Workshop 3- Asking Questions, Making Conjectures

The goal of this workshop was to show the students the mathematics in progress: how a mathematician comes with new questions after observing data and revisiting known facts, by using generalization, analogy, and, well, intuition and curiosity. Each student was asked to make her own questions and conjectures, and then the whole class discussed them and come with ideas, solutions, conclusions.

Here is how some of the students in the class viewed this workshop:

Samantha:
I was actually really glad that you chose to make "Research in Number Theory" a workshop topic. Primarily because as students we often get consumed in trying to learn and understand preexisting findings, often forgetting that there is an entire unknown realm of information to explore.
And secondly because the method of generating questions and conjectures can be intimidating, but by looking at the three techniques (generalizing, analogies, and observing data pattens in numerical data) you have made this task feasible for even the college student.
When discussing the questions, which primes are differences of 2 squares or which primes are sums of 2 cubes, we were able to arrive at a concrete answer by applying our previous knowledge.
This just shows how mathematics continues to builds on itself and our current knowledge can be applied to a new area of study. The Goldbach conjecture and analogous question, which integers could be written as difference of two primes are the most exciting and frustrating aspects of mathematics, it keeps you interested because we have yet to find a conclusion but anxious to either prove or disprove for once and for all. And the truth is an infinite number of examples will never and can never replace the strength of a proof.

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Susan:
Workshop #3 was on research in number theory. Kalin approached this topic by first having us brainstorm about what we think it means to do research in mathematics. He was able to combine our ideas with his to form three overarching categories of math research: generalization, analogy, and observing patterns from data. For the first hour and a half of the workshop Kalin had us focusing on the first two methods of research by drawing analogous questions to ones we had been working with in class or in the lecture notes, and then turning these specific questions into more broad, generalizations. After some examples from Kalin and a few minutes for thinking, the class was able to compile a list of about a dozen number theoretical questions. Kalin then identified a few that we could explore together easily: his question about prime differences or two squares, Samantha's question about integer differences of two primes, and then mine about prime sums of two cubes. What was most interesting to me from this exercise was discovering that a small modification to a problem can make it simple to prove (like the prime sums of two cubes) or currently impossible to prove (like the possibility to infinitely many prime sums of the form k^2 +1). Unfortunately we then ran out of time to explore patterns in data, but I look forward to that for next class!
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Cicek:

Our third workshop was one of the most productive ones in terms of brainstorming. We tried to experience a mathematician's workday; asking questions and making conjectures with a mathematician's way of thinking. We came up with more than 15 questions, some were similar to each other, some challenging, and some that forced us to think in more than one dimension. Here are my answers to some of the questions that were discussed during the workshop:

Q-For which $a \in N$, there exists $n \in N$, $n > 1$, such that "a^n-1" is prime?
A-And I followed this way for the argument: $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \dots + a+1)$

Another question regarding prime numbers:
Q-Which primes are a difference of two squares?
A-Our thinking follows, $p= x^2 + y^2 = (x+y)(x-y)$
Say, $x-y=1$
$x=y+1 \rightarrow p=2y+1$ for every odd prime. This will not give us a prime number for every integer y, but we can certainly obtain infinitely many prime numbers this way.

Q- Let 2n be an even number. Are there primes p,q such that p-q = 2n?
A-Actually the result is going to be even for every prime as long as the condition, p or $q \neq 2$, is satisfied. However, when we pick a random even number, finding prime numbers makes our job harder. But we can start by creating this list:
2 = 7 – 5
4=7 – 3
6=19 – 13
8=11 – 3
:
20=31 – 1
:

Q- Are there primes p, q, r such that p+q=r ?
A-We can start by stating, for this condition to occur, we have to have one of p or q =2, because the sum of two odd numbers will give us an even number, which certainly will not be a prime, unless it is 2.
We can start our list:
2 + 5 = 7
2 + 11 = 13
2 + 17 = 19
2 + 71= 73
:
2 + 1427 = 1429
:
2 + 1697 = 1699
:
2 + 2141 = 2143

I have explored some of the questions from the workshop which I found most interesting and fun to do.

Brainstorming is always good, and healthy!

