Polygonal Numbers

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Introduction to Number Theory

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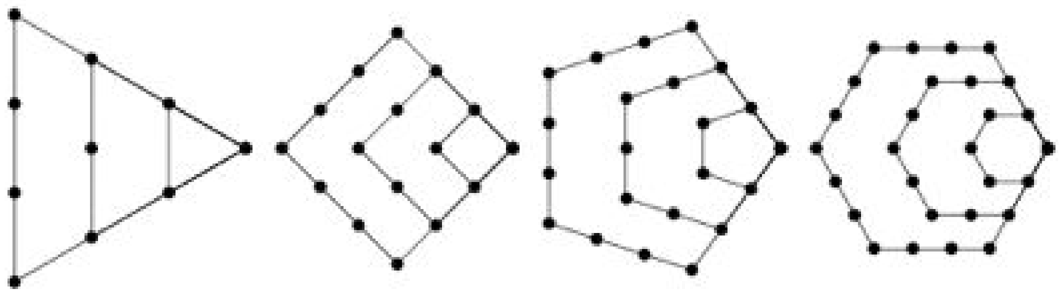
Introduction:

Polygonal numbers are number representing dots that are arranged into a geometric figure. Starting from a common point and augmenting outwards, the number of dots utilized increases in successive polygons. As the size of the figure increases, the number of dots used to construct it grows in a common pattern. The most common types of polygonal numbers take the form of triangles and squares because of their basic geometry. *Figure 1* illustrates examples of the first four polygonal numbers: the triangle, square, pentagon, and hexagon.

Figure 1:



http://www.trottermath.net/numthry/polynos.html

As seen in the diagram, the geometric figures are formed by augmenting arrays of dots. The progression of the polygons is illustrated with its initial point and successive polygons grown outwards. The basis of polygonal numbers is to view all shapes and sizes of polygons as numerical values.

History:

The concept of polygonal numbers was first defined by the Greek mathematician Hypsicles in the year 170 BC (Heath 126). Diophantus credits Hypsicles as being the author of the polygonal numbers and is said to have came to the conclusion that the n^{th} a-gon is calculated by the formula $\frac{1}{2}*n*[2 + (n - 1)(a - 2)]$. He used this formula to determine the number of elements in the n^{th} term of a polygon with a sides.

Before Hypsicles was acclaimed for defining polygonal numbers, there was evidence that previous Greek mathematicians used such figurate numbers to create their own theories. One example of the use of polygonal numbers even before Hypsicles is in Pythagorean's theorem. In his studies, Pythagoras established his famous theorem by discovering that the area of a square with the same length of a side of a right-triangle, plus the area of a square with the same length of the other adjacent side of a right-triangle, is equal to the area of a square with the same length of the hypotenuse of the same triangle (Heath and Diophantus 80). Thus he came up with the formula that $a^2 + b^2 = c^2$, where a and b are the sides of a triangle and c is the hypotenuse. By visualizing geometric shapes as numbers, he discovered one of the most utilized properties of a triangle vastly used in today's geometry.

Other important mathematicians who studied polygonal numbers included Theon of Smyrna, and Nicomachus. In their own ways, they determined how to obtain polygonal numbers of from the combination of other polygonal numbers of lower degree (Heath 126).

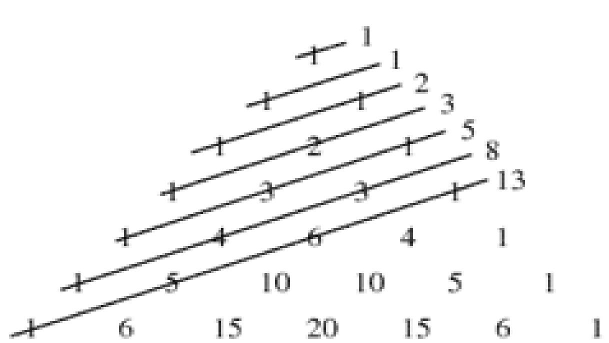
A modern use of the polygonal numbers is found in Pascal's triangle. It is an application that displays the coefficients that arise in binomial expansions in the form of a triangle. Pascal's triangle has clear roots in the triangular numbers in the way that the numbers are geometrically arranged. The only modification that it implements is that numbers are represented as the points in the arrays constructing the triangle instead of dots. The basic display of the triangle has the rows staggered to have it in the shape of a pyramid as seen in the picture below.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

