A NOTE ON THE SMOLUCHOWSKI-KRAMERS APPROXIMATION FOR THE LANGEVIN EQUATION WITH REFLECTION*

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Abstract

According to the Smoluchowski-Kramers approximation, the solution of the equation $\mu \ddot{q}_t^\mu = b(q_t^\mu) - \dot{q}_t^\mu + \Sigma(q_t^\mu)\dot{W}_t, q_0^\mu = q, \dot{q}_0^\mu = p$ converges to the solution of the equation $\dot{q}_t = b(q_t) + \Sigma(q_t)\dot{W}_t, q_0 = q$ as $\mu \to 0$. We consider here a similar result for the Langevin process with elastic reflection on the boundary.

Keywords: Smoluchowski-Kramers approximation, reflection, Langevin equation, Skorohod reflection problem.

1 Introduction

The well-known Smoluchowski-Kramers approximation ([9],[8]) implies that the solution of the stochastic differential equation (S.D.E.)

$$
\begin{aligned}
\mu \ddot{q}_t^\mu &= b(q_t^\mu) - \dot{q}_t^\mu + \Sigma(q_t^\mu)\dot{W}_t \\
q_0^\mu &= q \in \mathbb{R}^r \\
\dot{q}_0^\mu &= p \in \mathbb{R}^r
\end{aligned}
\tag{1}
$$

converges in probability as $\mu \to 0$ to the solution of the following S.D.E.:

$$
\begin{aligned}
\dot{q}_t &= b(q_t) + \Sigma(q_t)\dot{W}_t \\
q_0 &= q \in \mathbb{R}^r,
\end{aligned}
\tag{2}
$$

where $b = (b_1, ..., b_r)'$ (the transpose of $(b_1, ..., b_r)$) with $b_j : \mathbb{R}^r \to \mathbb{R}, j = 1, .., r$, $\Sigma = [\sigma_{ij}]_{i,j}^r$ with $\sigma_{ij} : \mathbb{R}^r \to \mathbb{R}, i, j = 1, .., r$ have bounded first derivatives and $W_t = (W_t^1, ..., W_t^r)'$ is the standard r-dimensional Wiener process. In other

words, one can prove that for any $\delta, T > 0$ and $q, p \in \mathbb{R}^r$ (see, for example, Lemma 1 in [6]),

$$\lim_{\mu \downarrow 0} P(\max_{0 \leq t \leq T} |q_t^\mu - q_t| > \delta) = 0. \tag{3}$$

Equation (1) describes the motion of a particle of mass μ in a force field $b(q) + \Sigma(q)\dot{W}_t$ with a friction proportional to velocity. The Smoluchowski-Kramers approximation justifies the use of equation (2) to describe the motion of a small particle.

It is easy to see now that (1) can be equivalently written as:

$$\begin{aligned}
\dot{q}_t^\mu &= p_t^\mu \\
\mu \dot{p}_t^\mu &= b(q_t^\mu) - p_t^\mu + \Sigma(q_t^\mu)\dot{W}_t \\
q_0^\mu &= q \in \mathbb{R}^r, \ \dot{q}_0^\mu = p \in \mathbb{R}^r.
\end{aligned} \tag{4}$$

Let us define $\mathbb{R}_+ = \{q^1 \in \mathbb{R} : q^1 \geq 0\}$ and let the configuration space be $D = \mathbb{R}_+ \times \mathbb{R}^{r-1}$. In this paper we examine the behavior of the process with elastic reflection on the boundary $\partial D \times \mathbb{R}^r = (\partial \mathbb{R}_+ \times \mathbb{R}^{r-1}) \times \mathbb{R}^r$ of the phase space $D \times \mathbb{R}^r$ that is governed by (4), i.e. of the Langevin process with reflection, as $\mu \to 0$ when Σ is the unit matrix. We will show that the first component (the q component) of the Langevin process with reflection at $q^1 = 0$, that is governed by equation (4), converges in distribution to the diffusion process with reflection on ∂D that is governed by (2). The method is based on properties of the Skorohod reflection problem and in techniques developed in [2] and in [3]. In section 2 we define the Langevin process with reflection for general diffusion matrx Σ with inputs that have bounded first derivatives, in section 3 we describe the Skorohod reflection problem and in section 4 we consider the limit $\mu \to 0$ when the diffusion matrix is the unit matrix. We note here that the limit when $\mu \to 0$ for a general diffusion matrix as above can be examined similarly.

