KPP Wave Front Propagation in Narrow Tubes and Thin Layers with Non-linear Boundary Conditions

Konstantinos Spiliopoulos
University of Maryland, College Park
USA

September 2007, Rome, Italy

Introduction and Motivating Examples

KPP wave fronts in reaction-diffusion equations

Consider the problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{D}{2}\frac{\partial^2 u}{\partial x^2} + c(u)u,\ t > 0, x \in \mathbb{R}^1 \quad (1) \\ u(0,x) &= I_{x\le 0}, \end{aligned}$$

where $D > 0$, $c(u)$ is Lipshitz continuous in u, positive for $u < 1$, negative for $u > 1$ and $c = c(0) = \max_{0\le u\le 1} c(u)$. It was proven in the paper of K.P.P. (1937) that for $m(t)$ such that $u(t, m(t)) = \frac{1}{2}$ we have $u(t, m(t) + z) \to v(z)$ as $t \to \infty$, where $v(\cdot)$ satisfies the equation

$$\frac{D}{2}v_{zz}(z) + av_z(z) + c(v(z))v(z) = 0 \qquad (2)$$

$$\lim_{z\to\infty} v(z) = 0,\ \lim_{z\to-\infty} v(z) = 1, v(0) = 1/2$$

for $a = \sqrt{2cD}$.

Thus u behaves like $v(x - at)$ for large t.

Definition. We say that a^* is the <u>asymptotic speed</u> as $t \to \infty$ for u if the following conditions are satisfied for every $h > 0$:

$$\begin{aligned}\lim_{t\to\infty} \sup_{x>t(a^*+h)} u(t,x) &= 0\\ \lim_{t\to\infty} \inf_{x<t(a^*-h)} u(t,x) &= 1\end{aligned}$$

Actually the convergence $u(t, m(t) + z) \to v(z)$ is a corollary of the asymptotics of the probabilities of large deviations for the Wiener process.

Sketch of the proof

By Feynman-Kac formula the solution to (1) is:

$$u(t,x) = EI_{x+\sqrt{D}W_t \le 0} e^{\int_0^t c(u(t-s,x+\sqrt{D}W_s))ds} \tag{3}$$

Now setting $x = at$ and taking into account that $c(u) \le c(0) = c$ we get:

$$0 \le u(t, at) \asymp \exp\left(t(c - \frac{a^2}{2D})\right) \tag{4}$$

Thus $\lim_{t\to\infty} \sup_{x>t(\sqrt{2cD}+h)} u(t,x) = 0$, which implies that $a^* \le \sqrt{2cD}$.

Using large deviations estimates for the one-dimensional Wiener process one can prove a lower bound for a^*. More specifically one can show:

$$\lim_{t\to\infty} \frac{1}{t} \ln u(t, t\sqrt{2cD}) = 0. \tag{5}$$

Lastly it remains to show that

$$\lim_{t\to\infty} \inf_{x<t(\sqrt{2cD}-h)} u(t,x) = 1. \tag{6}$$

First maximum principle implies $u(t,x) \leq 1$. It remains to show that

$$\lim_{t\to\infty} \inf_{x<t(\sqrt{2cD}-h)} u(t,x) \geq 1. \tag{7}$$

The proof of this goes as follows. Assume that there exists $\lambda > 0$, arbitrary large t and $x^* < (\sqrt{2cD} - h)t$ such that $u(t,x^*) < 1 - \lambda$.

Then Feynman-Kac formula and strong Markov property imply that

$$u(t,x^*) > 1 - \frac{3\lambda}{2}, \tag{8}$$

for t large enough. The latter contradicts the assumption $u(t,x^*) < 1 - \lambda$.

Using similar techniques one can consider wave front propagation in more general situations.

Consider the problem in the tube $\mathbb{R}^1 \times G$:

$$\begin{aligned}
\frac{\partial u(t,x,y)}{\partial t} &= \frac{D}{2}\triangle_{x,y} u - b\frac{\partial u}{\partial x} + c(u)u,\ t > 0 \\
u(0,x,y) &= I_{x\le 0} \qquad (9)\\
\frac{\partial u(t,x,y)}{\partial n(y)} &= 0,\ t > 0, x \in \mathbb{R}^1, y \in \partial G
\end{aligned}$$

If $b =$constant then by following the same steps as before we can conclude that $a^* = b + \sqrt{2cD}$. Moreover using the action functional and its properties one could also consider the case where $b = b(y)$.

