

The Pascal Triangle

History Blaise Pascal was born at Clermont-Ferrand, in the Auvergne region of France on June 19, 1623. In 1653 he wrote the Treatise on the Arithmetical Triangle which today is known as the Pascal Triangle. Although other mathematicians in Persia and China had independently discovered the triangle in the eleventh century, most of the properties and applications of the triangle were discovered by Pascal. This triangle was among many of Pascal's contributions to mathematics. He also came up with significant theorems in geometry, discovered the foundations of probability and calculus and also invented the Pascualine-calculator but he is best known for his contributions to the Pascal triangle¹.

Construction of The Triangle The easiest way to construct the triangle is to start at row zero and write only the number one. From there in order to obtain the numbers in the following rows, add the number directly above and left to the number directly above and right to acquire the new value. If there are no numbers on the left or the right just replace a zero for that missing number and proceed with the addition. Here is an illustration of rows zero to four:

$$\begin{array}{ccccccc}
 & & & & 1 & & & & \\
 & & & & 1 & & 1 & & \\
 & & & 1 & & 2 & & 1 & \\
 & & 1 & & 3 & & 3 & & 1 \\
 1 & & 4 & & 6 & & 4 & & 1
 \end{array}$$

Binomial Coefficients and Pascal's Rule This triangle represents the binomial coefficients in a geometric way. The relationship between the numbers in the triangle and the binomial coefficients is shown by Pascal's rule which states:

If $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the k th binomial coefficient in the binomial expansion

of $(x+y)^n$, where $n!$ is the factorial of n , then $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for any nonnegative integer n and any integer k between zero and n . Therefore, the coefficients in a binomial expansion correspond to the rows in Pascal's triangle²:

$$\begin{array}{lcl}
 (x+1)^0 = & & 1 \\
 (x+1)^1 = & & 1 + x \\
 (x+1)^2 = & & 1 + 2x + x^2 \\
 (x+1)^3 = & 1 + & 3x + 3x^2 + x^3 \\
 (x+1)^4 = & 1 + & 4x + 6x^2 + 4x^3 + x^4
 \end{array}$$

¹ <http://pages.csam.montclair.edu/~kazimir/history.html>

² http://en.wikipedia.org/wiki/Pascal's_triangle

$$(x+1)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \dots$$

There are $n+1$ terms in the expansion of $(x+y)^n$ and the degrees on each term is n . The powers of x begin with n and decrease to 0 whereas the powers on y begin with 0 and increase to n . The coefficients which are the rows on Pascal's triangle are always symmetric³.

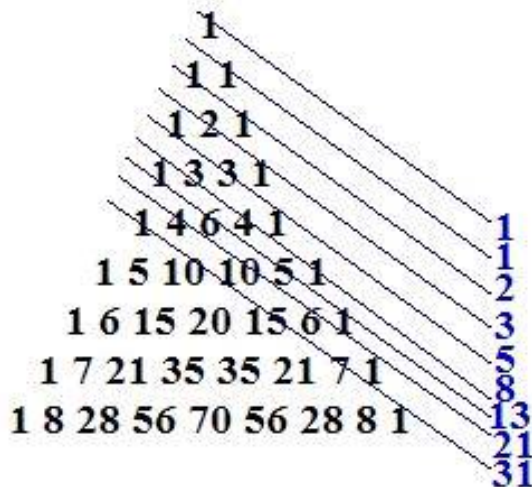
Patterns of the Pascal Triangle Pascal's triangle has various patterns and applications.

1) Addition of the Rows One of the interesting properties of the triangle is that the sum of its rows equal 2^n where n corresponds to the number of the row:

$$\begin{aligned} 1 &= 1 = 2^0 \\ 1 + 1 &= 2 = 2^1 \\ 1 + 2 + 1 &= 4 = 2^2 \\ 1 + 3 + 3 + 1 &= 8 = 2^3 \\ 1 + 4 + 6 + 4 + 1 &= 16 = 2^4 \end{aligned}$$

2) Prime Numbers in the Triangle Another pattern visible in the triangle deals with the prime numbers. If a row starts with a prime number or is a prime numbered row, all the numbers that are in that row (not counting the 1's) are divisible by that prime. If we look at row 5 (1 5 10 10 5 1) we see that 5 and 10 are divisible by 5. However for a composite numbered row, such as row 8 (1 8 28 56 70 56 28 8 1) 28 and 70 are not divisible by 8.

3) Fibonacci Sequence in the Triangle By adding the numbers in the diagonals of the Pascal triangle the Fibonacci sequence can be obtained:



There are various ways to show the Fibonacci numbers on the Pascal triangle. R. Knott was able to find the Fibonacci appearing as sums of "rows" in the Pascal triangle. He

³ <http://people.richland.edu/james/lecture/m116/sequences/binomial.html>

moved all the rows over by one place and here the sums of the columns would represent the Fibonacci numbers⁴.

