ABSTRACT

Title of scholarly paper:	THE SMOLUCHOWSKI-KRAMERS APPROXIMATION FOR THE LANGEVIN EQUATION WITH REFLECTION
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According to the Smoluchowski-Kramers approximation, the solution of the equation $\mu \ddot{q}_t^\mu = b(q_t^\mu) - \dot{q}_t^\mu + \sigma(q_t^\mu)\dot{W}_t, q_0^\mu = q, \dot{q}_0^\mu = p$ converges to the solution of the equation $\dot{q}_t = b(q_t) + \sigma(q_t)\dot{W}_t, q_0 = q$ as $\mu \to 0$. We consider here a similar result for the Langevin process with elastic reflection on the boundary. In particular we prove that the Langevin process with reflection converges in distribution to a standard diffusion process with reflection. This result is the main justification for using a first order equation, instead of a second order one, to describe the motion of a small mass particle that is restricted to move in the interior of some domain and reflects elastically on its boundary.

THE SMOLUCHOWSKI-KRAMERS APPROXIMATION
FOR THE LANGEVIN EQUATION WITH REFLECTION

by

Konstantinos Spiliopoulos

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Last but not least, I owe my deepest thanks to my mother, my father and my brother who have always stood by me and support my efforts. Words cannot express the gratitude I owe them.

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l.p.r.   Langevin process with reflection

Chapter 1

Introduction

1.1 Overview

According to the Smoluchowski-Kramers approximation ([4],[5]), the solution of the stochastic differential equation (S.D.E.)

$$\mu \ddot{q}_t^{\mu} = b(q_t^{\mu}) - \dot{q}_t^{\mu} + \sigma(q_t^{\mu})\dot{W}_t \tag{1.1}$$

$$q_0^{\mu} = q \in R^1$$

$$\dot{q}_0^{\mu} = p \in R^1$$

converges in probability as $\mu \to 0$ to the solution of the following S.D.E.:

$$\dot{q}_t = b(q_t) + \sigma(q_t)\dot{W}_t \tag{1.2}$$

$$q_0 = q \in R^1$$

More precisely, one can prove that for any $\delta, T > 0$ and $q, p \in R^1$,

$$\lim_{\mu \downarrow 0} P(\max_{0 \le t \le T} |q_t^{\mu} - q_t| > \delta) = 0, \tag{1.3}$$

where $b : R^1 \to R^1$ and $\sigma : R^1 \to R^1$ have bounded first derivatives and W_t is the standard one-dimensional Brownian motion (see, for example, Lemma 1 in [2]).

Equation (1.1) describes the motion of a particle of mass μ in a force field $b(q) + \sigma(q)\dot{W}_t$, with a friction proportional to velocity. The Smoluchowski-Kramers

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approximation justifies the use of equation (1.2) to describe the motion of a small particle.

Let $R_+^2 = \{x \in R^2 : x^1 \geq 0\}$. In this paper we examine the behavior of the process with elastic reflection on the boundary $\partial R_+^2 = \{x \in R^2 : x^1 = 0\}$ that is governed by (1.1) for $x^1 > 0$, i.e. of the Langevin process with reflection, as $\mu \to 0$. We will show that the first component of the Langevin process with reflection at $q = 0$ converges in distribution to the diffusion process with reflection on ∂R_+^1 that is governed by equation (1.2).

1.2 Outline of the scholarly paper

In chapter 2 we define the Langevin process with reflection and we prove that the definition is correct. In chapter 3 we consider the limit of the Langevin process with elastic reflection as $\mu \to 0$ and we prove that it converges in distribution to a diffusion process with reflection. We conclude with the bibliography.

Chapter 2

Construction of the Langevin Process with Reflection

2.1 Definition of the Langevin Process with Reflection

We begin with the construction of the Langevin process $(q_t^\mu; p_t^\mu)$ in R_+^2 with elastic reflection on the boundary. Let $b : R_+^1 \to R^1$ and $\sigma : R_+^1 \to R^1$ have bounded first derivatives and σ be non-degenerate. Let also $(q, p) \in R_+^2$ be the initial point (we assume that $q^2 + p^2 \neq 0$). Define the process $(q_t^\mu; p_t^\mu)$ as the solution of the following S.D.E.:

$$
\begin{aligned}
\dot{q}_t^\mu &= p_t^\mu \\
\mu \dot{p}_t^\mu &= -p_t^\mu + b(q_t^\mu) + \sigma(q_t^\mu)\dot{W}_t \\
q_0^\mu &= q,\ p_0^\mu = p,
\end{aligned}
\tag{2.1}
$$

for $t \in [0, \tau_1^\mu)$, where $\tau_1^\mu = \inf\{t > 0 : q_t^\mu = 0\}$. Then define $(q_t^\mu; p_t^\mu)$ for $t \in [\tau_1^\mu, \tau_2^\mu)$, where $\tau_2^\mu = \inf\{t > \tau_1^\mu : q_t^\mu = 0\}$, as the solution of (2.1) with initial conditions $(q_{\tau_1^\mu}^\mu; p_{\tau_1^\mu}^\mu) = (0; -\lim_{t \uparrow \tau_1^\mu} p_t^\mu)$. If $0 < \tau_1^\mu < \tau_2^\mu < \dots < \tau_k^\mu$ and $(q_t^\mu; p_t^\mu)$ for $t \in [0, \tau_k^\mu)$ are already defined, then define $(q_t^\mu; p_t^\mu)$ for $t \in [\tau_k^\mu, \tau_{k+1})$ as solution of (2.1) with initial conditions $(q_{\tau_k^\mu}^\mu; p_{\tau_k^\mu}^\mu) = (0; -\lim_{t \uparrow \tau_k^\mu} p_t^\mu)$ (see Figure 1 for an illustration).

This construction defines the process $(q_t^\mu; p_t^\mu)$ in R_+^2 for all $t \geq 0$. This follows from Theorem 2.4, whose proof however will be given at the end of this section. The

sequence $\{\tau_i^\mu\}$ is a strictly increasing sequence of Markov times. Therefore we have the following definition:

Definition 2.1 *We call the above recursively constructed process the Langevin process with elastic reflection on the boundary ∂R_+^2. This process has jumps on ∂R_+^2 and is continuous inside R_+^2.*

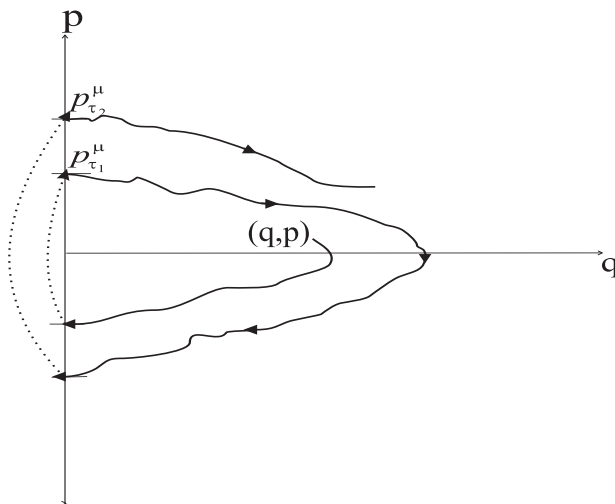Below we see an illustration of the construction above in the $q-p$ phase space.

Figure 2.1: Illustration of the Langevin process with reflection in the $q - p$ phase space

We will refer to the Langevin process with reflection as l.p.r.$(q_t^\mu; p_t^\mu)$. Moreover we will denote by $(q_t^{\mu,q}; p_t^{\mu,p})$ the solution to (2.1) with initial condition (q, p).

2.2 An equivalent construction of the Langevin process with reflection

Let us give now another construction of the Langevin process with reflection. This construction is equivalent to the first one and it will help us prove the above mentioned Theorem 2.4.

Let us consider the following S.D.E. in R^2:

$$\dot{q}_t^\mu = p_t^\mu$$

$$\mu \dot{p}_t^\mu = -p_t^\mu + \text{sgn}(q_t^\mu)b(|q_t^\mu|) + \text{sgn}(q_t^\mu)\sigma(|q_t^\mu|)\dot{W}_t \quad\quad (2.2)$$

$$q_0^\mu = q, p_0^\mu = p,$$

where $\text{sgn}(x)$ takes two values, 1 if $x \geq 0$ and -1 if $x < 0$.

Lemma 2.2 *Equation (2.2) has a weak solution which is unique in the sense of probability law.*

Proof. The existence follows from a Girsanov's Theorem on the absolutely continuous change of measures in the space of trajectories (b and σ are assumed bounded) and the fact that (2.2) with $b = 0$ has a weak solution. The uniqueness follows from Proposition 5.3.10 of [4].