KPP fronts in slowly varying media.

Consider the problem:

$$\begin{aligned} \frac{\partial u^\epsilon}{\partial t} &= \frac{\epsilon D}{2}\frac{\partial^2 u^\epsilon}{\partial x^2} + \frac{1}{\epsilon}c(x,u^\epsilon)u^\epsilon,\ t>0, x\in\mathbb{R}^1 \\ u^\epsilon(0,x) &= g(x) \text{ with } \mathrm{spt}(g) = G, \end{aligned} \tag{10}$$

Feynman-Kac formula implies that

$$u^\epsilon(t,x) = E_x g(X_t^\epsilon)\exp(\epsilon^{-1}\int_0^t c(X_s^\epsilon, u^\epsilon(t-s, X_s^\epsilon))ds) \tag{11}$$

Let $X_t^\epsilon = \sqrt{\epsilon D}W_t$ and $c(x,u) \le c(x,0) = c(x)$. Define also the following:

- $S_{0t}(\phi) = \int_0^t D^2 \dot\phi_s^\tau \dot\phi_s ds$ for absolutely continuous functions $\phi_s : [0,t] \to \mathbb{R}^r$.
- $R_{0t}(\phi_s) = \int_0^t c(\phi_s)ds - S_{0t}(\phi)$.
- $W(t,x) = \sup\{R_{0t}(\phi_s) : \phi_0 = x, \phi_t \in \bar{G}\}$.

Then one can show that

$$\lim_{\epsilon \to 0} \epsilon \ln E_x g(X_t^\epsilon) \exp(\epsilon^{-1} \int_0^t c(X_s^\epsilon, 0)) ds = W(t,x) \tag{12}$$

In particular we have the following theorem

Theorem. Suppose that for any $(t,x) \in \{(t,x) : W(t,x) = 0\}$:

$$\begin{aligned} W(t,x) &= \sup\{R_{0t}(\phi_s) : \phi_0 = x, \phi_t \in \bar{G}, \\ &\quad (t-s, \phi_s) \in \{(t,x) : W(t,x) < 0\}\}. \end{aligned} \tag{13}$$

Then we have

$$\lim_{\epsilon \downarrow 0} u^\epsilon(t,x) = \begin{cases} 1, & W(t,x) > 0 \\ 0, & W(t,x) < 0. \end{cases} \tag{14}$$

Action functional and its main properties.

Let (X, ρ) be a metric space. Consider ξ^ϵ to be a family of random variables on the probability spaces $(\Omega^\epsilon, \mathfrak{F}^\epsilon, P^\epsilon)$. Let also $\lambda(\epsilon)$ be a positive real-valued function such that $\lambda(\epsilon) \to +\infty$ as $\epsilon \to 0$ and let $S(x)$ be a positive function on X.

Definition. We say that $\lambda(\epsilon)S(x)$ is the action function for P^ϵ as $\epsilon \to 0$ if the following assertions hold:

(i). The set $\Phi(s) = \{x : S(x) \leq s\}$ is compact for every $s \geq 0$.

(ii). $\forall\ \delta > 0, \gamma > 0$ and $x \in X$, $\exists \epsilon_o > 0$ s.t.

$$P^\epsilon\{\rho(\xi^\epsilon, x) < \delta\} \geq \exp(-\lambda(\epsilon)[S(x)+\gamma])\,\forall\,\epsilon \leq \epsilon_o.$$

(iii). $\forall\ \delta > 0, \gamma > 0$ and $s > 0$, $\exists \epsilon_o > 0$ s.t.

$$P^\epsilon\{\rho(\xi^\epsilon, \Phi(s)) \geq \delta\} \leq \exp(-\lambda(\epsilon)[s-\gamma])\,\forall\,\epsilon \leq \epsilon_o.$$

Below are listed some of the properties of the action function.

