century in which he used them in order to solve linear equations. They re-emerged in Europe in the 15th and 16th centuries and Fibonacci attempted to define them in a general way. John Wallis was the first to use the term "continued fraction" in 1653 in the book *Arithmetica Infinitorum*. Their properties were also much studied by one of Wallis's English contemporaries, William Brouncker, who along with Wallis, was one of the founders of the Royal Society. At about the same time, the famous Dutch mathematical physicist, Christiaan Huygens made practical use of continued fractions in building scientific instruments, and later, in the eighteenth and early nineteenth centuries, Gauss and Euler explored many of their deep properties.

What is a continued fraction?

A continued fractions refers to all expressions of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

where a_0 is some integer (set of numbers including the natural numbers) and all the other numbers *an* are positive integers. The number of terms can either be finite or infinite. Simple continued fractions can also be written in a compact abbreviated notation as $x = [a_0, a_1, a_2, a_3, \dots]$.

While continued fractions are not the only possible representation of real numbers in terms of a sequence of integers, they are a very common such representation that arises most frequently in number theory.

Some care is needed, since some authors begin indexing the terms at a_1 instead of a_0, causing the parity of certain fundamental results in continued fraction theory to be reversed. The first n terms of the simple continued fraction of a number x can be computed in *Mathematica* using the command ContinuedFraction[*x*, *n*]. Continued fractions with closed forms are given in the following table (Euler 1775).

continued fraction	value	approximate	Sloane

$0+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\dots}}}}$	$\frac{I_1(2)}{I_0(2)}$	0.697774...	A052119
$1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9+\dots}}}}$	$(\sqrt{e}-1)^{-1}$	1.541494...	A113011

Starting the indexing of a continued fraction with a_0,

$$a_0 = \lfloor x \rfloor$$

is the integer part of x, where $\lfloor x \rfloor$ is the floor function,

$$a_1 = \left\lfloor \frac{1}{x - a_0} \right\rfloor$$

is the integral part of the reciprocal of $x - a_0$,

$$a_2 = \left\lfloor \frac{1}{\frac{1}{x-a_0} - a_1} \right\rfloor$$

is the integral part of the reciprocal of the remainder, etc. Writing the remainders according to the recurrence relation

$$\begin{aligned} r_0 &= x \\ r_n &= \frac{1}{r_{n-1} - a_{n-1}} \end{aligned}$$

gives the concise formula

$$a_n = \lfloor r_n \rfloor.$$

The quantities a_n are called partial quotients, and the quantity obtained by including n terms of the continued fraction is called the nth convergent.

$$c_n = \frac{p_n}{q_n} = [a_0, a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_n}}}}$$

Periodic Continued Fraction

A periodic continued fraction is a continued fraction whose terms eventually repeat from some point onwards. The minimal number of repeating terms is called the period of the continued fraction. All nontrivial periodic continued fractions represent irrational numbers. In general, an infinite simple fraction (periodic or otherwise) represents a unique irrational number, and each irrational number has a unique infinite continued fraction.

The square root of a square free integer has a periodic continued fraction of the form

$$\sqrt{n} = [a_0, \overline{a_1, a_2, a_3, \ldots, a_2, a_1, 2a_0}]$$

where the repeating portion (excluding the last term) is symmetric upon reversal, and the central term may appear either once or twice.

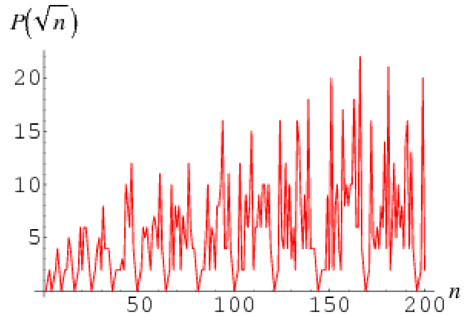
If D is not a square number, then the terms of the continued fraction of $\sqrt{D}$ satisfy

$$0 < a_n < 2\sqrt{D}.$$

An even stronger result is that a continued fraction is periodic if and only if it is a root of a quadratic polynomial. Calling the portion of a number x remaining after a given convergent the "tail," it must be true that the relationship between the number x and terms in its tail is of the form

$$x = \frac{a\,x + b}{c\,x + d},$$

which can only lead to a quadratic equation.



