Surprising Results of Decimal Representation

Final Project

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Congcong Li

Instructor: Kalin Kostadinov

I. Introduction

When we think about a number, each digit of the number, their sums and ordering give rich information about properties of the numbers and often produces surprising results. The information about a number is often hidden in its decimal representation. This research project focuses on two issues arise from observations: divisibility and the famous Kaprekar's operation. We will be able to answer the question of how and why the divisibility of a number is linked to each of the digits and find a general characteristic of numbers of the same digits. I will introduce the divisibility theorem and use the knowledge of modulo congruence to provide a formal proof for the theorem. I will also introduce the famous Kaprekar's Operation and examine some of the surprising results we observe by rearranging the order of the digits and applying simple arithmetic. In the end, we will get a better knowledge of how decimal representation links numbers and their mathematical properties in a unique way.

II. Divisibility Rules

1) Base systems and Conversions

Consider an integer a under the base 10 decimal system, where

$$a = a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10^1 + a_0$$

Generally speaking, we can expand this to integer base b and derive

$$a = (a)_b = c_k b^k + c_{k-1} b^{k-1} + \ldots + c_1 b^1 + c_0$$

Conversion between different bases is used in our method to test the divisibility of different numbers.

2) **Theorem of Divisibility**

Let b be an integer, then an integer a

1. is divisible by (b-1) if and only if the sum of digits in its expansion $(a)_b$ is divisible by (b-1)
2. is divisible by (b+1) if and only if the alternating sum of digits in its expansion $(a)_b$ is divisible by (b+1)

3) An Example

Let a = 284, is it divisible by 4?

By 1), let us take b = 5, according to the base system conversion rules, we have

$(284)_5 = 2 * 5^3 + 1 * 5^2 + 1 * 5 + 4 * 5^0 = (2114)_{5.}$

The sum of digits is 2+1+1+4 = 8 and is divisible by 4. Therefore, by theorem 1), 284 is divisible by 4.

On the other hand, by 2), let us take b = 3, according to the base system conversion rules, we have

$(284)_3 = 1 * 3^5 + 1 * 3^3 + 1 * 3^2 + 1 * 3^1 + 2 * 3^0 = (101112)_3$

cordi The alternating sum of digits is 1 – 0 + 1 – 1 + 1 – 2 = 0 and is divisible by 4. Therefore, by theorem 2), 284 is divisible by 4.

If we check, 284 / 4 = 71, so it is indeed divisible by 4.

4) Proof (Modulo Arithmetic)

1) We have $b - 1 = 0 \pmod{(b - 1)}$

$b = 1 \pmod{(b - 1)}$

Therefore, for every $i > 0$, $b^i = 1 \pmod{(b - 1)}$

Multiplying this by c_i and summing up for i = 0, 1, …, k, we have

$$c_k b^k + c_{k-1} b^{k-1} + ... + c_1 b^1 + c_0 = c_k + c_{k-1} + ... + c_1 + c_0 \pmod{(b - 1)}$$

So that we prove if a number is divisible by b – 1, the sum of digits in its expansion $(a)_b$ is divisible by b – 1.

2) We have $b + 1 = 0 \pmod{(b + 1)}$

$$b = -1 \pmod{(b + 1)}$$

Therefore, for every i > 0, $b^i = (-1)^i \pmod{(b+1)}$

Multiplying this by c_i and summing up for i = 0, 1, …, k, we have

$$c_kb^k + c_{k-1}b^{k-1} + ... + c_1b^1 + c_0 = c_k - c_{k-1}b^{k-1} + ... - c_1b^1 + c_0 \pmod{(b+1)}$$

So that we prove if a number is divisible by b +1, the sum of digits in its expansion $(a)_b$ is divisible by b +1.

III. Kaprekar's Operation

We now see that the sum of digits conveys information on the divisibility of the number. Let us try another operation: if we take a four-digit number, rearrange the digits to get the largest and the smallest number these digits can make; and finally subtract the smallest from the largest to get a new number and keep on this operation for each new number. For example, we take the number 5149, under the operation we just described, we find

9541 – 1459 = 8082

8820 – 288 = 8532

8532 – 2358 = 6174

7641 – 1467 = 6174

After three steps, we reach the value of 6174 and then the operation just repeats itself and we keep getting the value 6174. Try other numbers where the digits are not all the same (i.e. not a number like 1111, 2222), we will always get the same result, 6174. In fact, studies done by the mathematician Kaprekar found that all four digits number where the digits are not all the same reach the value 6174. The number theorists are still working on the mathematical theory that this phenomenon reveals. Let us do some more detailed analysis on this mathematical process.