Property 1. Let a family of Markov processes X_t^ϵ with infinitesimal generator $A^\epsilon = \sum_i b_i(x)\frac{\partial f(x)}{\partial x_i} + \frac{\epsilon^2}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j}$ for functions $f \in \mathcal{C}^2$. Then it can be shown that for absolutely continuous $\phi_t, T1 \leq t \leq T2$

$$S_{T1T2}(\phi) = \frac{1}{2}\int_{T1}^{T2} \sum_{ij} a_{ij}(\phi_t)(\dot{\phi}_t^i - b^i(\phi_t))(\dot{\phi}_t^j - b^j(\phi_t))dt$$

Property 2. For any continuous function F

$$\lim_{\epsilon \downarrow 0} \lambda^{-1}(\epsilon) \ln E e^{\lambda(\epsilon) F(X^\epsilon)} = \sup\{F(x) - S(x) : x \in X\}. \tag{15}$$

Property 3. Let P^ϵ be a family of probability measures on $\mathbb{R}^r$ and let us define $H^\epsilon(a) = \ln E^\epsilon(\exp\{(a,x)\})$ and assume that the limit $H(a) = \lim_\epsilon \lambda(\epsilon)^{-1} H^\epsilon(\lambda(\epsilon)a)$ exists for all a. Then for $x \in \mathbb{R}^r$ we have that

$$S(x) = \sup_a [(a,x) - H(a)].$$

Property 4. Let $\lambda(\epsilon)S^P(x)$ be the action function for a family of probability measures P^ϵ on a space (X, ρ_X) as $\epsilon \to 0$. Let ϕ be a continuous mapping of (X, ρ_X) into a space (Y, ρ_Y) and let a probability measure Q^ϵ on (Y, ρ_Y) be given by the formula $Q^\epsilon = P^\epsilon(\phi^{-1}(A))$. Then the action function for Q^ϵ is $\lambda(\epsilon)S^Q(x)$, where $S^Q(x) = \min\{S^P(x) : x \in \phi^{-1}(y)\}$.

KKP Fronts in Thin Layers and Narrow Tubes

Let $D^\epsilon = \{(x,y) \in \mathbb{R}^{n+m} : -\epsilon k_i(x) \leq y_i \leq \epsilon h_i(x)$ $, x \in \mathbb{R}^n, i = 1, \cdots, m\}$.

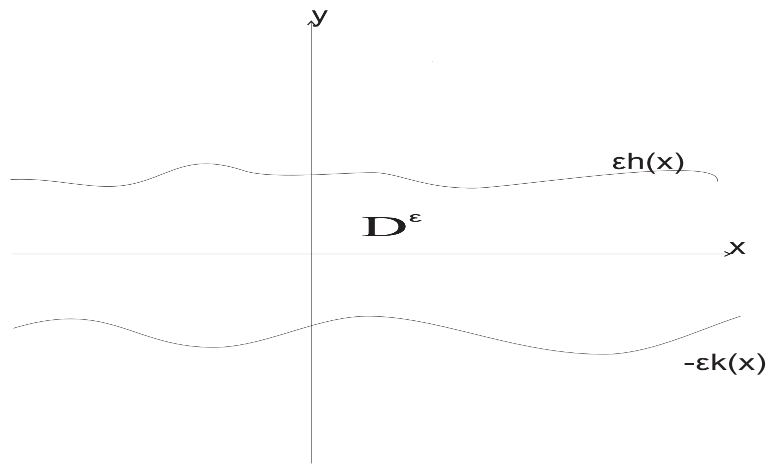

Figure 1.

Consider the quasi-linear parabolic P.D.E.:

$$\begin{aligned}
\frac{\partial u^\epsilon}{\partial t} &= \frac{1}{2}\triangle u^\epsilon && , \text{ in } (0,T) \times D^\epsilon \\
u^\epsilon(0,x,y) &= f(x) && , \text{ on } \{0\} \times D^\epsilon \\
\frac{\partial u^\epsilon}{\partial \gamma^\epsilon} + \epsilon c(x,y,u^\epsilon)u^\epsilon &= 0 && , \text{ on } (0,T) \times \partial D^\epsilon
\end{aligned} \tag{16}$$

where γ^ϵ is the inward normal vector to ∂D^ϵ.

Remark: If $n = 1$ the domain D is a tube in $\mathbb{R}^{1+m}$ and if $n > 1$ then it is a layer in $\mathbb{R}^{n+m}$.

Assumptions:

(i). The initial function f and the boundary function c are smooth and bounded. Moreover c is assumed Lipshitz continuous in u^ϵ variable.

(ii). Function c is decreasing and of KPP type, i.e., it is positive for $u < 1$, negative for $u > 1$ and $c(x, y) = c(x, y, 0) = \max_{0 \le u \le 1} c(x, y, u)$

(iii). $D^1_x = \{y \in \mathbb{R}^m : (x, y) \in D^1\}$ is a bounded connected domain and that the normal vector to ∂D^1 is not parallel to $\mathbb{R}^n$.

