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The Fibonacci numbers first appeared in the 6th century AD with the Indian mathematician Virahanka's analysis of metres with long and short syllables. In the West, the sequence was first studied by Leonardo of Pisa, known as Fibonacci, in his Liber Abaci (1202). The Fibonacci numbers are the product of his analysis of the growth of an idealized rabbit population. He posed the following question:

A certain man put a pair of rabbits in a place surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

This scenario implies that in the "zeroth" month, there is one pair of rabbits (additional pairs of rabbits=0). In the first month, the first pair begets another pair (additional pairs of rabbits=1). In the second month, both pairs of rabbits have another pair, and the first pair dies (additional pairs of rabbits=1). And in the third month, the second pair and the new two pairs have a total of three new pairs, and the older second pair dies. (additional pairs of rabbits=2). Thus the laws of this are that each pair of rabbits has 2 pairs in its lifetime, and dies.

If we let the population at month n be $F(n)$. At this time, only rabbits who were alive at month $n-2$ are fertile and produce offspring, so $F(n-2)$ pairs are added to the current population of $F(n-1)$. Thus the total is $F(n) = F(n-1) + F(n-2)$. And this is the formula for the Fibonacci numbers.

To clarify, the first number of the sequence is 0, the second number is 1, and each subsequent number is equal to the sum of the previous two numbers of the sequence itself. In mathematical terms, it is defined by the following recurrence relation:

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n > 1. \end{cases}$$

That is, after two starting values, each number is the sum of the two preceding numbers. Here is a list of the first twenty Fibonacci numbers:

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_{20}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181	6765

Thus far, we have only defined the nth Fibonacci number in terms of the two before it: the n-th Fibonacci number is the sum of the (n-1)th and the (n-2)th. So to calculate the 100th Fibonacci number, for instance, we need to compute all the 99 values before it first - quite a task, even with a calculator. However, like every sequence defined by linear recurrence, the Fibonacci numbers have a closed-form solution. It has become known as Binet's formula:

$$F\left(n\right) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}},$$

where φ is the golden ratio: $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887\ldots$

Here is a proof of this explicit method by induction:

Any root of the equation above satisfies $x^2 = x + 1$, and multiplying by x^{n-1} shows:

$$x^{n+1} = x^n + x^{n-1}$$

By definition φ is a root of the equation, and the other root is $1 - \varphi = -1/\varphi$. Therefore:

$\varphi^{n+1} = \varphi^n + \varphi^{n-1}$ and $(1-\varphi)^{n+1} = (1-\varphi)^n + (1-\varphi)^{n-1}$.

Both φ^n and $(1-\varphi)^n = (-1/\varphi)^n$ are geometric series (for *n* = 1, 2, 3, ...) that satisfy the Fibonacci recursion. The first series grows exponentially; the second exponentially tends to zero, with alternating signs. Because the Fibonacci recursion is linear, any linear combination of these two series will also satisfy the recursion. These linear combinations form a two-dimensional linear vector space, and the original Fibonacci sequence can be found in this space.

Linear combinations of series φ^n and $(1-\varphi)^n$, with coefficients *a* and *b*, can be defined by

$F_{a,b}(n) = a\varphi^n + b(1-\varphi)^n$ for any real a, b.

All of these defined series satisfy the Fibonacci recursion

$$\begin{aligned} F_{a,b}(n+1) &= a\varphi^{n+1} + b(1-\varphi)^{n+1} \\ &= a(\varphi^n + \varphi^{n-1}) + b((1-\varphi)^n + (1-\varphi)^{n-1}) \\ &= a\varphi^n + b(1-\varphi)^n + a\varphi^{n-1} + b(1-\varphi)^{n-1} \\ &= F_{a,b}(n) + F_{a,b}(n-1)\,. \end{aligned}$$

Requiring that $F_{a,b}(0) = 0$ and $F_{a,b}(1) = 1$ yields a = $1/\sqrt{5}$ and b = $-1/\sqrt{5}$, resulting in the Binet formula we started with. It has been shown that this formula satisfies the Fibonacci recursion. Furthermore, an explicit check can be made:

$$F_{a,b}(0) = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0$$

and $$F_{a,b}(1) = \frac{\varphi}{\sqrt{5}} - \frac{(1-\varphi)}{\sqrt{5}} = \frac{-1+2\varphi}{\sqrt{5}} = \frac{-1+(1+\sqrt{5})}{\sqrt{5}} = 1,$$

establishing the base cases of the induction, proving that

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$ for all n .

