Cicek Penpecioglu

MA341

Final Project Draft: Diophantine Approximations

In this project, we will be exploring Diophantine Approximations while understanding transcendental numbers, simultaneous approximations, and several important theorems concerning the subject. We will try to see how accurate one can be while approximating real numbers by rational numbers. The most important subjects we will be focusing include Dirichlet's Principle and the Liouville number.

We will first go into the approximations of irrational numbers by rational number with the help of Dirichlet's box principle, which is also known as the Pigeon-hole Principle. We start by thinking, given a real number $\boldsymbol{\alpha}$, how closely can it be approximated by rational numbers? To make the question more precise, for any given positive $\boldsymbol{\acute{\varepsilon}}$, is there a rational number a/b within $\boldsymbol{\acute{\varepsilon}}$ of $\boldsymbol{\alpha}$, so that the inequality

$$(1)\ \left|\alpha - \frac{a}{b}\right| < \varepsilon$$

is satisfied? The answer is yes because the rational numbers are dense on the real line. In other words, for every real number r, we can find numbers $s \in$ S, where S is the subset of real numbers, that are as close as to r. In fact, this established that for any real number α and any positive $\boldsymbol{\acute{\varepsilon}}$, there are infinitely many rational numbers a/b satisfying the above inequality.

Another way of approaching this problem is to consider all rational numbers with a fixed denominator b, where b is a positive integer. The real number α can be located between two such rational numbers, say

$$\frac{c}{b} \le \alpha < \frac{c+1}{b} ,$$

and so we have $|\ \alpha - c/b\ | < 1/b$ In fact, we can write

$$(2)\ \left|\alpha - \frac{a}{b}\right| \le \frac{1}{2b}$$

By choosing a = c or a = c+1. The inequality would be strict, that is to say, equality would be excluded if α was not only real but irrational.

For the following Theorem, we will rely on Dirichlet's Box Principle, or the Pigeonhole Principle, which states that if n+1 objects are distributed into n boxes, then at least one box contains at least two objects. In other words, given two natural numbers *n* and *m* with $n > m$, if *n* items are put into *m* pigeonholes, then at least one pigeonhole must contain more than one item.

Theorem 1: (Dirichlet's Approximation Theorem) :

If α is irrational, then there exist infinitely many rational numbers p/q such that

$$(3) \qquad \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

The qualitative question whether a given real number α is irrational or not, depends on a quantitative property of α, namely on the quality of rational approximations to α. Multiplying this inequality (3) with *q* shows that there are infinitely many best approximations to a given irrational α. To prove Theorem 1, we will rely on the pigeonhole principle.

Proof 1:

We denote the *integral part* and the *fractional part* of a real number x by

[x] = max {z € Z : $z \leq x$ } and {x} = x =[x], respectively.

Let Q be a positive integer. The numbers

$$0, \{\alpha\}, \{2\alpha\}, \ldots, \{Q\alpha\}$$

define Q+1 points distributed among the Q disjoint intervals

$$\left[\frac{j-1}{Q}, \frac{j}{Q}\right) \qquad \text{for } j=1, \ldots, Q.$$

By the pigeonhole principle there has to be at least one interval which contains at least two numbers $\{k\alpha\} \geq \{l\alpha\}$, say, with $0 \leq k, l \leq Q$ and $k \neq l$. It follows that

(1.1) $\{k\alpha\} - \{l\alpha\} = k\alpha - [k\alpha] - l\alpha + [l\alpha]$

$$= \{(k-l)\alpha\} + \underbrace{[(k-l)\alpha] + [l\alpha] - [k\alpha]}_{\in Z}$$

Since $\{k\alpha\} - \{l\alpha\}$ lies in the interval [0, 1/Q), the integral parts of the above equation add up to zero. Setting $q = k - l$ we obtain, $\{q\alpha\} = \{k\alpha\} - \{l\alpha\} < 1 / Q$.

With p:= [qα] it follows that

(1.2) $$\left|\alpha - \frac{p}{q}\right| = \frac{|q\alpha - p|}{q} = \frac{\{q\alpha\}}{q} < \frac{1}{qQ},$$

which implies the estimate (2), since $q < Q$.

Now, to contradict (1.2), we suppose that α is irrational and that there exist only finitely many solutions $p_1/q_1, \ldots, p_n/q_n$ to statement (3). Since α does not belong to Q, we can find a Q such that:

$$\left|\alpha - \frac{p_j}{q_j}\right| > \frac{1}{Q} \quad \text{for } j=1,\ldots,n \quad \text{contradicting (1.2).}$$

Finally we assume that α is rational, say α = a/b with a € Z ad b € N.