(iv). $\partial D^\epsilon \in \mathcal{C}^{3+\alpha}(\mathbb{R}^m)$ where $\alpha \in (0, 1)$.

Main result

Define $S(x)$ = surface area of ∂D_x, $V(x)$ = volume of D_x and $\overline{c}(x,u) = \frac{S(x)}{V(x)} c(x,0,u)$.

Let also u be the solution to:

$$\begin{aligned} u_t &= \frac{1}{2}\triangle_x u + \frac{1}{2}\nabla(\log V(x))\nabla_x u + \overline{c}(x,u)u, \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{in}\,(0,T)\times\mathbb{R}^n \\ u(0,x) &= f(x), \qquad\qquad\qquad\qquad\quad \text{on}\,\{0\}\times\mathbb{R}^n. \end{aligned} \tag{17}$$

Then we have the following theorem:

Theorem 1. If $u^\epsilon(t,x,y), u(t,x)$ are the solutions to (16) and (17) respectively, we have

$u^\epsilon(t,x,y) \to u(t,x)$ as $\epsilon \to 0$, uniformly in compacts.

Remark: Using now Theorem 1, we can study wave front propagation for u^ϵ for small enough $\epsilon > 0$.

Sketch of proof of Theorem 1.

Steps:

(i). Using Feynman-Kac formula we write the probabilistic representation of the solution u^ϵ to (16).

(ii). Consider the limit of the underlying stochastic process as $\epsilon \downarrow 0$.

(iii). Using the compactness of the family $\{u^\epsilon\}$ we prove the claim.

Underlying stochastic process.

Let W_t^1 be a Wiener process in $\mathbb{R}^n$, W_t^2 be a Wiener process in $\mathbb{R}^m$ and (x, y) a point inside D^ϵ. Consider $(X_t^\epsilon, Y_t^\epsilon)$ to be the solution to the stochastic differential equation with normal reflection on ∂D^ϵ :

$$
\begin{aligned}
X_t^\epsilon &= x + W_t^1 + \int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon) dL_s^\epsilon \\
Y_t^\epsilon &= y + W_t^2 + \int_0^t \gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon) dL_s^\epsilon, \quad (18)
\end{aligned}
$$

where if $\gamma^\epsilon = (\gamma_1^\epsilon, \gamma_2^\epsilon)$ is the unit inward normal vector to ∂D^ϵ, then γ_1^ϵ is the projection of γ^ϵ on $\mathbb{R}^n$ and γ_2^ϵ is the projection of γ^ϵ on $\mathbb{R}^m$.

Moreover L_t^ϵ is the local time process for the process $(X_t^\epsilon, Y_t^\epsilon)$ on ∂D^ϵ, i.e. it is a continuous, non-decreasing process that increases only when $(X_t, Y_t^\epsilon) \in \partial D^\epsilon$ such that the Lebesque measure $\Lambda\{t > 0 : (X_t, Y_t^\epsilon) \in \partial D^\epsilon\} = 0$.

We need a preliminary result, concerning:

averaging of stochastic integrals in local time:

Lemma 2. Let $Q(x) = \frac{1}{V(x)} \int_{\partial D_x} P(x,y) dS_x$. Then under the above hypotheses we have that as $\epsilon \to 0$:

$$\sup_{0 \le t \le T} E \left| \int_0^t \epsilon P(X_s^\epsilon, Y_s^\epsilon/\epsilon) dL_s^\epsilon - \int_0^t \frac{1}{2} Q(X_s^\epsilon) ds \right|^2 \to 0. \tag{19}$$

Sketch of proof of Lemma 2.

We consider the following elliptic problem:

$$\begin{aligned} \triangle_y v(y) &= Q(x),\ y \in D_x \subset \mathbb{R}^m \\ \frac{\partial_y v(y)}{\partial n(y)} &= -P(x,y),\ y \in \partial D_x. \end{aligned} \tag{20}$$

A necessary condition for the solvability of this problem is that

$$Q(x) = \frac{1}{V(x)} \int_{\partial D_x} P(x,y) dS_x, \tag{21}$$

The solution to (20), $v(y)$, also depends on x and we will write $v(x, y)$. However the existence, uniqueness and the differentiability in x of the solution to problem (20) is not immediate since v depends on x not only through functions P and Q but also through the varying boundary ∂D_x. They follow however from the main result of:

M. Bochniak, 2003, Linear elliptic boundary value problems in varying domains, Math. Nachr. 250, pp. 17-24.

The discussion above implies that we can apply Itô formula to the function $\epsilon v(x, y/\epsilon)$. Doing so and using the fact that the local time is of order $1/\epsilon$ as $\epsilon \to 0$, we get the desired result.