http://daugerresearch.com/vault/parallelpascalstriangle.shtml

The triangle is broken down into rows starting with the number one in row one. The elements in the rows grow in increments of one with two elements in the second row, three elements in the third row and so forth. Each number is generated by adding the two numbers directly above it. Each element in the triangle is indexed by nCr, where *n* is the row starting with the number zero and *r* is the element in the row chosen, also starting with the number zero. The purpose of the triangle is to determine the coefficients of a binomial expression such as $(x + y)^2$. This expression is expanded out to $x^2 + 2xy + y^2$. It is clear that the coefficients for the equation are one, two, and one which is exactly the third row of Pascal's triangle.

Pascal's triangle has been carefully studied since it's been created. One great discovery made from the triangle is the Fibonacci numbers. The Fibonacci numbers are the sequence of numbers where the next element of the sequence is the sum of the previous two elements starting from the number one. This sequence can be obtained by adding the shallow diagonals viewed in the picture below:



http://mathworld.wolfram.com/PascalsTriangle.html

More importantly, the triangular numbers can be seen in the triangle as well. The diagonal that connects the first element in row three and the third element in row five is the exact series of numbers that make up the successive triangular numbers. The third diagonal consists of 1, 3, 6, 10, 15 which so happens to be the first five triangular numbers. This is the case because the diagonal can also be represented as the series of 2C2, 3C2, 4C2, etc.

Fermat was another great mathematician that developed theories in the world of polygonal numbers. He proposed that all whole numbers could be represented as the sum of at three triangular numbers or less. The sequence of triangular numbers can bee seen in the chart and goes from 1, 3, 6, 10, etc. An example of Fermat's theorem is the number 100 being represented with three triangular numbers.

(1) $100 = 91 + 6 +3 = T_{13} + T_3 + T_2$

(2) $100 = 45 + 55 = T_{10} + T_9$

As seen in the example above, the number 100 can be broken down in many different ways. With the use of triangular numbers, 100 can be represented as the sum of three triangular numbers seen in the first example or even two triangular numbers, as seen in the second example.

<u>Theory</u>:

As mentioned previously, the basic formula for deriving the n^{th} a-gonal number is:

$$p_a(n) = \frac{n*[2 + (n - 1)(a - 2)]}{2} \qquad (1)$$

This implies that the formula for a triangular number is:

$$p_3(n) = \frac{n*(n+1)}{2}, \qquad (2)$$

and the first 5 triangular numbers are:

$p_3(1) = 1$
$p_3(2) = 3$
$p_3(3) = 6$
$p_3(4) = 10$
$p_3(5) = 15$

These numbers can be represented as figures, by starting at one point and augmenting out, as shown in *figure 2.*