Therefore, for any two starting values, a combination a,b can be found such that the function $F_{a,b}(n)$ is the exact closed formula for the series.

The Fibonacci sequence has countless fascinating mathematical properties. These properties are what set it apart as an exceptional series. I'm going to discuss just a brief few of them.

Every 3rd number of the sequence is even and more generally, every kth number of the sequence is a multiple of F_k.

The sequence extended to negative index n satisfies $F_n = F_{n-1} + F_{n-2}$ for *all* integers n, and $F_{-n} = (-1)^{n+1}F_n$: .., -8, 5, -3, 2, -1, 1, followed by the sequence above.

If we look at the Fibonacci numbers mod 2, then we get either 0 or 1 as these are the only two remainders we can have on dividing by 2. However, if the number is *even* the remainder is 0 and if it is *odd* the remainder is 1. This is called the *parity* of a number x and so **x mod 2** tell us if x is even or odd.

For the Fibonacci numbers we have:

0	1	1	2	3	5	8	13	21	...
Even	Odd	Odd	Even	Odd	Odd	Even	Odd	Odd	...
0	1	1	0	1	1	0	1	1	...

and the cycle repeats with a cycle length of 3.

Actually, we will *always* get a cycle that repeats. The reason is that there are only a finite number of remainders when we divide by N: the values 0, 1, 2, ..., N-1 (N of them) and therefore there are a finite number of *pairs* of remainders (N^2) so when we keep adding the latest two Fibonacci numbers to get the next, we *must* eventually get a pair of remainders that we have had earlier and the cycle will repeat from that point on.

Here are the Fibonacci numbers mod 3:

0, 1, 1, 2, 0, 2, 2, 1, (0, 1, 1, 2, …)

so the cycle-length of the Fibonacci numbers mod 3 is 8.

For the divisors (moduli) 2, 3, 4, 5, and so on we have the following cycle lengths for the Fibonacci Numbers:

2	3	4	5	6	7	8	9	10	11	12
3	8	6	20	24	16	12	24	60	10	24

These cycle lengths are also called the Pisano periods. Some interesting facts about Pisano periods are that the longest Pisano periods are 6 times the modulus and are only achieved for numbers that are twice a power of 5:

$2\times5 = 10$ and the cycle length is 60

$2\times5^2 = 50$ with cycle length 300

$2\times5^3 = 250$ with cycle length 1500

Another interesting fact is that apart from modulo 2, all the cycle lengths are even. This begs the question: Is there a formula for the cycle lengths of Fibonacci numbers? Unfortunately, the answer is no, but this would be an interesting question to explore.

Another fascinating property of the Fibonacci numbers is their relation to right triangles. Starting with 5, every second Fibonacci number is the length of the hypotenuse of a right triangle with integer sides, or in other words, the largest number in a Pythagorean triple. The length of the longer leg of this triangle is equal to the sum of the three sides of the preceding triangle in this series of triangles, and the shorter leg is equal to the difference between the preceding bypassed Fibonacci number and the shorter leg of the preceding triangle.

The first triangle in this series has sides of length 5, 4, and 3. Skipping 8, the next triangle has sides of length 13, 12 (5 + 4 + 3), and 5 (8 − 3). Skipping 21, the next triangle has sides of length 34, 30 (13 + 12 + 5), and 16 (21 − 5). This series continues indefinitely. The triangle sides a, b, c can be calculated directly:

$$a_n = F_{2n-1}$$
$$b_n = 2F_nF_{n-1}$$
$$c_n = {F_n}^2 - {F_{n-1}}^2$$

These formulas satisfy $a_n^2 = b_n^2 + c_n^2$ for all n, but they only represent triangle sides when $n > 2$. Any four consecutive Fibonacci numbers F_n, F_{n+1}, F_{n+2} and F_{n+3} can also be used to generate a Pythagorean triple in a different way:

$$a = F_nF_{n+3}\,;\; b = 2F_{n+1}F_{n+2}\,;\; c = F_{n+1}^2 + F_{n+2}^2\,;\; a^2 + b^2 = c^2 \,.$$

