Diophantus and the *Arithmetica*

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"Knowing, my most esteemed friend Dionysius, that you are anxious to learn how to investigate problems in numbers, I have tried, beginning from the foundations on which the science is built up, to set forth to you the nature and power subsisting in numbers." Thus Diophantus begins his great work *Arithmetica* (Heath, 129) and gives the world what would become its most ancient text on algebra. The *Arithmetica* "essentially teach[es] the solution of those computational problems which are known today as determinate and indeterminate equations of the first and second degree" (Klein, 126). Today, these equations are called Diophantine equations, and the methods of finding their solutions comprise an entire branch of mathematics known as Diophantine analysis.

Biography:

Very little is known about the life of Diophantus. We know he must have lived later than 150BCE since he quotes from Hypsicles in his book on Polygonal Numbers. On the other hand, Diophantus is quoted around 350CE by Theon of Alexandria, (Heath, 2) giving us a possible interval of about five hundred years. Heath argues that Diophantus is contemporary to Anatolius, who was the Bishop of Laodicea around 280CE. However, W. R. Knorr believes this conclusion to be based on an incorrect translation of the Greek, insisting instead that "we must entertain the possibility that Diophantus lived earlier than the third century, possibly even earlier than Heron in the first century" (qtd. in O'Connor and Robertson). Even less is known of Diophantus' personal life. Indeed, the only extant reference we have comes from the Greek Anthology, a collection of numerical riddles compiled around 500CE: "...his boyhood lasted $1/6^{th}$ of his life; he married after $1/7^{th}$ more; his beard grew after $1/12^{th}$ more, and his son was born 5 years later; the son lived to half his father's age, and the father died

4 years after the son" (qtd. in O'Connor and Robertson). This gives Diophantus' age at death as 84 years, but scholars doubt whether this account of his life is not entirely fictional. Paul Tannery, using the same shaky evidence as Heath to connect Diophantus with Anatolius, identifies the Dionysius to whom the *Arithmetica* is dedicated as Saint Dionysius, the bishop of Alexandria, and argues that Diophantus therefore must have been a Christian (Klein, 129).

Genealogy and Translation of the *Arithmetica*:

The *Arithmetica* as written by Diophantus originally contained thirteen books. Around 400CE Hypatia of Alexandria wrote a commentary on the first six of these, and the remaining seven were eventually forgotten and are now believed to be lost (Heath, 5). Later copyists would mistake some of Hypatia's and other later commentators' explications and additional solutions as original Diophantine material, and in this way some interpolations have entered the manuscripts (Heath, 14). The first Latin translation of the original Greek did not appear until 1575, published by the German Wilhelm Holzmann under the Graecised form of his name, Xylander (Heath, 22). This first translation, based on a single Greek manuscript that was itself full of errors, was later corrected and revised by the French mathematician Bachet in 1621 (Heath, 26). The *Arithmetica* finally appeared in English in 1885 thanks to Sir Thomas Heath.

Sometime before 1000CE, the Greek was translated into Arabic from an early, accurate manuscript in excellent condition (Sesiano, 67). In 1968, Fuat Sezgin discovered a manuscript in the Holy Shrine Library in Meshed, Iran, with a title claiming it to be a translation by Qusṭā ibn Lūqa (who died in 912CE) of Books IV to VII of Diophantus' *Arithmetica* (O'Connor and Robertson). Some scholars argue that several of these may be missing books; however, most agree that the texts which differ from the extant Greek are probably translations not of Diophantus but of Hypatia or one of his other commentators. In any case, this Arabic manuscript now also exists in English and grants us another glimpse of the type of mathematics taught by Diophantus.

Toward Modern Algebra:

Diophantus is often found described as the “Father of Algebra.” This is, of course, an oversimplification. For one thing, the Greeks of Diophantus' time were well acquainted with *the hau-*calculations of the Egyptians. “*Hau*, meaning a *heap*, is the term used to denote the unknown quantity, and the calculations in terms of it are equivalent to the solutions of simple equations with one unknown quantity” (Heith, 112). Heith cites three examples from the Papyrus Rhind, which dates to about 1650BCE:

$$\frac{1}{7}x+x=19 \quad , \qquad \frac{2}{3}x+\frac{1}{2}x+\frac{1}{7}x+x=33 \quad , \qquad (x+\frac{2}{3}x)-\frac{1}{3}(x+\frac{2}{3}x)=10 \quad .$$

(Of course, these are our modern symbolic representations of the Papyrus Rhind problems.) Thus, it is clear that Diophantus did not “invent” algebra but rather collected, expanded, and generalized the work of the earlier algebraists.