Let the triple $(X^\epsilon, Y^\epsilon, L^\epsilon)$ in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1_+$ satisfy (18). Then we have the following Feynman-Kac representation of u^ϵ:

Proposition 3. Problem (16) has a unique classical solution in $[0, T) \times D^\epsilon$ which satisfies:

$$u^\epsilon(t, x, y) = E_{x,y} f(X^\epsilon_t) e^{\int_0^t \epsilon c(X^\epsilon_s, Y^\epsilon_s, u^\epsilon(t-s, X^\epsilon_s, Y^\epsilon_s)) dL^\epsilon_s}. \tag{22}$$

We want to consider the limit as $\epsilon \downarrow 0$ of $u^\epsilon(t, x, y)$.

We need first to examine the asymptotic behavior of X_t^ϵ as $\epsilon \to 0$.

Using Lemma 2, it is not difficult to see that X_t^ϵ converges as $\epsilon \downarrow 0$ to X_t, where X_t is the solution to

$$X_t = x + W_t^1 + \int_0^t \frac{1}{2} \nabla (\log V(X_s)) ds. \qquad (23)$$

More precisely we have the following lemma:

Lemma 4. For any $T > 0$ we have

$$\sup_{0 \le t \le T} E_x |X_t^\epsilon - X_t|^2 \to 0 \text{ as } \epsilon \to 0. \qquad (24)$$

The next step now is to show compactness for the family $\{u^\epsilon\}$ of solutions to (16).

By maximum principle and standard techniques for parabolic type equations we can derive the following:

Proposition 5. Let us define $U_T^\epsilon = (0,T) \times D^\epsilon$ and $\overline{U}_T^\epsilon = [0,T) \times \overline{D}^\epsilon$. There is a constant C, independent of ϵ, and an open set $I \subset (0,1)$ such that for any $\alpha \in I$:

$$\begin{aligned} \|u^\epsilon\|_{\overline{U}_T^\epsilon} + \|H^\alpha u^\epsilon\|_{\overline{U}_T^\epsilon} &+ \|u_t^\epsilon\|_{\overline{U}_T^\epsilon} \\ &+ \|Du^\epsilon\|_{\overline{U}_T^\epsilon} + \|D^2 u^\epsilon\|_{U_T^\epsilon} \leq C. \end{aligned} \tag{25}$$

where u^ϵ is a classical solution to (16).

For more details see for example:

A. Friedman, 1964, Partial Differential Equations of Parabolic Type, Prentice Hall.

Therefore Proposition 5 and the well-known Ascoli-Arzela theorem imply the compactness of $\{u^\epsilon\}$.

Using now again Lemma 2 and taking into account Proposition3, Lemma 4 and Proposition 5 we derive the desired result, i.e that $u^\epsilon \to u$ uniformly as $\epsilon \to 0$, where u is the solution to:

$$\begin{aligned} u_t &= \frac{1}{2}\triangle_x u + \frac{1}{2}\nabla(\log V(x))\nabla_x u + \overline{c}(x,u)u, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{in}\,(0,T)\times\mathbb{R}^n \\ u(0,x) &= f(x), \qquad\qquad\qquad\qquad\quad \text{on}\,\{0\}\times\mathbb{R}^n. \end{aligned}$$

KPP Wave Propagation for u^ϵ for small $\epsilon > 0$ in Slowly Varying Tubes.

Let us in addition assume that the functions $\overline{c}(\cdot, u)$, $f(\cdot)$, $h_i(\cdot)$ and $k_i(\cdot)$ ($i = 1, \ldots, n$) depend on δx where $\delta > 0$ is a small parameter, i.e. they are slowly varying in x-variable.

This is a natural assumption for the domain D^ϵ, since we do not expect it to have rapid changes. Moreover we take T increasing to infinity and obviously Theorem 1 still holds. So if we set $u^\delta(t, x) = u(t/\delta, x/\delta)$, then u^δ is the solution to the following parabolic problem:

$$\begin{aligned} u_t^\delta &= \frac{\delta}{2}\triangle u^\delta + \frac{\delta}{2}\nabla \log V(x) \nabla u^\delta + \\ &\qquad + \frac{1}{\delta}\overline{c}(x, u^\delta(t, x))u^\delta \\ u^\delta(0, x) &= f(x) > 0, \qquad \text{on } \{0\} \times \mathbb{R}^n. \end{aligned} \tag{26}$$

Let us now define $\overline{c}(x) = \overline{c}(x, 0)$.