$\square$

Using the processes $(q_t^{\mu,q}; p_t^{\mu,p})$ and $(q_t^{\mu,-q}; p_t^{\mu,-p})$ we can construct the Langevin process with reflection as follows. Assume that $p > 0$ and $q > 0$. Then the graphs of $p_t^{\mu,p}$ and of $p_t^{\mu,-p}$ will be exactly symmetric with respect to zero. The same will be

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true also for the graphs of $q_t^{\mu,q}$ and of $q_t^{\mu,-q}$. Let $\tau_0^\mu = 0, \tau_k^\mu = \inf\{t > \tau_{k-1} : q_t^{\mu,q} = 0\}$ and $(\widehat{q}_t^\mu; \widehat{p}_t^\mu)$ be a stochastic process, which is defined as follows:

$$(\widehat{q}_t^\mu; \widehat{p}_t^\mu) = (q_t^{\mu,q}; p_t^{\mu,p}) \text{ for } \tau_{2k}^\mu \le t \le \tau_{2k+1}^{\mu,-}$$

$$(\widehat{q}_t^\mu; \widehat{p}_t^\mu) = (q_t^{\mu,-q}; p_t^{\mu,-p}) \text{ for } \tau_{2k+1}^\mu \le t \le \tau_{2k+2}^{\mu,-}, k = 0, 1, 2, ... \qquad (2.3)$$

Process $(\widehat{q}_t^\mu; \widehat{p}_t^\mu)$ is a process with reflection on ∂R_+^2 and it is easy to see that $(\widehat{q}_t^\mu; \widehat{p}_t^\mu)$ defined by (2.3) and l.p.r.$(q_t^\mu; p_t^\mu) = (|q_t^\mu|; \frac{d}{dt}|q_t^\mu|)$ coincide.

In the figures below we see an illustration of this construction of the Langevin process with reflection. The first figure illustrates with thick continuous and dotted lines $\widehat{q}_t^\mu$ versus t. The continuous line is $q_t^{\mu,q}$ versus t and the dotted $q_t^{\mu,-q}$ versus t. The second figure illustrates with thick continuous and dotted lines $\widehat{p}_t^\mu$ versus t. The continuous line is $p_t^{\mu,p}$ versus t and the dotted $p_t^{\mu,-p}$ versus t.
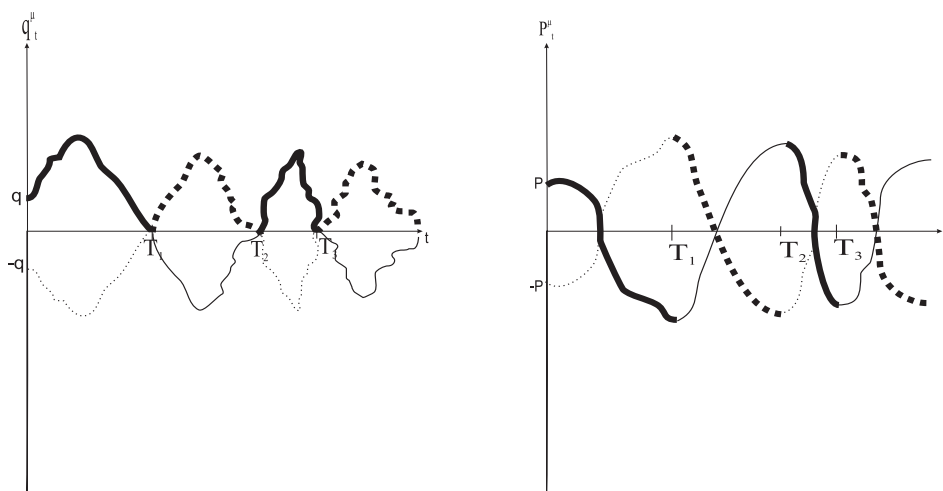

Figure 2.2: Illustration of the process with reflection

2.3 Proof that the Langevin process with reflection is well defined

Theorem 2.4 uses Lemma 2.3 to show that in any finite time T, the process l.p.r.$(q_t^\mu; p_t^\mu)$, does not have infinitely many jumps.

