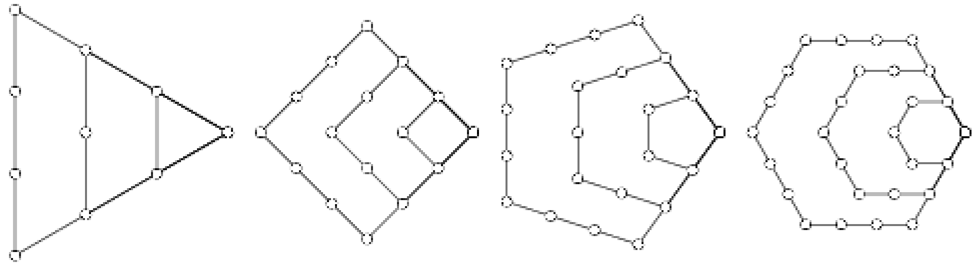
# An Exploration of Polygonal Numbers

David Plotkin
MA 341: Number Theory

## Introduction

We are all familiar with square numbers, and probably with triangular numbers as well: these are numbers such that, given a square or triangular number n and n dots, we can make form a lattice structure of a square or an equilateral triangle, respectively. We might reasonably expect polygonal numbers in general to be defined like this; however, for m-gonal numbers with m>4, we cannot make a lattice structure of an m-gon given m dots. Instead, polygonal numbers and the shapes they form look like:

Figure 1:



m=3 m=4 m=5 m=6.

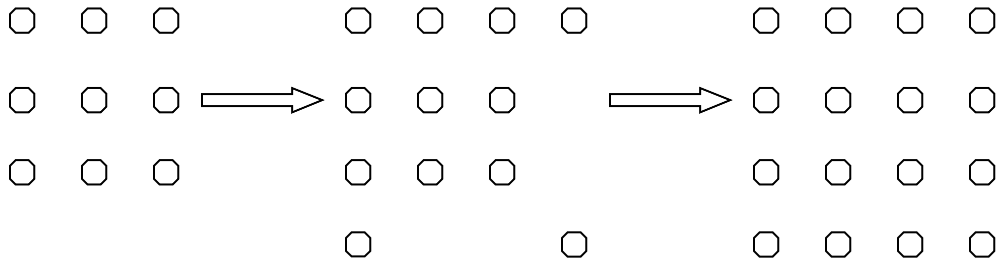
Much as we expect, the figures formed are all regular polygons, but those with m>4 look very different that the square or triangular numbers.

Polygonal numbers were first studied by the ancient Greeks (the Pythagoreans), who made several important observations that were later proved by more modern mathematicians. For instance, they observed that the common difference in the sequence of differences of adjacent n-gonal numbers was n-2, i.e. that in the sequence of triangular numbers – 1, 3, 6, 10, 15, 21, etc – the differences between adjacent numbers are 1, 2, 3, 4, 5, 6, etc, and the common difference in that sequence is 3-2 = 1. This will become more obvious when we discuss the general form of an n-gonal number. The Greeks also began to notice something that would lead to a conjecture by Fermat and then a proof by Gauss: that every natural number can be written as a sum of n n-gonal numbers. We will return to this in the last section.

## General Information: Geometric Approaches

Let us now construct the polygonal numbers: given the $n^{th}$ m-gonal number, how must we modify it to make the $n+1^{st}$ m-gonal number. We can use one of our old vertices as a vertex for the new polygon, leaving us with m-1 vertices to add. We then need to fill in the sides in between these vertices; since we have two sides, we do this m-2 times, adding n-2 dots each time.

Figure 2:

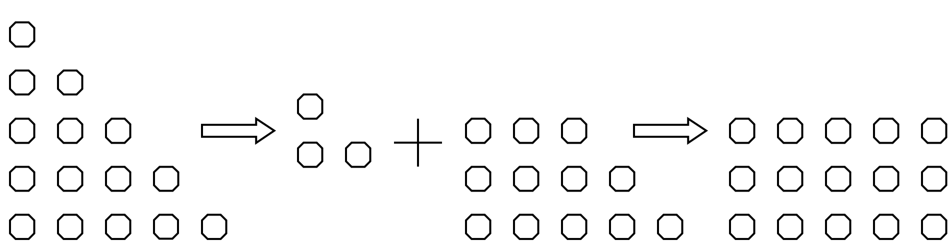


Thus, for each iteration, we add (m-1)+(n-2)(m-2) dots. Since m and m-2 are both constant, the only term that changes is (n-2). This leads to the common difference observation: each iteration, we add m-2 more dots than we added the previous iteration because the coefficient of the (m-2) term has increased by 1.