	0	1	2	3	4	5	6	7	8	9
0	1									
1		1	1							
2			1	2	1					
3				1	3	3	1			
4					1	4	6	4	1	
5						1	5	10	10	5
6							1	6	15	20
7								1	7	21
8									1	8
9										1
	1	1	2	3	5	8	13	21	34	55

4)Combination and Probabilities Pascal's triangle is also useful for understanding combinations and calculating their probability. For example it can show us how many ways heads and tails combine⁵:

Tosses	Possible Results (Grouped)	Pascal's Triangle
1	H T	1, 1
2	HH HT TH TT	1, 2, 1
3	HHH HHT, HTH, THH HTT, THT, TTH TTT	1, 3, 3, 1
4	HHHH HHHT, HHTH, HTHH, THHH HHTT, HTHT, HTTH, THHT, THTH, TTHH HTTT, THTT, TTHT, TTTH TTTT ... etc ...	1, 4, 6, 4, 1

In this example we can see that if we toss a coin four times, there's only one combination that gives us four heads and only one combination that gives us four tails. However, we get 4 combinations that give us three tails and one head and four combinations that give us three heads and one tail. There are 6 combinations that give us two heads and two tails. The number of combinations is determined by the numbers in the corresponding

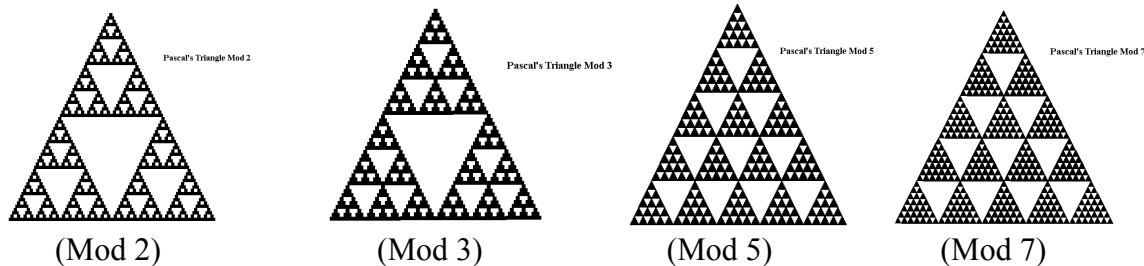
⁴ <http://milan.milanovic.org/math/english/fibo/fibo2.html>

⁵ <http://www.mathsisfun.com/pascals-triangle.html>

row of the Pascal triangle. In this case, in order to calculate the probability of two tails we need to consider the number of possibilities ($1+4+6+4+1=16$) in which 6 of them would give exactly two tails so the probability is $6/16$ or 37.5%. The number of ways of selecting r objects from a total of n is written as nC_r where n represents the number of the row and r represents the number of the element in the corresponding row. The Formula for it is: ${}^nC_r = n! / (n - r)! r!$ ⁶. For example, there are 10 ways of selecting 2 books from 5 ${}^5C_2 = 10$ and we can clearly see that the second element in the fifth row is 10. We can also calculate this number using the formula: ${}^5C_2 = 5!/3!2!=10$.

5) Visual Patterns of Different Modulas on the Triangle By

calculating the different modula's on each row of the Pascal triangle we get some interesting visual patterns. Let the empty space in the following images represent the zeros in the rows. First we try to find the results for the prime modules:



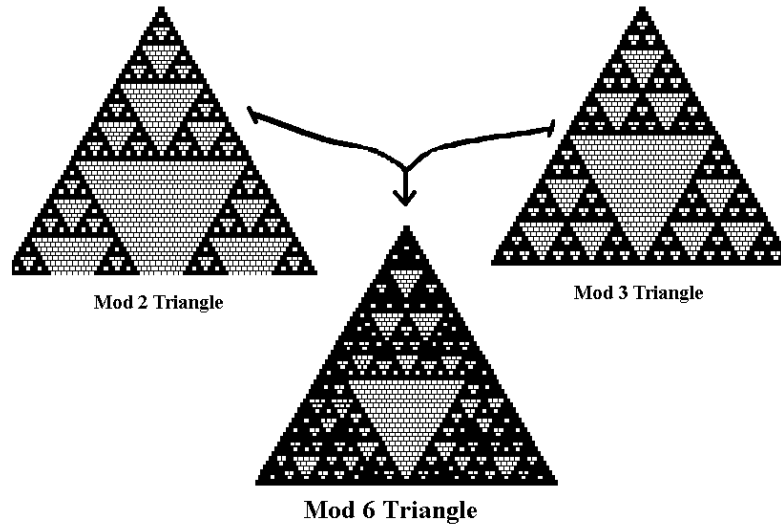
When the modulus is prime the triangular building block is the triangle that is built between row zero and p^2 . For example, the building block for modulus 2 is the triangle formed between row zero and row 4. There are a number of smaller triangles within each building block⁷. The number of these small triangles within the building blocks correspond to the element in the triangle which is after the prime number in the prime numbered row. For example the number of the small triangles within the building blocks of the triangle mod 7 can be found on row 7 after the number 7 which is 21.