Lemma 2.3 *Let $T > 0$. The process $(q_t^\mu; p_t^\mu)$, starting at a point different from $(0,0)$, that satisfies system (2.2), does not reach the origin $O = (0,0)$ in finite time T, i.e. $P(\exists t \leq T \text{ s.t. } (q_t^\mu; p_t^\mu) = O) = 0$.*

Proof. Let $\delta \ll 1$ be a small number. Define the rectangle $\Delta = \{(q, p) \in R^1 \times R^1 : |q| \leq \frac{\delta^2}{2}, |p| \leq \frac{\delta}{2}\}$ and suppose that the trajectory starts from some point outside the rectangle Δ, say from $(q, 0) \in R^2 \setminus \Delta$.
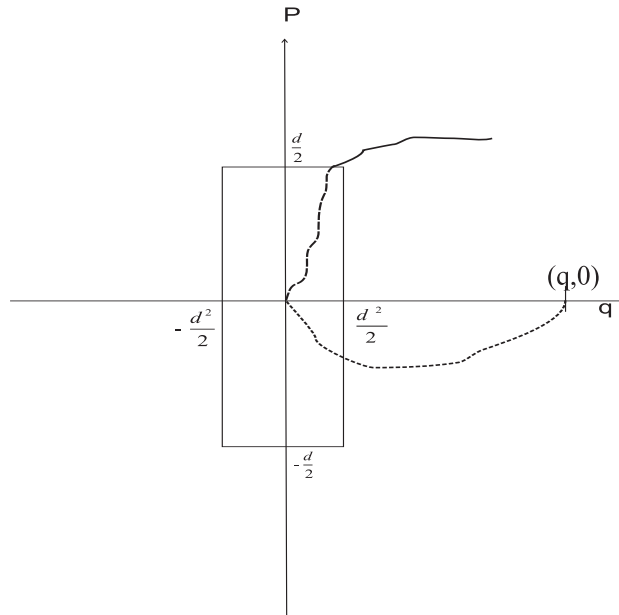

Figure 2.3: The particle does not hit the origin with positive probability

Let also $\chi_\Delta(x)$ denote the indicator function of the set Δ, which takes value 1 if $x \in \Delta$ and 0 otherwise. If $b = 0$ and $\sigma = 1$, $(q_t^\mu; p_t^\mu)$ is a Markovian Gaussian

process. One can write down its density explicitly (see equation (2.2)), which we denote by $\rho(\cdot)$, and obtain the bound

$$E^{(q,0)} \int_0^T \chi_\Delta(q_s^\mu; p_s^\mu) ds = \int_\Delta \int_0^T \rho(s, (q, 0), y) ds dy \le A(T, q) \delta^3, \qquad (2.4)$$

where $A(T, q)$ is a constant that depends on T and q, and $E^{(q,0)} \int_0^T \chi_\Delta(q_s^\mu; p_s^\mu) ds$ is the expected value of the time ,during $[0, T]$, that the process $(q_t^\mu; p_t^\mu)$ with initial point $(q, 0)$ spends inside the rectangle Δ. The general case can be reduced to the case with $b = 0$ and $\sigma = 1$ by an absolutely continuous change of measures in the space of trajectories and a random time change.

We will establish now a lower bound for the quantity $E^{(q,0)} \int_0^T \chi_\Delta(q_s^\mu; p_s^\mu) ds$ under the assumption that the Markov process $(q_t^\mu; p_t^\mu)$ will reach $(0, 0)$ before time T with positive probability. This will lead to a contradiction.

Again by Girsanov's theorem on the absolute continuity of measures in the space of trajectories it is enough to consider the solution of the following S.D.E:

$$\begin{aligned}
\dot{q}_t^\mu &= p_t^\mu \\
\dot{p}_t^\mu &= \frac{1}{\mu} \sigma(|q_t^\mu|) \dot{\overline{W}}_t \qquad\qquad (2.5) \\
q_0^\mu &= q, p_0^\mu = 0,
\end{aligned}$$

where $\overline{W}_t = \int_0^t \operatorname{sgn}(q_u^\mu) dW_u$.

By the self similarity properties of the Wiener process one can find a Wiener process W_t^* such that $\int_0^t \frac{1}{\mu} \sigma(|q_s^\mu|) d\overline{W}_s = W_{\theta(t)}^*$, where $\theta(t) = \int_0^t \frac{1}{\mu^2} \sigma^2(|q_s^\mu|) ds$. So $\int_0^t \frac{1}{\mu} \sigma(|q_s^\mu|) d\overline{W}_s$ can be obtained from W_t^* via a random time change.