2   Langevin process with reflection and preliminary results

We begin with the construction of the Langevin process $(q_t^\mu; p_t^\mu)$ in $D \times \mathbb{R}^r$ with elastic reflection on the boundary. Let $b = (b_1, ..., b_r)'$ with $b_j : D \to \mathbb{R}, j = 1, .., r$ and $\Sigma = [\sigma_{ij}]$ with $\sigma_{ij} : D \to \mathbb{R}, i, j = 1, .., r$ have bounded first derivatives and Σ be non-degenerate. Let $(q, p) \in D \times \mathbb{R}^r$ be the initial point (we

assume that $(q^1)^2 + (p^1)^2 \neq 0$). Then $(q_t^\mu; p_t^\mu)$ is the right-continuous Markov process in $D \times \mathbb{R}^r$ defined as follows. Consider the following system of S.D.E.'s:

$$\dot{q}_t^{i,\mu} = p_t^{i,\mu}$$

$$\mu \dot{p}_t^{i,\mu} = -p_t^{i,\mu} + b_i(q_t^\mu) + \sum_{j=1}^{r} \sigma_{ij}(q_t^\mu) \dot{W}_t^j \tag{5}$$

$$q_0^{i,\mu} = q^i, \ p_0^{i,\mu} = p^i, \ i = 1, ..., r.$$

We define $(q_t^\mu; p_t^\mu)$ to be the solution to (5) for $t \in [0, \tau_1^\mu)$, where $\tau_1^\mu = \inf\{t > 0 : q_t^{1,\mu} = 0\}$. Then define $(q_t^\mu; p_t^\mu)$ for $t \in [\tau_1^\mu, \tau_2^\mu)$, where $\tau_2^\mu = \inf\{t > \tau_1^\mu : q_t^\mu = 0\}$, to be the solution of (5) with initial conditions

$$(q_{\tau_1^\mu}^\mu; p_{\tau_1^\mu}^\mu) = (0, \lim_{t \uparrow \tau_1^\mu} q_t^{2,\mu}, ..., \lim_{t \uparrow \tau_1^\mu} q_t^{r,\mu}; -\lim_{t \uparrow \tau_1^\mu} p_t^{1,\mu}, \lim_{t \uparrow \tau_1^\mu} p_t^{2,\mu}, ..., \lim_{t \uparrow \tau_1^\mu} p_t^{r,\mu}).$$

If $0 < \tau_1^\mu < \tau_2^\mu < ... < \tau_k^\mu$ and $(q_t^\mu; p_t^\mu)$ for $t \in [0, \tau_k^\mu)$ are already defined, then define $(q_t^\mu; p_t^\mu)$ for $t \in [\tau_k^\mu, \tau_{k+1}^\mu)$ as solution of (5) with initial conditions

$$(q_{\tau_k^\mu}^\mu; p_{\tau_k^\mu}^\mu) = (0, \lim_{t \uparrow \tau_k^\mu} q_t^{2,\mu}, ..., \lim_{t \uparrow \tau_k^\mu} q_t^{r,\mu}; -\lim_{t \uparrow \tau_k^\mu} p_t^{1,\mu}, \lim_{t \uparrow \tau_k^\mu} p_t^{2,\mu}, ..., \lim_{t \uparrow \tau_k^\mu} p_t^{r,\mu})$$

(see Figure 1 for an illustration).

This construction defines the process $(q_t^\mu; p_t^\mu)$ in $D \times \mathbb{R}^r$ for all $t \geq 0$. This follows from Theorem 2.4, which states that the process that we constructed above does not have infinitely many jumps in any finite time interval $[0, T]$. Therefore we have the following definition:

Definition 2.1. *We call the above recursively constructed process, the Langevin process with elastic reflection on the boundary $\partial D \times \mathbb{R}^r$. This process has jumps on $\partial D \times \mathbb{R}^r$ and is continuous inside $D \times \mathbb{R}^r$.*

We will refer to the Langevin process with reflection as l.p.r.$(q_t^\mu; p_t^\mu)$. Moreover we will denote by $(q_t^{\mu,q}; p_t^{\mu,p})$ the trajectories of $(q_t^\mu; p_t^\mu)$ with initial position (q, p). For easy of notation we also define $-x = (-x^1, x^2, \ldots, x^r)$ and $|x| = (|x^1|, x^2, \ldots, x^r)$ for $x \in \mathbb{R}^r$.

Below we see an illustration of the construction above in the $(q^1 - p^1)$ phase space.