We can now look try to find a formula for the polygonal numbers: we will derive it for triangles and then state the general case. It is tempting to try something along the lines of square numbers, i.e. the n-th square = n*n, but does not bear much relation to the general case, owing to the specific geometry of square numbers, i.e. the fact that they form rectangular arrays. Let us now graphically develop the formula for triangle numbers:

1. Letting m be odd, start with the $m^{th}$ triangular number, in our case, we let m=7.
2. Separate the triangle into two parts by "cutting" underneath the $2^{nd}$ row (in general, the $(n-1)/2^{nd}$ row).
3. Rotate the top part 180 degrees and fit into bottom part.
4. You are left with a rectangle of length n and height ½(n+1). Thus the formula for the $m^{th}$ triangle number, with odd m, is ½(n)(n+1).

Figure 3:

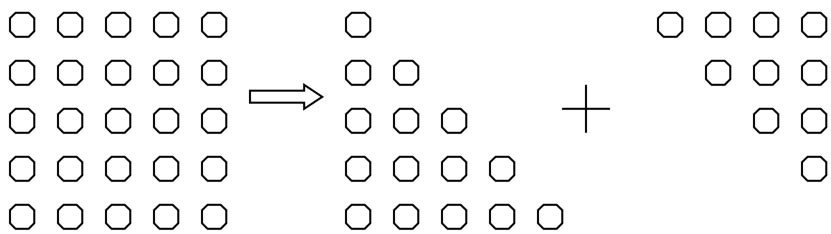


Our strategy can easily be extended to cases where m is even by cutting the triangle along the n/2nd row and yielding a rectangle of height n/2 and width n+1, which evidently yields the same formula.

A similar geometric strategy can be used to show that all square numbers are the sum of two adjacent triangular numbers, i.e. 4 = 1 + 3, 9 = 3 + 6, 16 = 6 + 10, and so on:

1. Start with the nth square number.
2. Cleave it into two triangles just on either side of the main diagonal.
3. The two triangles you are left with are not equilateral; however, since each side of one triangle has n dots and each side of the has n-1 dots, they can be rearranged to form equilateral triangular lattices.

Figure 4:



Geometry can also be used to find some (but not all) Pythagorean triples. Consider figure 5, which shows that if we subtract the nth square number from the n+1st square number, we are left with 2n + 1. This leads means that $2n + 1 = (n + 1)^2 – n^2$, $n^2 + 2n + 1 = (n+1)^2$. Thus, if 2n + 1 is a perfect square, we have produced an integral solution to the Diophantine equation $a^2 + b^2 = c^2$.

Unfortunately, our geometric methods are rather complicated for giving a general expression for polygonal numbers, so to save space we will state the expression for the nth m-gonal number (but thinking about it is a good exercise). This expression proves will prove useful later on, when we discuss which triangular numbers are also pentagonal.

$$\mathbf{p = \tfrac{1}{2}\, m[(n-2)m - (n-4)]}$$

**Sums of Polygonal Numbers**

In the 17th century, the French lawyer (and avid amateur mathematician) Pierre de Fermat conjectured that any number could be written as a sum of n n-gonal numbers. This was eventually proved by Cauchy in 1813; however, here we will only deal with the square case, i.e. that any number is the sum of four square numbers – also known as Lagrange's four-square theorem. We prove this by descent, much as we proved that any prime of the form 4k + 1 is a sum of two squares in lecture. We first need three lemmas:

<u>Lemma 1</u>: The Euler four-square identity:

Given integers a,b,c,d,q,r,s,t, then

$$(a^2 + b^2 + c^2 + d^2)(q^2 + r^2 + s^2 + t^2) = (aq + br + cs + dt)^2$$
$$+ (ar - bq - ct + ds)^2$$
$$+ (as + bt - cq - dr)^2$$
$$+ (at - bs + cr - dq)^2$$

<u>Lemma 2</u>: We proved this one in class:

If $2m = x^2 + y^2$, then there exist natural numbers u,v, such that $m = u^2 + v^2$

Proof: Since 2m is even, x and y must be of the same parity, (x+y)/2 and (x-y)/2 are both integers, and the identity

$M = ((x-y)/2)^2 + ((x+y)/2)^2$

<u>Lemma 3:</u> More sums of squares:

For any odd prime p, a2 + b2 + 1 = kp for some natural number k, 0 < k < p

Proof: Let p = 2n +1, and take 2 sets, $A = \{a^2 \mid a = 0, 1,\dots, n\}$

$B = \{-b^2 - 1 \mid b = 0, 1,\dots, n\}$, with the properties:

1. No two elements of A are congruent mod p. Proof: let two different elements, $a^2$ and $b^2$ in A, be congruent. This means that $p|(a^2 - b^2)$, so p|(a – b), which is obviously impossible unless a = b, which contradicts our assumption. By similar reasoning, no two elements of B are congruent to each other mod p.
2. The intersect of A and B = null set, because all elements of B are negative while all elements of A are non-negative.
3. Thus set C = A U B has 2n + 2 = p + 1 elements.
4. Thus, by the pigeonhole principle, two elements of C, a and b, must be congruent two each other mod p. Since none of the elements in either set are congruent, mod p, to any other elements of their respective set, if a is in A, then b must be in B.

