Boston University
Summer I 2009

Number Theory
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Homework No.X

due 06/22/2009

Problem A: Compute the first 7 digits of the p-adic number x if
a) $p = 11$ and $5x = 1$;
b) $p = 13$ and $x^2 = -1$;

Problem B: Working in the ring of Gaussian integers, find exponents n such that
a) $(1+i)^n \equiv 1 \pmod 3$;
b) $(1+i)^n \equiv 1 \pmod 5$;

Problem C: In this exercise we will be investigating the equation $y^2 = x^3 - 2x$. Write a program, that takes as an input two positive integers, x and y, and a prime p, and then prints YES or NO depending on whether (x, y) is solution of the congruence $y^2 \equiv x^3 - 2x \pmod{p}$. (2 points)
For example, given $x = 2, y = 3$ and $p = 7$ your program should print "NO" since $3^2 \not\equiv 2^3 - 2 * 2 \pmod{7}$. Or, given $x = 1, y = 5$ and $n = 13$ it should print "YES", since $5^2 \equiv 1^3 - 2 * 1 \pmod{13}$.
Modify this program to make a second program which takes as an input a prime number p and then goes through all the pairs $(x, y), 0 \leq x, y \leq p - 1$ to count the number of solutions of the congruence $y^2 \equiv x^3 - 2x \pmod{p}$. (2 points)
Use this program and make a table with two rows, in the first one put all the primes less than a 100, and in the second one put the number of solutions you get from the second program for the relevant prime. (3 points)
Use your data to make conjectures about the values you see. (3 points)
Send me by e-mail the source code of the two programs, the table with the data, and the text of your conjectures.

Problem D: Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ be the Riemann zeta function. Prove the formal identity

$$\zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}$$

where $\sigma(n)$ is the sum of all the divisors of the natural number n.
Use the Euler identity $\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod (1 - p^{-s})^{-1}$, where the product is over all prime natural numbers, and the fact that $\sigma(n)$ is a multiplicative function.