Let us give now another construction of the Langevin process with reflection. Consider the following S.D.E. in $\mathbb{R}^{2r}$:

$$\dot{q}_t^{1,\mu} = p_t^{1,\mu}$$

$$\mu \dot{q}_t^{1,\mu} = -p_t^{1,\mu} + \text{sgn}(q_t^{1,\mu}) b_1(|q_t^\mu|) + \sum_{j=1}^{r} \text{sgn}(q_t^{1,\mu}) \sigma_{1j}(|q_t^\mu|) \dot{W}_t^j$$

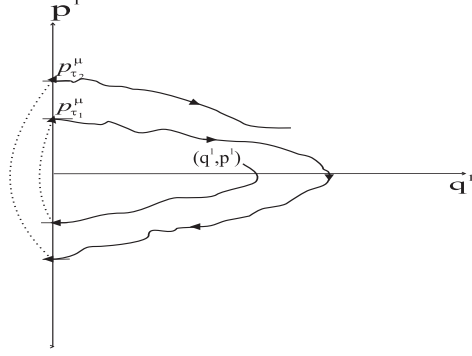$$q_0^{1,\mu} = q^1, p_0^{1,\mu} = p^1,$$

$$\dot{q}_t^{i,\mu} = p_t^{i,\mu} \tag{6}$$

Figure 1: Illustration of the Langevin process with reflection in the $(q^1 - p^1)$ phase space

$$\mu \dot{p}_t^{i,\mu} = -p_t^{i,\mu} + b_i(|q_t^\mu|) + \sum_{j=1}^r \sigma_{ij}(|q_t^\mu|)\dot{W}_t^j$$

$$q_0^{i,\mu} = q^i, p_0^{i,\mu} = p^i, i = 2, ..., r,$$

where $\text{sgn}(x)$ takes two values, 1 if $x \geq 0$ and -1 if $x < 0$.

Lemma 2.2. *Equation (6) has a weak solution which is unique in the sense of probability law.*

Proof. The existence follows from the Girsanov's Theorem on the absolute continuous change of measures in the space of trajectories (b and Σ are assumed bounded) and the fact that (6) with $b = 0$ has a weak solution. The uniqueness follows from Proposition 5.3.10 of [7].

$\square$

Using the processes $(q_t^{\mu,q}; p_t^{\mu,p})$ and $(q_t^{\mu,-q}; p_t^{\mu,-p})$ we can give another construction of the Langevin process with reflection, as follows. Assume that $q^1 > 0$ and $p^1 > 0$, The graphs of p_t^{1,μ,p^1} and $p_t^{1,\mu,-p^1}$ will be exactly symmetric with respect to zero. The same will be true also for the graphs of q_t^{1,μ,q^1} and of $q_t^{1,\mu,-q^1}$. Let $\tau_0^\mu = 0, \tau_k^\mu = \inf\{t > \tau_{k-1}^\mu : q_t^{1,\mu,q^1} = 0\}$ and $(\widehat{q}_t^\mu; \widehat{p}_t^\mu)$ be a stochastic process, which is defined as follows:

$$(\widehat{q}_t^\mu; \widehat{p}_t^\mu) = (q_t^{\mu,q}; p_t^{\mu,p}) \text{ for } \tau_{2k}^\mu \leq t \leq \tau_{2k+1}^{\mu,-}$$
$$(\widehat{q}_t^\mu; \widehat{p}_t^\mu) = (q_t^{\mu,-q}; p_t^{\mu,-p}) \text{ for } \tau_{2k+1}^\mu \leq t \leq \tau_{2k+2}^{\mu,-}, k = 0, 1, 2, ... \quad (7)$$

Process $(\widehat{q}_t^\mu; \widehat{p}_t^\mu)$ is a process with reflection and it can be seen that $(\widehat{q}_t^\mu; \widehat{p}_t^\mu)$, which is the same as $(|q_t^{1,\mu}|, q_t^{2,\mu}, \cdots, q_t^{r,\mu}; \frac{d}{dt}|q_t^{1,\mu}|, \dot{q}_t^{2,\mu}, \cdots, \dot{q}_t^{r,\mu})$, and l.p.r.$(q_t^\mu; p_t^\mu)$ coincide.