Consider a function $\phi_s \in \mathbb{R}^n$ and introduce the functional

$$R_{0,T}(\phi) = \int_0^T [\overline{c}(\phi_s) - \frac{1}{2}\sum_{i=1}^n |\dot{\phi}_s^i|^2] ds, \qquad (27)$$

and the function

$$W(t,x) = \sup\{R_{0,t}(\phi) : \phi \in \mathcal{C}_{0,t}(\mathbb{R}^n), \phi_0 = x, \phi_t \in F_o\}. \qquad (28)$$

Definition. *We say that the condition (N) is satisfied if for any* $t > 0$ *and* $(t,x) \in \{(t,x) : W(t,x) = 0\}$:

$$\begin{aligned} W(t,x) \;&=\; \sup\{R_{0,t}(\phi) : \phi_0 = x, \phi_t \in F_o, \\ &\qquad (t-s, \phi_s) \in \{(t,x) : W(t,x) < 0\}\}. \end{aligned}$$

This definition is not empty, as the following lemma shows:

Lemma 6. Assume that $\overline{c}(x)$ is an increasing function. Condition (N) is fulfilled for $\overline{c}(x)$.

Under the assumptions above, the following theorem states that $W(t,x)$ determines the motion of the wave front for $u^\delta(t,x)$ under condition (N):

Theorem 7.[M.F.] We have:

$$\lim_{\delta \downarrow 0} u^\delta(t,x) = \begin{cases} 1, & W(t,x) > 0 \\ 0, & W(t,x) < 0. \end{cases} \tag{29}$$

Let us consider now equation (16) when ϵ is small and let $u^{\epsilon,\delta}(t,x,y) = u^\epsilon(t/\delta, x/\delta, y)$. Under the assumptions above, Theorems 1 and 7 imply that $W(t,x)$ will determine the behavior of the wave front in this case too, as follows:

Theorem 8. The following statement holds:

$$\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} u^{\epsilon,\delta}(t,x,y) = \begin{cases} 1, & W(t,x) > 0 \\ 0, & W(t,x) < 0. \end{cases} \tag{30}$$

Jumps of the Wave Front of u^ϵ.

It is easy to see that $W(t,x)$ is an increasing function of t. The domain $G_t = \{x \in R^n : W(t,x) > 0\}$ may not expand in a continuous way though. In such case have "ignition" of new sources ahead of the wave front.

Define the following quantities:

- $\phi_s^x : \int_{\phi_s^x}^x \frac{dy}{\sqrt{2\overline{c}(x)-2\overline{c}(y)}} = s.$

- $t_o^x : t_o^x = \int_0^x \frac{dy}{\sqrt{2\overline{c}(x)-2\overline{c}(y)}}.$

- $R_{0,t_o^x}(\phi^x) = \int_0^{t_o^x} [2\overline{c}(\phi_s^x)]ds - \overline{c}(x)t_o^x.$

In particular we have the following proposition:

Proposition 9. Let $x > 0$ such that $\overline{c}(x) > \overline{c}(w)$ for all $w \in (0, x)$ and $\phi^{1,x}$ be the extremal such that $W(t_1, x) = R_{0,t_1}(\phi^{1,x}) = 0$ for some $t_1 > 0$. Assume that

$$R_{0,t_o^x}(\phi^x) < 0 \Rightarrow \int_0^{t_o^x} [2\overline{c}(\phi_s^x)] ds < \overline{c}(x) t_o^x \quad (31)$$

is satisfied. Then for small enough $\epsilon > 0$ new sources will be igniting ahead of the position at x of the wave front of u^ϵ from points $z \in (x, \max_{0 \le s \le t_1} \phi_s^{1,x})$ such that

$$\overline{c}(z) > \overline{c}(w) \text{ for all } w \in (0, z).$$

Sketch of the proof for the case $x \in \mathbb{R}^1$

The Euler equation of the extremals of the functional $R_{0,t}(\phi)$ in the case $\phi \in \mathbb{R}^1$ is

$$\ddot{\phi}_s = -\overline{c}'(\phi_s).$$

This equation has first integral

$$\frac{1}{2}\dot{\phi}_s^2 + \overline{c}(\phi_s) = K_\phi = \text{constant}. \tag{32}$$