By the law of iterated logarithm we get that for any $k \in [0, 1]$ there exists a $t_o(k)$ small enough, such that $P(t^{\frac{1}{2}+k} \le |W_t^*| \le t^{\frac{1}{2}-k} \text{ for } t \in [0, t_o(k)]) \ge 1 - k$.

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Observe that if $t \in [0, t_o(k)]$ then $\theta(t) \in [0, ct_o(k)]$, where $c = \frac{1}{\mu^2} \sup_{x \in R^1} |\sigma^2(x)|$.

Define also $t'_o(k) = \min\{t_o(k), \frac{t_o(k)}{c}\}$. Then with probability very close to 1, as $k \to 0$, and for any $t \in [0, t'_o(k)]$, it must hold that $|p_t^\mu| \leq c_1 t^{\frac{1}{2}-k}$ and $q_t^\mu = \int_0^t p_s ds \leq \int_0^t c_1 s^{\frac{1}{2}-k} ds < 2c_1 t^{\frac{3}{2}-k}$, for a constant c_1.

Let τ be the first time, after the time that the Markov process reached the origin, that it exits from the rectangle Δ, i.e. $\tau = \inf\{t > 0 : (q_t^\mu; p_t^\mu) \in R^2 \setminus \Delta\}$. Then it follows that

$$E^{(q,0)} \int_0^T \chi_\Delta(q_s^\mu; p_s^\mu) ds > E\{\tau\} \times P(\exists t \leq T \ s.t. \ (q_t^\mu; p_t^\mu) = O) \qquad (2.6)$$

Define $\tau_q = \inf\{t > 0 : |q_t^\mu| > \frac{\delta^2}{2}\}$ and $\tau_p = \inf\{t > 0 : |p_t^\mu| > \frac{\delta}{2}\}$. By the above bounds for q_t^μ and p_t^μ we get that $\tau_q > c_q \delta^{\frac{4}{3}}$ and $\tau_p > c_p \delta^2$, where c_q, c_p are some constants independent of δ. So the trajectory exits the rectangle faster in the direction of p than in the direction of q and the exit time is of order δ^2. Therefore, by this and by (2.4), we have that

$$B\delta^2 < E^{(q,0)} \int_0^T \chi_\Delta(q_s^\mu; p_s^\mu) ds \leq A\delta^3, \qquad (2.7)$$

which cannot hold for constants A and B and small enough δ. So we have a contradiction and therefore it is true that $P(\exists t \leq T \ s.t. \ (q_t^\mu; p_t^\mu) = O) = 0$.

$\square$

Theorem 2.4 *The following two statements are true:*

1. *Let $T > 0$. The Markov process l.p.r.$(q_t^\mu; p_t^\mu)$[1] does not reach the origin, $O =$*

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[1]We remind the reader that l.p.r. represents the Langevin process with reflection.

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$(0,0)$, *in finite time T, i.e.*

$$P(\exists t \leq T \ s.t. \ l.p.r.(q_t^\mu; p_t^\mu) = O) = 0.$$

2. *The sequence of Markov times $\{\tau_k^\mu\}$ converges to $+\infty$ as $k \to +\infty$, i.e.*

$$P(\lim_{k \to +\infty} \tau_k^\mu = +\infty) = 1.$$

Proof. The Langevin process with reflection, l.p.r.$(q_t^\mu; p_t^\mu)$, coincides at any time t either with $(q_t^{\mu,q}; p_t^{\mu,p})$ or with $(q_t^{\mu,-q}; p_t^{\mu,-p})$. Therefore we have that:

$$P(\exists t \leq T \ s.t. \ l.p.r.(q_t^\mu; p_t^\mu) = O) \ \leq \ P(\exists t \leq T \ s.t. \ (q_t^{\mu,q}; p_t^{\mu,p}) = O)$$