In the figures below we give an illustration of the construction of $(\widehat{q}_t^{1,\mu}; \widehat{p}_t^{1,\mu})$. The first figure illustrates with thick continuous and dotted lines $\widehat{q}_t^{1,\mu}$ versus t. The continuous line is q_t^{1,μ,q^1} versus t and the dotted is $q_t^{1,\mu,-q^1}$ versus t. The second figure illustrates with thick continuous and dotted lines $\widehat{p}_t^{1,\mu}$ versus t. The continuous line is p_t^{1,μ,p^1} versus t and the dotted is $p_t^{1,\mu,-p^1}$ versus t.
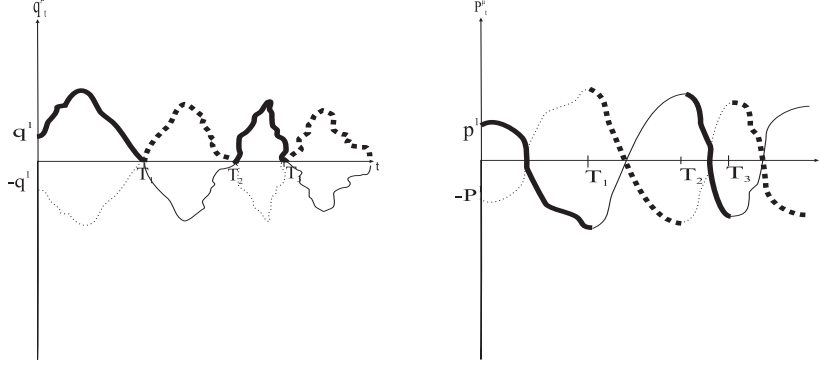
Figure 2: Illustration of the process with reflection

Lemma 2.3. *Let $T > 0$. The Markov process $(q_t^\mu; p_t^\mu)$ starting at a point (q, p) different from the origin $O = (0, ..., 0; 0, ..., 0)$, that satisfies system (6), does not reach the origin O in finite time T, i.e.*

$$P(\exists t \leq T \ s.t. \ (q_t^\mu; p_t^\mu) = O) = 0.$$

Proof. We easily see that it is actually enough to consider only $(q_t^{1,\mu}; p_t^{1,\mu})$. Let $\delta \ll 1$ be a small number. Define the rectangle $\Delta = \{(q, p) \in \mathbb{R} \times \mathbb{R} : |q| \leq \frac{\delta^2}{2}, |p| \leq \frac{\delta}{2}\}$ and suppose that the trajectory starts from some point outside the rectangle Δ, say from $(q, 0) \in \mathbb{R}^2 \setminus \Delta$. Let also $\chi_\Delta(x)$ denote the indicator function of the set Δ. Then $E^{(q,0)} \int_0^T \chi_\Delta(q_s^1, p_s^1) ds$ is the expected value of the time ,during time $[0, T]$, that the process (q_t^1, p_t^1) with initial point $(q, 0)$ spends inside the rectangle Δ. If $b = 0$ and Σ is a matrix with constant entries, (q_t^1, p_t^1) is a Gaussian process. One can write down its density explicitly (see equation (6)), which we denote by $\rho(\cdot)$, and obtain the bound

$$E^{(q,0)} \int_0^T \chi_\Delta(q_s^1, p_s^1) ds = \int_\Delta \int_0^T \rho(s, (q, 0), y) ds dy \leq A(T, q) \delta^3 \qquad (8)$$

where $A(T, q)$ is a constant that depends on T and q. The general case can be reduced to the case with $b = 0$ and Σ constant by an absolutely continuous change of measures in the space of trajectories and by a random time change.

We will establish now a lower bound for the quantity $E^{(q,0)} \int_0^T \chi_\Delta(q_s^1, p_s^1) ds$ under the assumption that the process $(q_t^{1,\mu}, p_t^{1,\mu})$ will reach $(0, 0)$ before time T with positive probability. This will lead to a contradiction.

Again by Girsanov's theorem on the absolute continuity of measures in the

space of trajectories it is enough to consider the solution of the following S.D.E:

$$\dot{q}_t^1 = p_t^1$$

$$\dot{p}_t^1 = \frac{1}{\mu} \sum_{j=1}^{r} \sigma_{1j}(q_t^\mu)\dot{\overline{W}}_t^j \qquad (9)$$

$$q_0^1 = q^1, p_0^1 = p^1,$$

where $\overline{W}_t^j = \int_0^t \operatorname{sgn}(q_u^{1,\mu})dW_u^j$.

By the self similarity properties of the Wiener process one can find a Wiener process $W_t^{1,*}$ such that $\int_0^t \frac{1}{\mu} \sum_{j=1}^{r} \sigma_{1j}(q_t^\mu)\dot{\overline{W}}_t^j = W_{\theta(t)}^{1,*}$, where $\theta(t) = \int_0^t \frac{1}{\mu^2}\alpha_{11}(q_s^\mu)ds$ and $\alpha_{11} = \sum_{j,k=1}^{r} \sigma_{1j}\sigma_{1k}$. So $\int_0^t \frac{1}{\mu} \sum_{j=1}^{r} \sigma_{1j}(q_t^\mu)\dot{\overline{W}}_t^j$ can be obtained from $W_t^{1,*}$ via a random time change.