Let now a point $x > 0$ be such that $\overline{c}(x) > \overline{c}(w)$ for all $w \in (0, x)$. Consider the extremal ϕ_s^x starting from point x with zero initial velocity, i.e. $\dot{\phi}_0 = 0$. This extremal satisfies (32) with $K_\phi = \overline{c}(x)$. We easily derive that ϕ_s^x is determined by the integral equation:

$$\int_{\phi_s^x}^{x} \frac{dy}{\sqrt{2\overline{c}(x) - 2\overline{c}(y)}} = s. \tag{33}$$

Moreover ϕ_s^x reaches the point $x = 0$ at time:

$$t_o^x = \int_0^x \frac{dy}{\sqrt{2\overline{c}(x) - 2\overline{c}(y)}}. \tag{34}$$

Equation (32) imply the following chain of equalities:

$$\begin{aligned} R_{0,t_o^x}(\phi^x) &= \int_0^{t_o^x} [\overline{c}(\phi_s^x) - \frac{1}{2}|\dot{\phi}_s^x|^2] ds \\ &= \int_0^{t_o^x} [2\overline{c}(\phi_s^x)] ds - \overline{c}(x) t_o^x \end{aligned} \quad (35)$$

We see that for $x > 0$ such that:

$$R_{0,t_o^x}(\phi^x) < 0 \Rightarrow \int_0^{t_o^x} [2\overline{c}(\phi_s^x)] ds < \overline{c}(x) t_o^x \quad (36)$$

then new sources will be igniting ahead of the wave front. Indeed, since $R_{0,t_o^x}(\phi^x) < 0$ there exists a $t_1 > t_o^x$ such that $W(t_1, x) = 0$. Let $\phi^{1,x}$ be the extremal such that $R_{0,t_1}(\phi^{1,x}) = 0$. Let now $z \in (x, \max_{0 \le s \le t_1} \phi_s^{1,x})$ and $\eta(z) \in (0, t_1)$ such that:

$$\begin{aligned} \overline{c}(z) &> \overline{c}(w) \text{ for all } w < z \\ \phi_{\eta(z)}^{1,x} &= z \text{ and } \phi_s^{1,x} < z \text{ for } 0 \le s \le \eta(z) \end{aligned} \quad (37)$$

Next we define

$$\phi_s^{2,z} = \begin{cases} z, & 0 \le s \le \eta(z) \\ \phi_s^{1,x}, & \eta(z) < s \le t_1. \end{cases} \quad (38)$$

Then we easily see that

$$0 = R_{0,t_1}(\phi^{1,x}) < R_{0,t_1}(\phi^{2,z}). \qquad (39)$$

Thus there is a $t < t_1$ such that $W(t,z) = 0$, which implies that a new source is igniting at z.

Moreover one can show that for $x > 0$ such that

$$\int_0^{t_o^x} 2\overline{c}(\phi_s^x)ds \geq \overline{c}(x)t_o^x$$

the wave front propagates in a continuous way and thus no new sources arise.

A simple example:

We conclude with a simple example that allows to give an explicit formula for the time that the jump of the wave front will appear and an explicit relation for $V(z)$ and $V(x)$.

Let us assume that $\overline{c}(x) = \frac{S(x)}{V(x)}$, i.e. $c(x, 0, 0) = 1$, is an increasing smooth function such that

$$\overline{c}(x) = \begin{cases} c_1, & \text{for } x < \bar{x} = \frac{1}{2}(z + T_o\sqrt{2c_1}) + \delta < z \\ c_2, & x > z, \end{cases}$$

where $c_2 > 2c_1$ are two constants, δ is a small enough positive number and $T_o = \frac{z\sqrt{2(c_2 - c_1)}}{c_2}$. Therefore $\overline{c}(x)$ is close to a step function. Using the analysis before, it is not difficult to see that the excitation reaches the region $\{x > \bar{x} + \delta\}$ before it reaches the point $\bar{x}$ and moreover the new source arises at point z at time T_o.