$$+ \ P(\exists t \leq T \ s.t. \ (q_t^{\mu,-q}; p_t^{\mu,-p}) = O).$$

Hence, by recalling Lemma 2.3, we deduce that

$$P(\exists t \leq T \ s.t. \ l.p.r.(q_t^\mu; p_t^\mu) = O) = 0.$$

Part (ii) is an easy consequence of part (i). It is easy to see that $\{\tau_k^\mu\}$ is an unbounded, strictly increasing sequence of Markov times. Indeed, if on the contrary we assume that there exists N such that $\tau_k^\mu \leq N$ for all k then the trajectories of l.p.r.$(q_t^\mu; p_t^\mu)$ will have limit points. The only possible limit point however is the origin $(0,0)$. But by part (i) the probability that within any time T the trajectory will reach the origin is 0. So $\{\tau_k^\mu\}$ is an unbounded strictly increasing sequence of Markov times. Therefore we have that $P(\lim_{k \to +\infty} \tau_k^\mu = +\infty) = 1$.

$\square$

Therefore the Langevin process with reflection has only finitely many jumps in any time interval $[0, T]$ with probability 1. Hence our definition for the Langevin process with reflection is valid.

Chapter 3

Convergence of the Langevin Process with Reflection

In this section we consider the limit of l.p.r.(q_t^μ) as $\mu \to 0$. Below we will always assume that $t \leq T$, where T is a fixed positive real number.

Consider first the following S.D.E.s in R^2 and R^1 respectively:

$$\dot{\widetilde{q}}_t^\mu = \widetilde{p}_t^\mu$$

$$\mu \dot{\widetilde{p}}_t^\mu = -\widetilde{p}_t^\mu + \sigma(|\widetilde{q}_t^\mu|)\dot{\widetilde{W}}_t \tag{3.1}$$

$$\widetilde{q}_0^\mu = q, \widetilde{p}_0^\mu = p,$$

and

$$\dot{\widetilde{q}}_t = \sigma(|\widetilde{q}_t|)\dot{\widetilde{W}}_t \tag{3.2}$$

$$\widetilde{q}_0 = q,$$

where $\widetilde{W}_t$ is the standard one-dimensional Wiener process.

Lemma 3.1 *For every $\delta > 0$ we have that $E \int_0^T \chi_{\{|\widetilde{q}_s| \leq \delta\}} ds \leq c\delta$, where c is a constant.*

Proof. If $\sigma = 1$, $\widetilde{q}_t$ is a Gaussian process. By writing down its transition density we obtain the bound above, i.e.,

$$E \int_0^T \chi_{\{|\widetilde{q}_s| \leq \delta\}} ds \leq \widetilde{c}\delta.$$

The general case can be reduced to the case with $\sigma = 1$ by a random time change.

$\square$

Let us consider now the following S.D.E.'s:

$$\dot{\overline{q}}_t^\mu = \overline{p}_t^\mu$$

$$\mu \dot{\overline{p}}_t^\mu = -\overline{p}_t^\mu + \mathrm{sgn}(\overline{q}_t^\mu)b(|\overline{q}_t^\mu|) + \sigma(|\overline{q}_t^\mu|)\dot{\widetilde{W}}_t \qquad (3.3)$$

$$\overline{q}_0^\mu = q, \overline{p}_0^\mu = p$$

and

$$\dot{\overline{q}}_t = \mathrm{sgn}(\overline{q}_t)b(|\overline{q}_t|) + \sigma(|\overline{q}_t|)\dot{\widetilde{W}}_t \qquad (3.4)$$

$$\overline{q}_0 = q$$

By the same way that we proved Lemma 2.2, one can show that equations (3.3) and (3.4) have weak solutions, which are unique in the sense of probability law.

Lemma 3.2 *For the time interval $[0,T]$, $\overline{q}^\mu \to \overline{q}_.$, weakly as $\mu \to 0$, where $\overline{q}_t^\mu$ satisfies (3.3) and $\overline{q}_t$ satisfies (3.4).*

Proof. We must prove that for any bounded and continuous functional f that

$$E_\mu f(\overline{q}^\mu) \to E f(\overline{q}_.) \text{ as } \mu \to 0. \qquad (3.5)$$

Let us define

$$R^\mu(T) = e^{Z^\mu(T)}, Z^\mu(T) = \int_0^T \mathrm{sgn}(\widetilde{q}_u^\mu)\phi(\widetilde{q}_u^\mu)d\widetilde{W}_u - \frac{1}{2}\int_0^T |\phi(\widetilde{q}_u^\mu)|^2 du, \qquad (3.6)$$

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and

$$R(T) = e^{Z(T)}, Z(T) = \int_0^T \mathrm{sgn}(\tilde{q}_u)\phi(\tilde{q}_u)d\widetilde{W}_u - \frac{1}{2}\int_0^T |\phi(\tilde{q}_u)|^2 du, \qquad (3.7)$$

where $\tilde{q}_t^\mu$ and $\tilde{q}_t$ satisfy (3.1) and (3.2) respectively and $\phi(\cdot) = \frac{b(|\cdot|)}{\sigma(|\cdot|)}$.