By the law of iterated logarithm we get that for all $k \in [0,1]$ there exists a $t_o(k)$ small enough, such that

$$P(t^{\frac{1}{2}+k} \le |W_t^{1,*}| \le t^{\frac{1}{2}-k} \text{ for } t \in [0, t_o(k)]) \ge 1 - k.$$

Observe that if $t \in [0, t_o(k)]$ then $\theta(t) \in [0, ct_o(k)]$, where $c = \frac{1}{\mu^2}\sup_{x \in \mathbb{R}}|\alpha_{11}(x)|$. Define also $t_o'(k) = \min\{t_o(k), \frac{t_o(k)}{c}\}$. Then with probability very close to 1, as $k \to 0$, and for all $t \in [0, t_o'(k)]$ it must hold that $|p_t^{1,\mu}| \le c_1 t^{\frac{1}{2}-k}$ and $q_t^{1,\mu} = \int_0^t p_s^{1,\mu}ds \le \int_0^t c_1 s^{\frac{1}{2}-k}ds < 2c_1 t^{\frac{3}{2}-k}$, for a constant c_1.

Let τ be the first time, after the time that the Markov process reached the origin, that it exits from the rectangle Δ, i.e. $\tau = \inf\{t > 0 : (q_t^1, p_t^1) \in \mathbb{R}^2 \setminus \Delta\}$. Then it follows that

$$E^{(q,0)} \int_0^T \chi_\Delta(q_s^1, p_s^1)ds > E\{\tau\} \times P(\exists t \le T \text{ s.t. } (q_t^{1,\mu}; p_t^{1,\mu}) = (0,0)) \qquad (10)$$

Define $\tau_q = \inf\{t > 0 : |q_t^{1,\mu}| > \frac{\delta^2}{2}\}$ and $\tau_p = \inf\{t > 0 : |p_t^{1,\mu}| > \frac{\delta}{2}\}$. By the above bounds for $q_t^{1,\mu}$ and $p_t^{1,\mu}$ we get that $\tau_q > c_q \delta^{\frac{4}{3}}$ and $\tau_p > c_p \delta^2$, where c_q, c_p are some constants independent of δ. So the trajectory exits the rectangle faster in the direction of p than in the direction of q and the exit time is of order δ^2. Therefore, by this and by (8), we have that

$$B\delta^2 < E^{(q,0)} \int_0^T \chi_\Delta(q_s^1, p_s^1)ds \le A\delta^3, \qquad (11)$$

which cannot hold for constants A and B and small enough δ. So we have a contradiction and hence it is true that $P(\exists t \le T \text{ s.t. } (q_t^{1,\mu}; p_t^{1\mu}) = (0,0)) = 0$.

$\square$

Theorem 2.4. *We have the following two statements:*

1. *Let $T > 0$. The Markov process l.p.r.$(q_t^\mu; p_t^\mu)$ (with arbitrary b) does not reach the origin $O = (0, ..., 0; 0, ..., 0)$ in finite time T, namely*

$$P(\exists t \leq T \ s.t. \ l.p.r.(q_t^\mu; p_t^\mu) = O) = 0.$$

2. *The sequence of Markov times $\{\tau_k^\mu\}$ converges to $+\infty$ as $k \to +\infty$, i.e.*

$$P(\lim_{k \to +\infty} \tau_k^\mu = +\infty) = 1$$

Proof. The Langevin process with reflection, l.p.r.$(q_t^\mu; p_t^\mu)$, coincides at any time t either with $(q_t^{\mu,q}; p_t^{\mu,p})$ or with $(q_t^{\mu,-q}; p_t^{\mu,-p})$. Therefore we have that:

$$
\begin{aligned}
P(\exists t \leq T \ s.t. \ \text{l.p.r.}(q_t^\mu; p_t^\mu) = O) \quad &\leq \quad P(\exists t \leq T \ s.t. \ (q_t^{\mu,q}; p_t^{\mu,p}) = O) \\
&+ \quad P(\exists t \leq T \ s.t. \ (q_t^{\mu,-q}; p_t^{\mu,-p}) = O).
\end{aligned}
$$

Hence Lemma 2.3 implies that

$$P(\exists t \leq T \ s.t. \ \text{l.p.r.}(q_t^\mu; p_t^\mu) = O) = 0.$$

Part (ii) is an easy consequence of part (i). It is easy to see that $\{\tau_k^\mu\}$ is an unbounded, strictly increasing sequence of Markov times. Indeed, if on the contrary we assume that there exists a N such that $\tau_k^\mu \leq N$ for all k with positive probability, then the trajectories of l.p.r.$(q_t^\mu; p_t^\mu)$ will have limit points. The only possible limit point however is the origin $(0, ..., 0; 0, ..., 0)$. But by part (i) the probability that within any time T the trajectory will reach the origin is 0. So $\{\tau_k^\mu\}$ is an unbounded strictly increasing sequence of Markov times. Therefore we have that $P(\lim_{k \to +\infty} \tau_k^\mu = +\infty) = 1$.