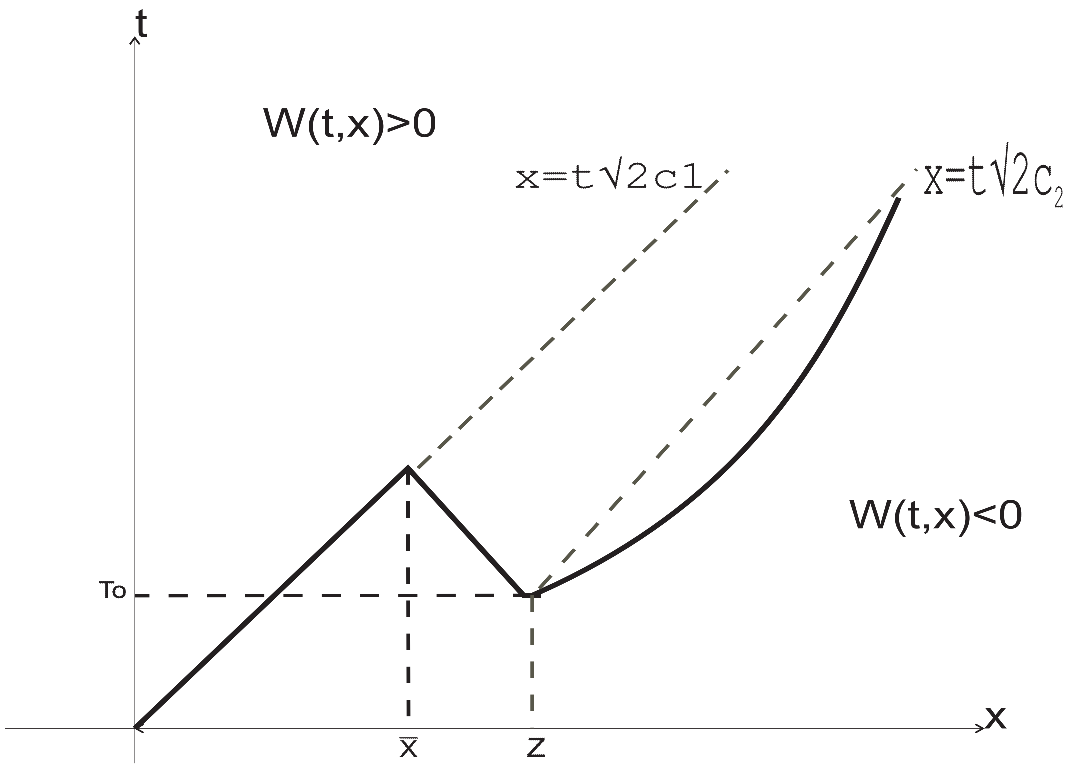

Figure 2.

Let us further assume for simplicity that the boundary of the tube D^ϵ satisfies $\partial D^\epsilon \subset \mathbb{R}^2$ and that

$$h_1(x) = h_2(x) \text{ and } k_1(x) = k_2(x) = 0.$$

Then it is easy to see that if $V(y) = V_1 =$ constant is the volume of the tube for $y < \bar{x}$ and $V(y) = V_2 =$ constant is the volume of the tube for $y > z$, then the relation $c_2 > 2c_1$ implies

$$V_2 < \frac{1}{4} V_1.$$

Namely the new sources are igniting, ahead of points $\bar{x}$, from points z where the volume is a quarter of the volume at point $\bar{x}$.

M. Bochniak, 2003, Linear elliptic boundary value problems in varying domains, Math. Nachr. 250, pp. 17-24.

S.N. Ethier, T.G. Kurtz, 1986, Markov processes: Characterization and Convergence, Wiley, New York.

L.C. Evans, P.E. Souganidis, 1989, A PDE approach to geometric optics for certain semilinear parabolic equations, Indiana U. Math, Vol. 52, pp. 43-80.

M. Freidlin, 1985, Functional Integration and Partial Diffferential Equations, Princeton University Press.

M. Freidlin, 1995, Wave Front Propagation for KPP-Type Equations, Survey in Applied Mathematics, 2, pp. 1-62.

M. Freidlin, 1996, Markov Processes and Differential Equations: Asymptotic Problems, Birkhäuser Verlang.

M.I. Freidlin, A.D. Wentzell, 1998, Random Perturbations of Dynamical Systems, Second Edition, Springer.

A. Friedman, 1964, Partial Differential Equations of Parabolic Type, Prentice Hall.

A. Kolmogorov, I. Petrovskii, N. Piskunov, 1937, Étude de l'èquation de la diffusion avec croissence de la matière et son application a un problème biologique, Moscov University Bull. Math., Vol. 1, pp. 1-25.

J. Xin, Front Propagation In Heterogeneous Media, Siam Review, Vol. 42, No.2, pp. 161-230.