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Joe:

In this workshop we tried to come up with a few conjectures to test. One thing I had mentioned is that the quadratic residue group of Z_pq, where p and q are prime, should be one quarter of the size of Z_pq (which is the multiplicative group Z_pq, not the additive one). This makes sense intuitively if you view the Chinese Remainder Theorem as a ring isomorphism between Z_p + Z_q -> Z_pq, where here + denotes the direct sum: an element in Z_p + Z_q looks like (a, b) with a in Z_p and b in Z_q. The CRT will map (a, b) to a unique element in Z_pq. Since multiplication is performed coordinate-wise, it is easy to see that (a, b) is a square if and only if both a and be are squares. Thus, we should have (a, b) in QR_pq exactly when a in QR_p and b in QR_q. Since we know that |QR_p| = (p - 1)/2 and |QR_q| = (p - 1)/2, there should be exactly

$$(p - 1)/2 * (q - 1)/2 = (p - 1)(q - 1) / 4 \qquad (*)$$

choices for pairs of squares a and b; and each of these squares maps to a unique element of Z_pq by the CRT. But we know that |Z_pq| = phi(pq), where phi() denotes the Euler totient function. Thus, we know that |Z_pq| = phi(pq) = (p - 1)(q - 1).

Comparing this to the number of choices for pairs of squares in eq. (*), we see that exactly one quarter of the elements in Z_pq map to pairs of squares in Z_p + Z_q. But since the CRT is an /isomorphism/ between the two groups, we know that these must be exactly the squares of Z_pq. Thus we conclude that |QR_pq| * 4 = |Z_pq|.

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David:

In workshop on Monday we posed our own original (sometimes) questions and subsequently attempted to answer them, occasionally even with some success. For instance, we generalized the Chinese remainder theorem to moduli that are not relatively prime. In this case, there is a unique solution if and only if for every d that divides any pair of moduli, m1 and m2, d divides a1 and a1-a2, with a1 and a2 in the equations x=a1(m1) and x=a2(m2). We also showed that every odd prime can be written as a difference of two squares and that no primes except for two can be written as the sum of two cubes. The only other question we answered conclusively was that a^n-1 can only be a prime if a=2.

I conjectured that every even number can be written as a difference of two primes and another student guessed that there are infinitely many primes that are the sum of three consecutive primes, but we couldn't prove either conjecture.

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Tina:

In this workshop we tried to explore some mathematical questions and came up with some answers and made some conjectures.

Some of the questions were

1. Could we get rid of the coprime restriction in Euler's theorem;
2. Could we get rid of the coprime restriction of the mods in the Chinese remainder theorem?
3. Let 2n be an even number. Are there primes p,q such that p-q=2n
4. Are there primes p,q,r such that p+q=r
5. Let p be a prime, when is the period of the decimal fraction 1/p a prime as well?
6. Which primes are a difference of two squares
7. Which primes are the sum of two cubes
8. Which primes are the sum of a square and a cube

We were also able to come up with some answers:
For example we came up with an answer for question 6: and realized that it works for every odd prime
The answer to question 7 is that it only works for 2
There were other questions which we disregarded as too difficult, like question 5.
Overall we were able to question some of the restrictions on theorems to see if we could come up with something better that did not contain that restriction. Then we moved on to the prime numbers and brought up some questions. Some were solved, some were disproved by counter examples and some remained open.

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Brendan:

In Monday's workshop we came up with a list of good mathematical questions stemming from the topics and theorems we have discussed so far in the lectures.
Some of the questions looked at properties of prime numbers, some looked at the sum of squares, and some were more or less original in thought. The question I posed was basically this: Let p, q, r be consecutive primes. When is their sum a prime number itself? This question was difficult to answer, and we were able to come up with guidelines and some logical theories but not an explicit answer. Actually, the majority of the questions we posed were too difficult to answer in the last hour of class. This exercise was very useful, however, in that it exposed us to the fundamental approach to number theory. The process is a very simple but important one: come up with a question, and answer it. This is the most elemental form of exploring number theory, but it is doubtlessly the way in which many of the most famous and important mathematical theories we know came into existence. We were able to realize that many of these questions can take a long time to answer, and it is apparent that there are still infinitely many questions left to answer today.