$\square$

Therefore the Langevin process with reflection has only finitely many jumps in any time interval $[0, T]$ with probability 1. Hence our definition for the Langevin process with reflection is correct.

3 The Skorohod reflection problem

The convergence of the Langevin process with reflection that will be presented in section 4 relies on results about solutions of the Skorohod reflection problem, proven in [3] and [10].

Let us first recall that $D = \mathbb{R}_+ \times \mathbb{R}^{r-1}$, $\partial D = \partial \mathbb{R}_+ \times \mathbb{R}^{r-1}$ and let $N(q)$ be the set of inward normals at $q \in \partial D$. Denote also by $\mathbb{D}(\mathbb{R}_+, D)$ the space of *cadlág* (right continuous with left limits) functions with values in D, endowed with the Skorohod topology and by $\mathbb{B}.\mathbb{V}.(\mathbb{R}_+, D)$ the set of *cadlág* functions with bounded variation and values in D.

Definition 3.1. *Let w be a function in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^r)$ such that $w(0) \in D$. We say that the pair (q, ϕ) with $q \in \mathbb{D}(\mathbb{R}_+, D)$, $\phi \in \mathbb{B}.\mathbb{V}.(\mathbb{R}_+, \mathbb{R}^r)$ is a solution to the Skorohod problem for (D, N, w) if*

$$q_t = w_t + \phi_t$$

$$\phi_t = \int_0^t \nu(s) d|\phi|_s, \nu(s) \in N(q_s), d|\phi| - a.e.$$

$$d|\phi|(t : q_t \in D) = 0,$$

where $|\phi|$ denotes the total variation of ϕ and is called the local time of the solution.

The following theorem characterizes the continuity properties of solutions of the Skorohod reflection problem.

Theorem 3.2. *Let W be a compact subset of $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^r)$ in the Skorohod topology such that $w(0) \in D$ for every $w \in W$. Moreover let $\mathfrak{Q}$ be the set of $(q, \phi, |\phi|, w) \in \mathbb{D}(\mathbb{R}_+, D) \times \mathbb{B}.\mathbb{V}.(\mathbb{R}_+, \mathbb{R}^r) \times \mathbb{B}.\mathbb{V}.(\mathbb{R}_+, \mathbb{R}_+) \times \mathbb{D}(\mathbb{R}_+, \mathbb{R}^r)$ such that (q, ϕ) is the solution to the Skorohod problem for (D, N, w) for some $w \in W$ and q is continuous. The set D is convex and so $\mathfrak{Q}$ is a relatively compact subset of $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{3r+1})$ in the Skorohod topology and for every accumulation point of $(q, \phi, |\phi|, w)$ in $\mathfrak{Q}$ we have that (q, ϕ) is a solution to the Skorohod problem for (D, N, w).*

Proof. This is a special case of theorem 3.2 in [2].

$\square$

4 Convergence of the Langevin process with reflection

In this section we consider the limit of l.p.r.(q_t^μ) as $\mu \to 0$ when the diffusion matrix is the unit matrix. Below we will assume that $t \le T$, where T ia s positive real number.

Consider the stochastic process $(q_t^\mu; p_t^\mu)$ in $D \times \mathbb{R}^r$, which satisfies the following system of S.D.E.'s:

$$
\begin{aligned}
\dot{q}_t^\mu &= p_t^\mu \\
\mu \dot{p}_t^\mu &= -p_t^\mu + b(q_t^\mu) + \dot{W}_t + \nu(q_t^\mu) \cdot \dot{\Psi}_t^\mu \\
q_0^\mu &= q_0, p_0^\mu = p_0,
\end{aligned}
\tag{12}
$$

where $q_t^\mu = (q_t^{1,\mu}, \cdots, q_t^{r,\mu})'$, $p_t^\mu = (p_t^{1,\mu}, \cdots, p_t^{r,\mu})'$, $W_t = (W_t^1, \cdots, W_t^r)'$, $\nu(q)$ denotes the unit inward normal to D at $q \in \partial D$, $b(q) = (b_1(q), ..., b_r(q))'$ and $\Psi_t^\mu = \mu \sum_{s \le t} (-2p_{s-}^\mu \cdot \nu(q_s^\mu)) \cdot \chi_{\partial D}(q_s^\mu)$. It is easy to see that (12) is pathwise

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equivalent to the Langevin process with reflection in $D \times \mathbb{R}^r$ of Definition 2.1. and so it admits a unique weak solution.