By the Girsanov's Theorem on the absolutely continuous change of measure in the space of trajectories we have that

$$
\begin{aligned}
|E_\mu f(\overline{q}^\mu_\cdot) - Ef(\overline{q}_\cdot)| &= |E[f(\tilde{q}^\mu_\cdot)R^\mu(T) - f(\tilde{q}_\cdot)R(T)]| && (3.8)\\
&\leq |E[R^\mu(T)(f(\tilde{q}^\mu_\cdot) - f(\tilde{q}_\cdot))]| + |E[f(\tilde{q}_\cdot)(R^\mu(T) - R(T))]|.
\end{aligned}
$$

The Cauchy inequality, the boundedness of ϕ and the fact that $\tilde{q}_t^\mu \to \tilde{q}_t$ uniformly in $[0, T]$ in probability, imply that $|E[R^\mu(T)(f(\tilde{q}^\mu_\cdot) - f(\tilde{q}_\cdot))]| \to 0$, as $\mu \to 0$.

It remains to show that

$$|E[f(\tilde{q}_\cdot)(R^\mu(T) - R(T))]| \leq \sqrt{Ef^2(\tilde{q}_\cdot)}\sqrt{E[R^\mu(T) - R(T)]^2} \to 0, \text{ as } \mu \to 0.$$

If we take into account the basic inequality $|e^x - e^y| \leq \max\{e^x, e^y\}|x - y|$ for all $x, y \in R$, Cauchy inequality and the fact that ϕ is bounded, we get that there is a constant $\tilde{c} = \tilde{c}(\sup_{x \in R}|\phi(x)|, T)$, such that

$$E[R^\mu(T) - R(T)]^2 \leq \tilde{c}\sqrt{E[Z^\mu(T) - Z(T)]^4}.$$

So it is enough to show that $E[Z^\mu(T) - Z(T)]^4 \to 0$, as $\mu \to 0$. We have:

$$
\begin{aligned}
E[Z^\mu(T) - Z(T)]^4 &\leq cE\int_0^T [\mathrm{sgn}(\tilde{q}^\mu_s)\phi(\tilde{q}^\mu_s) - \mathrm{sgn}(\tilde{q}_s)\phi(\tilde{q}_s]^4 ds\\
&\quad + \frac{1}{2}E[\int_0^T [|\phi(\tilde{q}^\mu_s)|^2 - |\phi(\tilde{q}_s)|^2]ds]^4.
\end{aligned}
$$

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It is easy to see that the second term of the right hand side of the inequality above converges to 0 as $\mu \to 0$. Moreover because $\tilde{q}_t^\mu \to \tilde{q}_t$ uniformly in $[0, T]$ in probability, we get that for any $k \in [0, 1]$, there exists a $\mu_o > 0$ so small such that $|\tilde{q}_t^\mu - \tilde{q}_t| < k$, for all $\mu < \mu_o, t \leq T$ with probability at least $1 - k$. For given δ now choose k such that $2k < \delta$. It is easy to see that if $|\tilde{q}_t| > \delta$, then $\tilde{q}_t^\mu$ for $\mu < \mu_o$ and $\tilde{q}_t$ have the same sign with probability at least $1 - k$. Hence for $\mu < \mu_o$ we have:

$$
E \int_0^T [\operatorname{sqn}(\tilde{q}_s^\mu)\phi(\tilde{q}_s^\mu) - \operatorname{sqn}(\tilde{q}_s)\phi(\tilde{q}_s)]^4 ds
$$

$$
= \int_0^T E[\phi(\tilde{q}_s^\mu) - \phi(\tilde{q}_s)]^4 \chi_{\{|\tilde{q}_s|>\delta\}} ds + \int_0^T E[\operatorname{sqn}(\tilde{q}_s^\mu)\phi(\tilde{q}_s^\mu) - \operatorname{sqn}(\tilde{q}_s)\phi(\tilde{q}_s)]^4 \chi_{\{|\tilde{q}_s|\leq\delta\}} ds
$$