Where Diophantus does seem to have made headway in the advancement of algebra is in notation, but his notation is still very limited in comparison to our own. The earliest algebraists developed what George Nesselmann calls “*Rhetorical Algebra* or 'reckoning by complete words'” (Heath, 49). He cites a passage from Muhammad ibn Mūsā's *Algebra* (circa 830CE) as one example:

> A *square* and ten of its *roots* are equal to nine and thirty dirhems, that is, if you add ten *roots* to one *square*, the sum is equal to nine and thirty. The solution is as follows. Take half the number of *roots*, that is in this case five; then multiply this by itself, and the result is five and twenty. Add this to the nine and thirty, which gives sixty-four; take the square root, or eight, and subtract from it half the number of *roots*, namely five, and there remain three: this is the *root* of the *square* which was required, and the square itself is *nine*. (qtd in Heath, 50)

This problem corresponds to the following operations in what Nesselmann calls *Symbolic Algebra:*

$$x^{\text{٢}}+\text{١٠x}=\text{٣٩}$$
$$x^{\text{٢}}+\text{١٠x}+\text{٢٥}=\text{٦٤}$$
$$x+\text{٥}=\text{٨}$$
$$x=\text{٣}$$

Diophantus exists in an intermediate stage between *Rhetorical Algebra*, where calculations are carried out in prose, and our modern *Symbolic Algebra*, “which uses a complete system of notation by signs having no visible connexion with the words or things which they represent” (Heath, 49). Nesselmann calls this intermediate stage *Syncopated Algebra*.

Diophantus uses the Greek letter ς' to represent the unknown quantity. “This symbol in verbal description he calls *ὁ ἄριθμός*, 'the number'” (Heath, 32), and Heath argues that the symbol ς' is a contraction of the first two letters of this word. Likewise, the symbols for the first powers of the unknown quantity up to the sixth are abbreviations of the Greek words for the various powers (Sesiano, 43). The most obvious limitation of this notational system is that it allows for only one variable. Thus, when solving determinate systems of equations, we will see Diophantus effectively parameterizing each required unknown quantity in terms of one variable. However,

> when he handles problems which are by nature indeterminate and would lead with our notation to an indeterminate equation containing two of three unknowns, he is compelled by limitation of notation to assume for one or other of these some particular number arbitrarily chosen, the effect of the assumption being to make the problem a determinate one. (Heath, 51)

Only after completing the problem with the assumed values can he step back to comment on the general solution.

Methods for solving Determinate Equations:

The problems in Book I of the *Arithmetica* are determinate (ie, having a unique solution or a finite number of solutions) and relatively simple, mostly dealing with ratios and products. For example, consider I,4:

> To find two numbers in a given ratio and such that their difference is also given.
> Given ratio 5 : 1, given difference 20. Numbers 5x, x. Therefore 4x = 20, x = 5, and the numbers are 25, 5.

Similarly, consider I,11:

> Given two numbers, to add the first to, and subtract the second from, the same (required) number, so as to make the resulting numbers have to one another a given ratio.
> Given numbers 20, 100, given ratio 3 : 1.
> Required number x. Therefore 3x – 300 = x + 20, and x = 160.

Immediately preceding Book I, Diophantus gives the following definitions to solve these simple problems:

> [I]f a problem leads to an equation in which certain terms are equal to terms of the same species but with different coefficients, it will be necessary to subtract like from like on both sides, until one term is found equal to one term. If by chance there are on either side or on both sides any negative terms, it will be necessary to add the negative terms on both sides, until the terms on both sides are positive, and then again to subtract like from like until one term only is left on each side.

This is still the same method we use to introduce school children to algebra to this day.

Methods for solving Indeterminate Equations:

In Book II we encounter squares and cubes. Of particular note is problem 8, since it is to this problem which Fermat appended his famous "Last Theorem."

> To divide a given square number into two squares.
> Given square number 16.
> x^2 one of the required squares. Therefore $16-x^2$ must be equal to a square.
> Take a square of the form $(mx-4)^2$, m being any integer and 4 the number which is the square root of 16, *e.g.* take $(2\text{x}-4)^2$, and equate it to $16-x^2$.
> Therefore $4\text{x}^2-16\text{x}+16=16-x^2$, or $5\text{x}^2=16\text{x}$, and $x=\frac{16}{5}$.
> The required squares are therefore $\frac{256}{25}, \frac{144}{25}$.

Now, recall from our discussion on notation that Diophantus was only able to work with one unknown quantity at a time. Problem 8 illustrates how adroitly Diophantus is able to handle this limitation. He notes explicitly where he chooses an arbitrary integer (called *m* in Heath's translation), and following the method which follows an interested student could find any number of other solutions. In this way, although he cannot express a general solution, Diophantus can describe a method by which any number of solutions can be obtained. For example, it is easy to see that setting m = 4 produces the pair of squares $\frac{1024}{289}, \frac{3600}{289}$.

Now let us move on to Book IV, where Diophantus introduces the technique known as the "false hypothesis," perhaps the most modern of the many methods he introduces. This is problem IV,15:

> To find three numbers such that the sum of any two multiplied into the third is a given number.
> Let (first + second) x third = 35, (second + third) x first = 27 and (third + first) x second = 32.
> Let the third be x.
> Therefore, (first + second) = 35/x.
> Assume first = 10/x, second = 25/x; then
> $$\frac{250}{x^2}+10=27$$
> $$\frac{250}{x^2}+25=32$$
> *These equations are inconsistent; but they would not be if* 25 – 10 *were equal to* 32 – 27 *or* 5.
> Therefore we have to divide 35 into two parts (to replace 25 and 10) such that their difference is 5. The parts are 15, 20.
> We take therefore 15/x for the first number, 20/x for the second, and we have
> $$\frac{300}{x^2}+15=27$$
> $$\frac{300}{x^2}+20=32$$
> Therefore x = 5, and (3,4,5) is a solution.

Works Cited

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A note on citing from the *Arithmetica*: There is very little agreement in problems between the texts of Heath and Sesiano. Unless otherwise noted, *Arithmetica* citations refer to Heath.