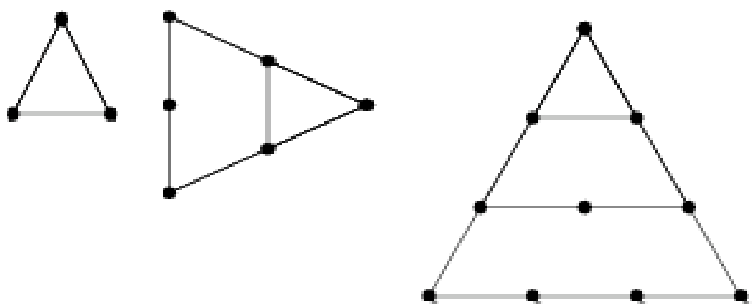


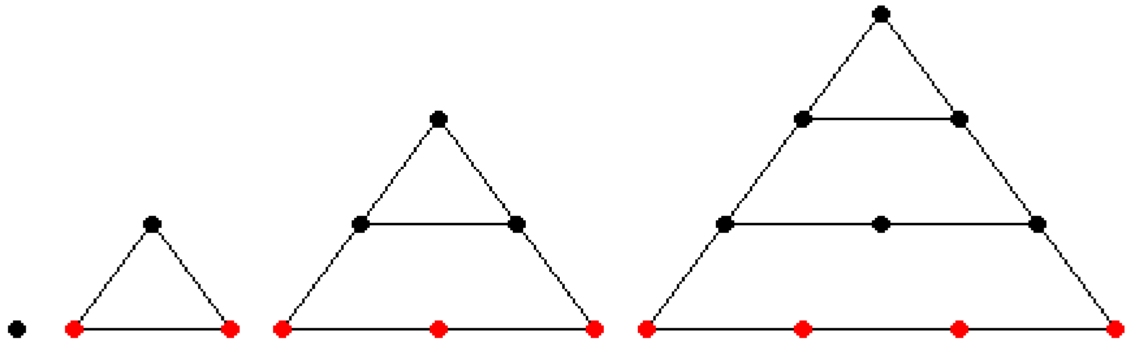
Figure 2: the figure illustrates the growth of a triangular number.

From left to right: $n = 2, n = 3, n = 4$.

Note that the total number of dots in each triangle, starting from the first row down to the n^{th}, equals $p_3(n)$. This general pattern holds for all $p_a(n)$. Polygonal numbers can also be described as sets of n terms rather than diagrams. Let S_a = {the first n a-gonal numbers}, so $|S_a| = n$, and the n^{th} term of S_a is $p_a(n)$.

Derivation of the general formula.

Polygonal numbers can be expressed as a sequence, where each element in the sequence is the number of dots to be added to the polygon as it is augmented. Take the triangular numbers, for example, starting with a single dot when $n = 1$. Next, two dots are added, then three, etc, as illustrated in *figure 3*.



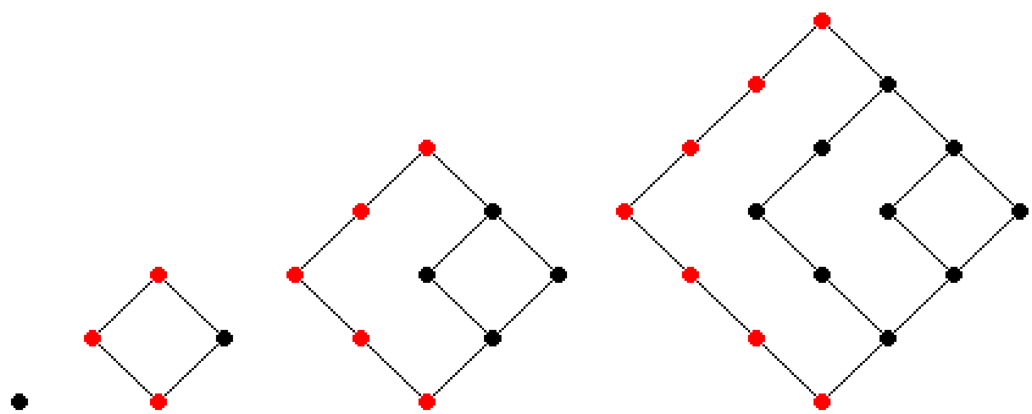
n =1 *n =2* *n = 3* *n = 4*

<u>Figure 3</u>: Augmentation of triangular numbers, from left to right

The red dots represent the elements of the sequence. This means that the total number of dots, $p_3(n)$, is the sum of the elements:

$$p_3(n) = \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + (n - 1) + n = \frac{n\,(n + 1)}{2}$$

Now, the same analysis can be made for square numbers: there is a single dot when $n = 1$. Next, three dots are added to form a square. The square augments as the length of each side increases by one dot as shown in *<u>figure 3</u>*:



n = 1 *n = 2* *n = 3* *n = 4*

<u>Figure 3</u>: Augmentation of square numbers, from left to right.

Notice how the length of each side of the squares increases by one dot as the value of n increases. The red dots represent the elements of the sequence. Therefore, the value of $p_4(n)$ is:

$$p_4(n) = \sum_{i=1}^{n} 2*i - 1 = 1 + 3 + 5 + ... + (2*(n - 1) - 1) + (2*n - 1) = n^2$$

Note that the sequences for triangular numbers ($a = 3$), and square numbers ($a = 4$) are represented by n, and $2*n - 1$, respectively. The triangular numbers can be rewritten as:

$$(a - 2)*n - (a - 3),$$

knowing that $a = 3$.

$$\begin{aligned} &(3 - 2)*n - (3 - 3) \\ &= n \end{aligned}$$

Similarly, when $a = 4$:

$$\begin{aligned} &(a - 2)*n - (a - 3) \\ &= (4 - 2)*n - (4 - 3) \\ &= 2*n - 1, \end{aligned}$$

which, in fact, is the formula for the square numbers.

