Spring Term 2008 MA412 Complex Variables
Boston University prof. T. Kaper

Solutions to Selected Homework Problems

written by: Kalin Kostadinov

Exercise 1. *Verify that* $\sqrt{2}|z| \geq |Re(z)| + |Im(z)|$.

Let $z = x + iy$ be an arbitrary complex number. Then $|z| = \sqrt{x^2 + y^2}$, and we have to show that

$$\sqrt{2}\sqrt{x^2 + y^2} \geq |x| + |y|$$

Since both parts of this inequality are non-negative, squaring it produces the equivalent inequality

$$2(x^2 + y^2) \geq x^2 + y^2 + 2|x| \cdot |y|$$

which readily simplifies to

$$(|x| - |y|)^2 \geq 0$$

The last one is always true, so we are done.
We remark that equality is reached precisely when $|x| = |y|$, i.e. when $\arg z$ is of the form $(2k+1)\frac{\pi}{4}$.
Note that the inequality we proved has a geometric interpretation as well: It says that in an arbitrary right triangle, the sum of the length of the legs never surpasses $\sqrt{2}$ times the length of the hypotenuses.

Exercise 2. *Using the fact that* $|z_1 - z_2|$ *is the distance between two points* z_1 *and* z_2, *give a geometric argument that*
(a) $|z - 4i| + |z + 4i| = 10$ *represents an ellipse whose foci are* $(0, \pm 4)$;
(b) $|z - 1| = |z + i|$ *represents the line through the origin whose slope is* -1.

Solution: An ellipse is the set of points in the plane whose sum of distances to two fixed points in the plane (called foci of the ellipse) is constant. The equation in (a) describes the set of points z in the complex plane such that the sum of distances from each such point to the points $z_1 = 4i$ and $z_2 = -4i$ is 10, a constant. So the set is the ellipse whose foci are $(0, \pm 4)$.
For part (b) note that the set of point equidistant from two points in the plane is a line that goes through the middle of the segment bounded by the points and perpendicular to that segment (the so called *perpendicular*

bisector.
Hence the equation $|z-1| = |z+i|$ describes the perpendicular bisector of the segment bounded by the points $z_1 = 1 = (1,0)$ and $z_2 = -i = (0,-1)$. Recall that the slopes of perpendicular lines multiply to -1 and notice that the slope of the line through 1 and i is 1. So the slope of the perpendicular bisector is -1, as was to be shown.

Exercise 3. *By factoring* $z^2 - 4z^2 + 3$ *into two quadratic factors and then using inequality* (8), *Sec.* (4), *show that if* z *lies on the circle* $|z| = 2$, *then*

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}$$

Solution: First we use a little algebra and the fact that the absolute value of a quotient equals the quotient of absolute values to transform the given inequality into

$$|z^4 - 4z^2 + 3| \geq 3$$

Then we factor the Left Hand Side (LHS) to get

$$|z^2 - 1| \cdot |z^2 - 3| \geq 3 \qquad (*)$$

Now we use inequality (8) on page 10 in the textbook:

$$|z_1 - z_2| \geq |\,|z_1| - |z_2|\,|,$$

to transform the LHS of (*):

$$|z^2 - 1| \cdot |z^2 - 3| \geq |\,|z|^2 - |1|\,| \cdot |\,|z|^2 - |3|\,| = |2^2 - 1| \cdot |2^2 - 3| = 3 \geq 3,$$

where for the first equality we used that $|z| = 2$.
This establishes the given inequality.