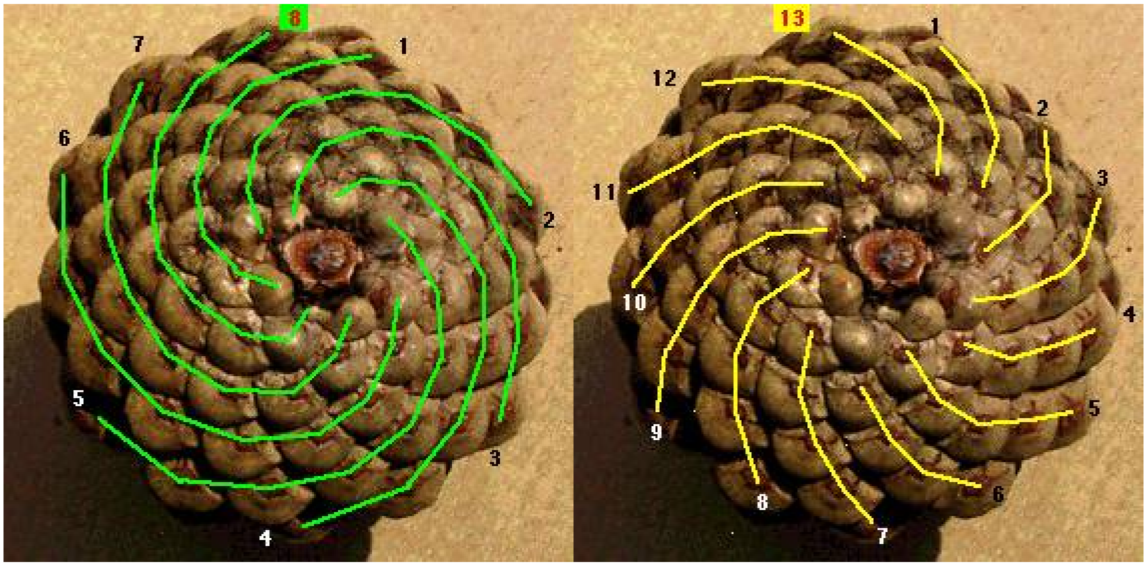
Example 1: let the Fibonacci numbers be 1, 2, 3 and 5. Then:

$$a = 1 \times 5 = 5$$
$$b = 2 \times 2 \times 3 = 12$$
$$c = 2^2 + 3^2 = 13$$
$$5^2 + 12^2 = 13^2 \,.$$

Example 2: let the Fibonacci numbers be 8, 13, 21 and 34. Then:

$$a = 8 \times 34 = 272$$
$$b = 2 \times 13 \times 21 = 546$$
$$c = 13^2 + 21^2 = 610$$
$$272^2 + 546^2 = 610^2 \,.$$

Perhaps the most intriguing characteristic of the Fibonacci numbers is its appearance in nature and biological settings. There are many examples of the appearance of two consecutive Fibonacci numbers in the description of some natural occurrences, such as branching in trees, the development of the fruitlets of a pineapple, the flowering of artichoke, the uncurling of a fern, the spirals of shells, and the curve of waves. The spiral arrangement of a pine cone show the Fibonacci numbers clearly. Here is a picture of an ordinary pine cone seen from its base where the stalk connects it to the tree.



Note that the number of spirals in each direction are 8 and 13, which are adjacent Fibonacci numbers.

The Fibonacci numbers are a truly fascinating mathematical phenomenon. From patterns within the series to applications in other areas of mathematics to their presence in nature and the world around us, there are countless uses of the Fibonacci numbers. There are still a number of problems unsolved and properties left to be proved. The Fibonacci numbers are a true gift to the field of mathematics and number theory.