We will follow the method introduced in [2]. The main idea is to represent q^μ as the first component of a solution to the Skorohod problem for $(D, N, H^\mu + X^\mu)$, where $H^\mu + X^\mu$ is a semimartingale. The family $\{H^\mu + X^\mu\}$ turns out to be tight and this enables us to use Theorem 3.2 to conclude that the family $\{q^\mu\}$ is tight as well.

We can suppose that there is a unique underlying complete probability space $(\Omega, \mathbb{F}, P)$. Let $\widehat{\mathbb{F}}$ denote the the $\sigma-$algebra of $\mathbb{F}$ of sets with $P-$ measure 0 or 1 and define the filtration

$$\mathbb{F}_t^\mu = \widehat{\mathbb{F}} \cup \sigma((q_s^\mu; p_s^\mu), s \leq t).$$

Lemma 4.1. *For every μ the pair of stochastic processes $(q_\cdot^\mu, \Phi_\cdot^\mu)$, where*

$$\Phi_t^\mu = \int_0^t \nu(q_s^\mu) d\Psi_t^\mu \tag{13}$$

is an almost surely solution to the Skorohod reflection problem for $(D, N, H^\mu + X^\mu)$, where

$$
\begin{aligned}
H_t^\mu &= q_0 + \mu p_0 - \mu p_t^\mu \\
X_t^\mu &= \int_0^t b(q_s^\mu) ds + W_t
\end{aligned}
\tag{14}
$$

Proof. Consider the integral form of (12). Taking into account that $\int_0^t p_s^\mu ds = q_t^\mu - q_0$ and solving for q_t^μ we see that:

$$q_t^\mu = H_t^\mu + X_t^\mu + \Phi_t^\mu$$

Then (q^μ, Φ^μ) verifies Definition 3.1 with probability 1.

$\square$

Lemma 4.2. *For every $T > 0$ we have that $\lim_{\mu \to 0} E[\sup_{t \leq T} |\mu p_t^\mu|^2] = 0$.*

Proof. Assume first that $b = 0$. Consider equations (12) and apply the Itô formula for semimartingales to the function $f(q, p) = |p|^2$ for every pair of times s, t such that $0 \leq s \leq t \leq T$. Doing that we get

$$|p_t^\mu|^2 = |p_s^\mu|^2 - \frac{2}{\mu} \int_s^t |p_u^\mu|^2 du + \frac{2}{\mu} \int_s^t p_u^\mu \cdot dW_u + \frac{1}{\mu^2} r(t - s) \tag{15}$$

It is interesting to observe that the local time Ψ_t^μ does not appear above. This comes from the fact that under elastic reflection $|p_t^\mu|^2 = |p_{t-}^\mu|^2$ for every $t > 0$.

Consider now a constant $c > 0$ and functions $x, g \in \mathbb{D}([0, T], \mathbb{R})$ with $g(0) = 0$ such that:

$$x_t \leq x_s - c \int_s^t x_u du + g_t - g_s, \ 0 \leq s \leq t \leq T \tag{16}$$

Then one can easily see that

$$x_t \leq e^{-ct}(x_0 + g_t) + c \int_0^t e^{-c(t-u)}(g_t - g_u)du, \ 0 \leq t \leq T \tag{17}$$

By taking expected value to (15) and applying (17) with $c = \frac{2}{\mu}$, $g_t = \frac{1}{\mu^2}rt$ and $x_t = |p_t^\mu|^2$, we get

$$
\begin{aligned}
E|p_t^\mu|^2 &\leq e^{-\frac{2}{\mu}t}(|p|^2 + \frac{1}{\mu^2}rt) + \frac{2}{\mu^3}\int_0^t e^{-\frac{2}{\mu}(t-u)}r(t-u)du \\
&= e^{-\frac{2}{\mu}t}|p|^2 + \frac{r}{\mu^2}(\frac{\mu}{2} - \frac{\mu}{2}e^{-\frac{2t}{\mu}})
\end{aligned}
\tag{18}
$$

This implies the statement of the Lemma for $b = 0$. The general case can be reduced to the case with $b = 0$ by an absolutely continuous change of measures in the space of trajectories.