If $\alpha = \frac{a}{b} \neq \frac{p}{q}$, then

$$\left|\alpha - \frac{p}{q}\right| = \frac{|aq - bp|}{bq} \geq \frac{1}{bq},$$

and (1.2) involves $q < b$. This proves that there are finitely many p/q with (1.2), concluding our proof 1.

In the second part of our paper, we will discuss Liouville's Theorem followed by Liouville's Number.

Theorem 2:

For any algebraic number α of degree d>1 there exists a positive constant c, depending only on α, such that:

(5) $$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^d}$$

for all rationals $\frac{p}{q}$ with $q > 0$.

Proof 2:

Let P(X) denote the minimal polynomial of α. Then, by the meanvalue theorem,

$$-P\left(\frac{p}{q}\right) = \underbrace{P(\alpha)}_{=0} - P\left(\frac{p}{q}\right) = \left(\alpha - \frac{p}{q}\right)P'(\varepsilon)$$

for some έ lying in between $\frac{p}{q}$ and α. Without loss of generality we may assume that the distance between α and $\frac{p}{q}$ is small:

$$\left|\alpha - \frac{p}{q}\right| < 1.$$

Then |έ| < 1 |α| and hence |P'(έ)| < 1/c for some positive c. While denoting this, we should not forget to indicate that polynomials are bounded on compact sets. In other words, the sequences in a set of real numbers have subsequences that converge to an element again contained in the set of real numbers. This is why the compact sets are always bounded, hence also the polynomials.

It follows that,

$$\left|\alpha - \frac{p}{q}\right| > c\left|P\left(\frac{p}{q}\right)\right|.$$

Since P(X) is irreducible of degree d ≥ 2, $\frac{p}{q}$ cannot be zero of P(X). Hence,

$$\left|q^d P\left(\frac{p}{q}\right)\right| \geq 1$$

Noting that P(x) has integer coefficients, we complete our proof.

A real number α is said to be a Liouville number if for every positive integer m, there exist integers a_m and $b_m > 1$ such that,

(4) $$0 < \left|\alpha - \frac{a_m}{b_m}\right| < \frac{1}{b_m^m}.$$

Since the right-hand side tends to zero as m → ∞, the rationals a_m / b_m approximate α better and better. In particular, it follows that the set of the numbers b_m is unbounded.

Theorem 3:

Every Liouville number is transcendental.

Proof 3:

We assume that the Liouville number α is algebraic of degree *d*. Combining (4) with the estimate in Liouville's theorem (5) implies

$$\frac{c}{b_m^d} < \frac{1}{b_m^m},$$

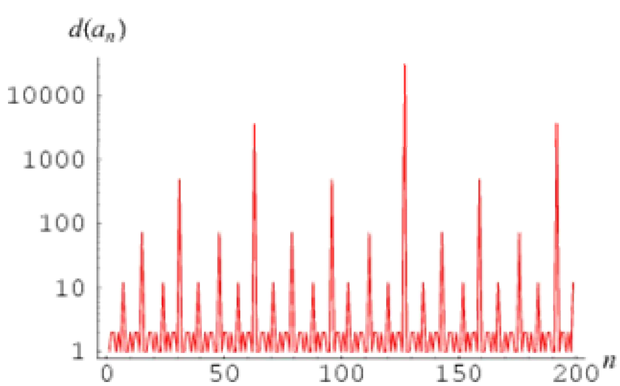
where c is a positive constant depending only on α. Thus it follows that $c < b_m^{d-m}$ and since the set of the b_m is unbounded, this gives the desired contradiction.

We can define Liouville's number or Liouville's constant (L) as:

$$\sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000\ldots$$

where the *k*th instance of the digit 1 is separated from the previous by *k*! – (*k* – 1)! – 1 instances of the digit 0. Liouville's constant nearly satisfies the equation: **10 x⁶ – 75 x³ – 190 x + 21 = 0**, which has solution **x= 0.1100009999...** , but plugging x = L into this equation gives -0.0000000059… instead of 0.

Here is a semilog plot, where the numbers of digits $d(a_n)$ in the nth term is plotted showing a nested structure.



Interestingly, the nth incrementally largest term (considering only those entirely of 9s in order to exclude the term $a_2 = 11$) occurs precisely at position $2^n - 1$, and this term consists of $(n-1)\,n!$ 9s.