DIFFUSIONS WITH RANK-BASED CHARACTERISTICS

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Abstract

Imagine you run two Brownian-like particles on the real line. At any given time, you assign drift g and dispersion σ to the laggard; and you assign drift $-h$ and dispersion ρ to the leader. Here g, h, ρ and σ are given nonnegative constants with $\rho^2 + \sigma^2 = 1$ and $g + h > 0$.

Is the martingale problem for the resulting infinitesimal generator

$$\mathcal{L} = \mathbf{1}_{\{x_1 > x_2\}} \left(\frac{\rho^2}{2} \frac{\partial^2}{\partial x_1^2} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x_2^2} - h \frac{\partial}{\partial x_1} + g \frac{\partial}{\partial x_2} \right) + \mathbf{1}_{\{x_1 \le x_2\}} \left(\frac{\sigma^2}{2} \frac{\partial^2}{\partial x_1^2} + \frac{\rho^2}{2} \frac{\partial^2}{\partial x_2^2} + g \frac{\partial}{\partial x_1} - h \frac{\partial}{\partial x_2} \right)$$

well-posed? If so, what is the probabilistic structure of the resulting two-dimensional diffusion process? What are its transition probabilities? How does it look like when time is reversed? Questions like these arise in the context of systems of diffusions interacting through their ranks; see, for instance, [1], [6], [8]. They become a lot more interesting, if one poses them for several particles instead of just two.

The construction we carry out involves features of Brownian motion with "bang-bang" drift [7], as well as of "skew Brownian motion" [4], [2]. Surprises are in store when one sets up a system of stochastic differential equations for this planar diffusion and then tries to decide questions of strength and/or weakness (cf. [2] for a one-dimensional analogue); also when one looks at the time-reversal of the diffusion. There are also very strong connections with the recent work [9] on the so-called "perturbed TANAKA equations".

I'll try to explain what we know about all this, then pose a few open questions.

(This talk covers joint work with E. Robert Fernholz, Tomoyuki Ichiba, Vilmos Prokaj and Mykhaylo Shkolnikov.)

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