It is a simple operation, but in 1949, mathematician Kaprekar first discovered the surprising result of this operation, that the number always reaches 6174. In fact, based on the studies of Yukata Nishiyama, the number reaches 6174 in at most 7 steps.
There are 8991 four-digit numbers from 1000 to 9999 where the digits were not all the same. With the help of computing software, we can calculate the frequency of all the iterations. The table is shown as following: All 3 digit numbers reach 495 under this operation and all four digit numbers reach 6174 under this operation. There are 8991 four digit numbers from 1000 to 9999 where the digits are not all equal, e.g. not 1111, not 2222. Studies have shown that all numbers reach the value of 6174 under the Kaprekar's operation in seven steps. With a table of frequency as following:

Iteration	Frequency
0	1
1	356
2	519
3	2124
4	1124
5	1379
6	1508
7	1980

(Diagram taken from Nishiyama, *Mysterious Number 6174*)

The study done by Malcolm Lines has used a different approach to analyze the characters of the numbers in these operations. As it turns out, it is enough to check only 30 numbers out of all the possible four digit numbers and the same conclusion was derived. Let us suppose four numbers a, b, c, d, where

$9 \geq a \geq b \geq c \geq d \geq 0$

The maximum number is 1000a + 100b + 10c + d and the minimum number is 1000d + 100c + 10b + a. So the subtraction is

$(1000a + 100b + 10c + d) - (1000d + 100c + 10b + a)$

$= 1000 (a – d) + 100 (b - c) + 10 (c - b) + (d – a)$

$= 999 (a – d) + 90 (b – c)$

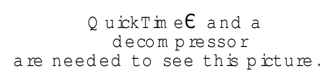
All the possible value of (a – d) is from 1 to 9, and (b - c) is from 0 to 9. Running through all values of (a - d) and (b - c), we can get 90 numbers. For example, we know that when (a – d) equal 2 and when (b – c) equal 4 will yield a four digit number we need, so we can get 999*2 + 90*4 = 2358, so that 2358 is a possible solution.

But given that a ≥ b ≥ c ≥ d, (a –d) should be greater than (b - c), we can eliminate all the values where (a - d) > (b – c), so we have the following diagram.

		999X(a-d)								
		1	2	3	4	5	6	7	8	9
	0	9990	9981	9972	9963	9954	9954	9963	9972	9981
	1	9810	8820	8730	8640	8550	8640	8730	8820	9810
	2		8721	7731	7641	7551	7641	7731	8721	9711
	3			7632	6642	6552	6642	7632	8622	9621
90X	4				6543	5553	6543	7533	8532	9531
(b-c)	5					5544	6444	7443	8442	9441
	6						6543	7533	8532	9531
	7							7632	8622	9621
	8								8712	9711
	9									9801

(Diagram taken from Nishiyama, *Mysterious Number 6174*)

The grey areas are the values that are repetitive, so eliminate those values as well. Now when we begin the second subtraction, the numbers under study have converged to a smaller set of numbers. It is reasonable to expect that after 7 iterations, all numbers would converge to the value where (a - d) = 6 and (b - c) = 2, so that 999 * 6 + 90 * 2 = 6174. What happens behind the scene is that in each iteration the possible numbers converge and resemble a smaller and smaller set of numbers. After seven iterations, all numbers converge to a single value, that is, the mysterious number 6174.



(Diagram taken from Nishiyama, *Mysterious Number 6174*)

All four digit numbers reach 6174 after at most 7 iterations, with each iteration yielding a smaller and smaller set of numbers. Does the small result hold for numbers that are consisted of other numbers of digits? For example, what would happen if the same operation is applied to a random three-digit number?

As it turns out, a similar outcome is found for numbers of three digits. They converge to a single value after a limited number of iterations, with the value being 495. Take a random number where the digits are not all the same, 679, as an example:

976 – 679 = 297

972 – 279 = 693

963 – 369 = 594

954 - 459 = 495

954 - 459 = 495

Studies done by Nishiyama have shown that the same result only holds for numbers consist of three and four digits. For numbers of more digits and numbers of two digits, the same operation fails to produce a single number.

Current studies on this issue have been limited and have not yet been successful in deriving a proof. The question of whether a big theorem is behind this operation or it is merely a coincidence is not answered.

IV. Bibliography

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