This yields a2 + b2 = kp for some natural number k. Further,

$p^2 = (2n+1)^2 > 2n^2 + 1 >= a^2 + b^2 + 1 = kp$, so p>k

So kp = sum of four squares, $a^2 + b^2 + 1^2 + 0^2$

Proof of Lagrange's four-square theorem:

Lemma 1 reduces our problem to showing only that any arbitrary prime is a sum of four squares, since 2 is obvious, let prime p be odd, i.e. $p = 2n + 1$.
By lemma 3, $a^2 + b^2 + c^2 + d^2 = kp$ for natural numbers a,b,c,d,m, and $0 < m < p$
Now we use the descent strategy: if $m = 1$, we're happy, if $m > 1$, we show that there exists n, $1 >= n < m$, such that np is a sum of four squares.

Case 1: m is even:

This is easy. M is even means that that either all four of a,b,c,d are of the same parity, or, without loss of generality, a and b are of the same parity and c and d are of the other parity, i.e. a and b are even while c and d are odd. We can group a with b and c with d, and then by lemma 2 $n = m/2$.

Case 2: m is odd (and not = 1):

Let w, x, y, z be congruent to a, b, c, d mod k, respectively, and $-k/2 >=$ w, x, y, $z >= k/2$, thus:
$w^2 + x^2 + y^2 + z^2 < k^2$ ($= 4 * k^2/4$) and
$w^2 + x^2 + y^2 + z^2$ congruent to 0 mod k, i.e. $w^2 + x^2 + y^2 + z^2 = nm$ for some natural number n, with $0 < n < k$
Now consider $k^2np.= (aw + bx + cy + dz)^2$
$+ (ax – bw – cz + dy)^2$
$+ (ay + bz – cw – dx)^2$
$+ (az – by + cx – dw)^2$
The last three are obviously multiples of $k^2$ (for instance, take the square root of the second and then group (ax – bw) and (-cz + dy); since a is congruent to w and b is congruent to x, the first part is congruent to 0 mod k, so it is a multiple of k. Similar reasoning works for analogous groupings in the last three squares as well. The first is also a multiple of m because:

$aw + bx + cy + dz$ is congruent to $w^2 + x^2 + y^2 + z^2$ is congruent to 0 (k), thus:
$k \mid aw + bx + cy + dz$

So both sides are divisible by $k^2$, thus we can divide through by $k^2$ to get a new expression for np, with $0 < n < k$.
Done!

**Triangular Pentagons: Diophantine Equations**

Using our knowledge of Diophantine equations and sums of squares, we can now try to figure when an n-gonal number is also an m-gonal number. We take $m = 3$ and $n = 5$, so we want to know when triangular numbers are also pentagonal numbers. We refer back to the general formula, which states that triangular numbers are ½m(m+1) and pentagonal ones are ½n(3n+1). Setting them equal yields:

$$m(m + 1) = n(3n - 1)$$
$$3(4m^2 + 4m) = 36n^2 - 12n$$
$$3(2m + 1)^2 – 2 = (6n – 1)^2$$
$$(6n – 1)^2 – 3(2m + 1)^2 = -2$$

Now let $x = 6n – 1$ and $y = 2m +1$. We get:

$$\mathbf{x^2 – 3y^2 = -2}$$

Which we can let a computer program solve and give the solutions (x,y) = (5,3), (19,11), (71,41), (265,153), (13775,7953), and so on; however, we aren't quite done since we're looking integer values of m and n, which end up being (n,m) = (1,1), (12,20), (165,285), (2296, 3976). This means that the first pentagonal number is also the first triangular number, the 12$^{th}$ pentagonal number is the 20$^{th}$ triangular number, etc.

We now see one reason why sums of squares are so important in number theory. Were we to try to find triangular square numbers, we would end up having to solve the equation $(4n)^2 – 2(4m+1)^2 = -2$, or $x^2 – 2y^2 = -2$. It looks like there may be a pattern in our Diophantine equations, so a good exercise could be: what is the general expression for a k-gonal number that is also a triangular number?