If the modulus is a composite number there is an easy way to construct the triangle. We reduce the modulus to its prime factors. For example, $6=2*3$ so we know that the white spots, which represent the zeros, in the triangle mod 6 must also be white spots in the triangle mod 3 and mod 2 since any binomial coefficient divisible by 6 is also divisible by 2 and 3. The black spots in the triangles 2 and 3 are also black spots in the mod 6 triangle. Therefore in order to construct the mod 6 triangle we could put copies of the mod 2 and mod 3 triangles on top of each other. Here the only white spots would correspond to the binomial coefficients that are divisible by 2 and 3 and therefore by 6: ⁸

⁶ <http://mathforum.org/dr.math/faq/faq.comb.perm.html>

⁷ <http://www-math.cudenver.edu/~wcherowi/jcorn5.html>

⁸ <http://www-math.cudenver.edu/~wcherowi/jcorn5.html>



Additional Suggestions on the Triangle I was able to find an interesting pattern in the Pascal triangle by calculating the greatest common divisor (GCD) of the two different middle numbers in each row (starting from row 7). For example, in row 9 the $\gcd(126, 84) = 42$ and in row 20 the $\gcd(167680, 184436) = 4$. The results are summarized in the table below:

Number of Row	GCD of two different middle elements
7	7
8	4
9	42
10	42
11	66
12	132
13	429
14	429
15	715
16	30
17	2
18	2
19	2
20	4
21	18
22	18

The GCDs in the prime numbered rows always divide that prime number. So 7 divides 7, 429 divides 13, 4862 divides 17 and etc... This is obvious by the property of prime numbered rows on the triangle discussed above. Another interesting property that I was able to find was that the GCDs of the middle numbers in rows 9 and 10 are equal. Then two rows after that we see the same property between rows 13 and 14 and again two rows

after we see the same property for rows 17 and 18 and then again two rows after for 21 and 22. So in periods of two rows we get the same GCDs for the middle numbers in the rows.

I also took the GCDs of the first two element in each row starting with the 5th row and realized that the GCDs of the first two elements in odd numbered rows are equal to the number of the row. For example in row 9 the $\gcd(9,36)=9$ and in row 15 the $\gcd(15,105)=15$. This is because the second element in the odd numbered rows is always a multiple of the odd number. In the even numbered rows the GCDs of the first two elements is always half the number of the row. For example, in row 10 the $\gcd(10,45)=5$ and in row 12 the $\gcd(12,66)=6$. The results are shown in the table below:

Number of Row	GCD of first two elements
5	5
6	3
7	7
8	4
9	9
10	5
11	11
12	6
13	13
14	7
15	15
16	8
17	17
18	9
19	19

Another interesting property lies in the numbers that lie in the center line of the Pascal triangle. The even numbered rows have only one number in the center. For example in row four the number 6 is in the middle and row 8 has the number 70 in the middle. These numbers (disregarding the 1's) are all even because we get them by adding two odd numbers or two even numbers. There are two middle numbers in the odd numbered rows. For example in row 7 we have the two 35's as the middle numbers, in row 5 we have the two 10's as the middle number and so on. These middle numbers are not always even like the middle numbers of the even numbered rows. Sometimes they are not even because we get them by adding an odd and an even number. Rows 3, 7 and 15 have odd numbers in their center.

References

- <http://pages.csam.montclair.edu/~kazimir/history.html>
- http://en.wikipedia.org/wiki/Pascal's_triangle
- <http://people.richland.edu/james/lecture/m116/sequences/binomial.html>
- <http://milan.milanovic.org/math/english/fibo/fibo2.html>
- <http://www.mathsisfun.com/pascals-triangle.html>
- <http://www-math.cudenver.edu/~wcherowi/jcorn5.html>
- <http://pages.csam.montclair.edu/~kazimir/history.html>
- http://www.thocp.net/biographies/pascal_blaize.html
- <http://www.ideafinder.com/history/inventors/pascal.htm>
- <http://mathforum.org/dr.math/faq/faq.comb.perm.html>