$$
\leq \int_0^T E[\phi(\tilde{q}_s^\mu) - \phi(\tilde{q}_s)]^4 \chi_{\{|\tilde{q}_s|>\delta\}} ds + c_1 E \int_0^T \chi_{\{|\tilde{q}_s|\leq\delta\}} ds
$$

$$
\leq \int_0^T E[\phi(\tilde{q}_s^\mu) - \phi(\tilde{q}_s)]^4 \chi_{\{|\tilde{q}_s|>\delta\}} ds + c_2\delta,
$$

where we have used the boundedness of ϕ and Lemma 3.1.

If we let now $\delta, \mu \to 0$ we get $E[Z^\mu(T) - Z(T)]^4 \to 0$. Therefore (3.5) has been proven.

$\square$

Along with equation (2.2), consider lastly the following S.D.E. in R^1:

$$
\dot{q}_t = \operatorname{sgn}(q_t)b(|q_t|) + \operatorname{sgn}(q_t)\sigma(|q_t|)\dot{W}_t \tag{3.9}
$$

$$
q_0 = q,
$$

Because of the non-degeneracy of the diffusion coefficient near $x^1 = 0$, one can prove existence and uniqueness of the weak solution of (3.9), (see for example [3]).

Lemma 3.3 *For the time interval* $[0, T]$, $q_\cdot^\mu \to q_\cdot$ *weakly as* $\mu \to 0$, *where* q_t^μ *satisfies (2.2) and* q_t *satisfies (3.9).*

Proof. We must show that for any bounded and continuous functional f that

$$E_\mu f(q^\mu_\cdot) \to E f(q_\cdot) \text{ as } \mu \to 0. \qquad (3.10)$$

The latter follows immediately from Lemma 3.2. Indeed, let us define the Wiener processes $\widetilde{W}^\mu_t = \int_0^t \text{sqn}(q^\mu_s) dWs$ and $\widetilde{W}_t = \int_0^t \text{sqn}(q_s) dWs$. Then in terms of the new Wiener processes, equations (2.2) and (3.9) take the form of (3.3) and (3.4) respectively. Hence we will have that $E_\mu f(q^\mu_\cdot) = E_\mu f(\overline{q}^\mu_\cdot)$ and $Ef(q_\cdot) = Ef(\overline{q}_\cdot)$, where $\overline{q}^\mu_t$ satisfies (3.3) and $\overline{q}_t$ satisfies (3.4).

$\square$

Lastly by Lemma 3.3 we get that $|q^\mu_\cdot| \to |q_\cdot|$, weakly as $\mu \to 0$.

We sum up our result in

Theorem 3.4 *For the time interval* $[0, T]$*, the Langevin process with reflection* l.p.r.(q^μ_t) *converges, weakly as* $\mu \to 0$*, to* $|q_t|$*; i.e.*

$$\text{l.p.r.}(q^\mu_\cdot) \to |q_\cdot|, \text{ weakly as } \mu \to 0. \qquad (3.11)$$

Bibliography

[1] M. Freidlin, 1985, Functional Integration and Partial Differential Equations, Princeton University Press.

[2] M. Freidlin, 2004, Some Remarks on the Smoluchowski-Kramers Approximation, Journal of Statistical Physics, Vol. 117, No. 314, pp. 617-634.

[3] I. V. Girsanov, 1961, On a stochastic integral Ito's equation, Dokl.Acd.Nauk. SSSR, 138, 18-21 (Russian).

[4] I.Karatzas, S.E.Shreve, 1994, Brownian Motion and Stochastic Calculus, Second edition, Springer.

[5] H. Kramers, 1940, Brownian Motion in a field of force and the diffusion model of chemical reactions, Physica, Vol. 7, pp. 284-304.

[6] M. Smoluchowski, 1916, Drei Vortrage über Diffusion Brownsche and Koagulation von Kolloidteilchen, Phys. Z., Vol 17, pp. 557-585.

[7] D.W. Stroock, S.R.S. Varadhan 1979, Multidimensional Diffusion Processes, Springer.

[8] J. Spiliotis 2004, Stochastic Differential Equations (in Greek).