Therefore, in the general case:

$$\begin{aligned} p_a(n) &= \sum_{i=1}^{n} (a - 2)*i - (a - 3) \\ &\Leftrightarrow \sum_{i=1}^{n} (a - 2)*i - \sum_{i=1}^{n}(a - 3) \\ &\Leftrightarrow (a - 2) * \sum_{i=1}^{n} i - (a - 3) \sum_{i=1}^{n} 1 \\ &\Leftrightarrow (a - 2)*\frac{n*(n + 1)}{2} - (a - 3)*n \\ &\Leftrightarrow (a - 2)*\frac{(n^2 + n)}{2} - a*n - 3*n \\ &\Leftrightarrow \frac{a*n^2 + a*n - 2*n^2 + 2*n}{2} - a*n - 3*n \\ &\Leftrightarrow \frac{n*(2 + (n - 1)*(a - 2))}{2} = p_a(n) \end{aligned}$$

Analysis:
Taking a closer look at polygonal numbers reveals some interesting characteristics and patterns within them:

Triangular numbers are in all polygonal numbers.

Triangular numbers are the basis for other polygonal numbers of higher degree.

Consider the formula for $p_a(n)$:

$$p_a(n) = \frac{n*(2 + (n - 1)*(a - 2))}{2},$$

which can be rewritten as:

$$p_a(n) = n + \frac{n*(n - 1)}{2}*(a - 2)) . \qquad (4)$$

Note that $\frac{n*(n - 1)}{2} = p_3(n - 1)$, so (4) is the same as:

$$p_a(n) = n + p_3(n-1)*(a - 2). \qquad (5)$$

Table 1 shows the first 9 triangular to decagonal numbers. Notice that the numbers in each column increase by the same number. It is no coincidence that the elements of the n^{th} column increase by increments of $p_3(n - 1)$.

Table 1:

NAME	n=1	2	3	4	5	6	7	8	9
a = 3	1	3	6	10	15	21	28	36	45
a = 4	1	4	9	16	25	36	49	64	81
a = 5	1	5	12	22	35	51	70	92	117
a = 6	1	6	15	28	45	66	91	120	153
a = 7	1	7	18	34	55	81	112	148	189
a = 8	1	8	21	40	65	96	133	176	225
a = 9	1	9	24	46	75	111	154	204	261
a = 10	1	10	27	52	85	126	165	232	297

By expanding and rearranging formula (4):

$$p_a(n) = \frac{n + n*(n - 1)*(a - 2))}{2}$$

$$= \frac{2*n + a*n^2 - 2*n^2 - a*n + 2*n}{2}$$

$$= \frac{(5*n - n) + a*n^2 - (3*n^2 - n^2) - a*n}{2}$$

$$= \frac{(5*n + a*n^2 - 3*n^2 - a*n)}{2} + \frac{(n^2 - n)}{2}$$

$$= p_{a-1}(n) + p_3(n - 1)$$

Indeed, $p_a(n) - p_{a-1}(n) = p_3(n - 1)$.

If all polygonal numbers are related to triangular numbers, under what circumstances are they equal to each other? One trivial answer to this question is that all polygonal numbers appear at least twice. When $n = 1$, all a-gonal numbers equal a. This is simply because of the fact that when $n = 1$ the graph for that polygon is it's simplest one, with each side consisting of only one edge. However, since the sequence of polygonal numbers is infinite, any a-gonal number corresponding to $n > 1$ will equal the a^{th} a-gonal number when $n = 1$.

<u>Conclusion</u>:

Polygonal numbers has been meticulously studied since their very beginnings in ancient Greece. Numerous discoveries stemmed from these peculiar numbers and can be seen in the basic fundamental groundwork of number theory today. With finding such as Pascal's triangle and Fermat's triangular number theorem, polygonal numbers has become a popular field of research for mathematicians.

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