General identities for periodic continued fractions include

$$[\overline{a}] = \frac{a+\sqrt{a^2+4}}{2}$$

$$[1, \overline{a}] = \frac{2-a+\sqrt{a^2+4}}{2}$$

$$[a, \overline{2a}] = \sqrt{a^2+1}$$

$$[\overline{a, b}] = \frac{-a b+\sqrt{a b (a b+4)}}{2a}$$

$$[\overline{a_1, \ldots, a_n}] = \frac{-(q_{n-1}-p_n)+\sqrt{(q_{n-1}-p_n)^2+4 q_n p_{n-1}}}{2 q_n}$$

$$[a_0, \overline{b_1, \ldots, b_n}] = a_0 + \frac{1}{[\overline{b_1, \ldots, b_n}]}$$

$$[\overline{b_1, \ldots, b_n}] = \frac{[\overline{b_1, \ldots, b_n}]\, p_n + p_{n-1}}{[\overline{b_1, \ldots, b_n}]\, q_n + q_{n-1}}.$$

The first follows from

$$\alpha = a + \frac{1}{a + \frac{1}{a+\frac{1}{a+\ldots}}}$$

$$= a + \frac{1}{a + \left(\frac{1}{a+\frac{1}{a+\ldots}}\right)}.$$

Therefore,

$$\alpha - a = \frac{1}{a + \frac{1}{a+\frac{1}{a+\ldots}}},$$

so plugging in gives

$$\alpha = a + \frac{1}{a + (\alpha - a)} = a + \frac{1}{\alpha}.$$

Expanding

$$\alpha^2 - a\alpha - 1 = 0,$$

and solving using the quadratic formula gives

$$\alpha = \frac{a + \sqrt{a^2 + 4}}{2}.$$

The analog of this treatment in the general case gives

$$\alpha = \frac{\alpha\, p_n + p_{n-1}}{\alpha\, q_n + q_{n-1}}.$$

The Golden Ratio

The most basic of all continued fractions is the one using all 1's:

$$[1,1,1,1,\cdots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

To find the value of this continued fraction, we let x denote this value:

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

$$= 1 + \frac{1}{x}$$

In the equation above, notice that the part colored blue is, in fact, identical to *x*, giving the result that $x = 1 + \frac{1}{x}$. By multiplying the equation $x = 1 + \frac{1}{x}$ on both sides by *x* gives us $x^2 = x + 1$ or, equivalently, $x^2 - x - 1 = 0$. Thus, the value of the continued fraction

$$[1,1,1,1,\cdots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

must be a solution to the equation $x^2 - x - 1 = 0$. Since the value is positive (all of the numbers being added are positive), the quadratic formula gives us $x = \frac{1+\sqrt{5}}{2}$, the number known as the golden ratio or golden mean.

Discovered by the ancient Greeks in many geometric constructions and used greatly in their architecture, the golden ratio continues to pop up over and over in mathematics, usually unexpectedly. It occurs repeatedly in nature, as a limiting value of sequences of ratios in certain measurements of flowers and other plant life. Another interesting aspect of this unique number is that psychological tests have indicated that the most aesthetically pleasing size for a rectangle is one for which the ratio of length to width is the golden ratio. The golden ratio appears in art throughout the ages, from Leonardo da Vinci to Piet Mondrian. So it is certainly fitting that this same number should arise when considering the most basic of continued fractions.

Concluding Thoughts…

The following website is a great resource tool because it contains a Continued Fraction Calculator, which is an interactive guide that shows a full expansion and explanation of a continued fraction:

http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/cfCALC.html

At an education standpoint, this would be a great tool to use in the introduction of continued fractions in high school as well as in college courses. This calculator allows the student to create an interactive visual of the continued fraction, and allowing for the student to access this website at any time or anyplace.

Representing numbers in a continued fraction can be very useful in many different aspects of number theory including Euclid and the Greatest Common Divisor Algorithm, solving quadratic equations, and Fibonacci numbers, just to name a few. The uses seem to be endless with continued fractions, and it is very disappointing that this topic is not more highly covered in mathematics education. Certain aspects, especially in number theory, would be better understandable with the introduction of continued fractions and the unique properties that arise when a number is represented in this format. There seems to be no end in sight with the possibilities with continued fractions, as mathematicians continue to find unique uses and properties.

REFERENCES

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