$\square$

The following two theorems are restatements of theorems 3.8.6 and 3.10.2 respectively of [4].

Theorem 4.3. *Let $\{Y^n\}$ be a family of processes with sample paths in $\mathbb{D}(\mathbb{R}_+, D)$. Assuming that for every $\epsilon > 0$ and rational $t \geq 0$ there exist a compact set $\Gamma(\epsilon, t) \subset D$ such that $\liminf_n P(Y^n(t) \in \Gamma(\epsilon, t)) \geq 1 - \epsilon$, then the following are equivalent*

1. *$\{Y^n\}$ is relatively compact.*

2. *For each $T > 0$, there exists $\beta > 0$ and a family of nonnegative random variables $\{\gamma^n(\delta), 0 < \delta < 1\}$ satisfying*

$$E(|Y^n(t+u) - Y^n(t)|^\beta \ |\mathbb{F}_t^n) \leq E(\gamma^n(\delta)|\mathbb{F}_t^n),$$

for $t \in [0, T]$ and $u \in [0, \delta]$ and in addition $\lim_{\delta \to 0} \limsup_n E(\gamma^n(\delta)) = 0$.

Theorem 4.4. *Let $\{Y^n\}$ and Y be processes with sample paths in $\mathbb{D}(\mathbb{R}_+, D)$ such that Y_n converges in distribution to Y. Then Y is almost surely continuous if and only if $\int_0^\infty e^{-u}[\sup_{0 \leq t \leq u} |Y^n(t) - Y^n(t-)| \wedge 1]du \Rightarrow 0$.*

The following lemma shows that the family $\{H^\mu + X^\mu\}$ is tight in the Skorohod topology.

Lemma 4.5. *The family $\{H^\mu + X^\mu\}$ defined in (14) is relatively compact and all of its accumulation points are continuous.*

Proof. It is easily seen that $\{X^\mu\}$ is relatively compact and that all of its accumulation points are continuous.

Now Lemma 4.2 suggests that:

$$\lim_{\mu \to 0} E[\sup_{t \leq T} |H_t^\mu|^2] \leq c \qquad (19)$$

$$\lim_{\mu \to 0} E[\sup_{|t-s| \leq \delta} |H_t^\mu - H_s^\mu|] \leq c_1 \delta, \qquad (20)$$

where c, c_1 are positive constants independent of μ.

Chebychev's inequality and (19) imply that

$$\liminf_{n \to \infty} P(|H^{1/n}(t)| \leq \lambda) \geq 1 - \frac{c}{\lambda^2}.$$

Therefore by this and (20), Theorem 4.3. gives us that $\{H^\mu\}$ is relatively compact. Lastly (20) and Theorem 4.4 implies that all its accumulation points are continuous.

$\square$

Theorem 4.6. *The family $\{(q^\mu, \Phi^\mu, \Psi^\mu, H^\mu, X^\mu)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{4r+1})$.*

Proof. It follows from Lemma 4.5 and Theorem 3.2.

$\square$

Now that tightness has been established we will proceed with the identification of the stochastic differential equation with reflection that describes the behavior of q^μ as $\mu \to 0$.

Consider the following S.D.E. with reflection:

$$q_t = q_0 + \int_0^t b(q_s)ds + W_t + \Phi_t \qquad (21)$$

where $\Phi_t = \int_0^t \nu(q_s)d|\Phi|_s, \nu(s) \in N(q_s)$ and $d|\Phi|(\{t : q_t \in D\}) = 0$. It is known that (21) has a unique weak solution (q, Φ) ([1]).

Theorem 4.7. *The family $\{(q^\mu, \Phi^\mu)\}$ converges in distribution to the unique solution (q, Φ) of (21).*

Proof. By Theorem 4.6. we have that the five-tuple $\{(q^\mu, \Phi^\mu, H^\mu, X^\mu, W)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{5r})$. Hence it (or a subsequence) converges in distribution to a stochastic process $\{(q, \Phi, H, X, W)\}$. By the Skorohod representation theorem, one can find a probability space $(\widetilde{\Omega}, \widetilde{\mathbb{F}}, \widetilde{P})$ and realizations $\{(\widetilde{q}^\mu, \widetilde{\Phi}^\mu, \widetilde{H}^\mu, \widetilde{W}^\mu)\}$ and $\{(\widetilde{q}, \widetilde{\Phi}, \widetilde{H}, \widetilde{X}, \widetilde{W})\}$ of $\{(q^\mu, \Phi^\mu, H^\mu, X^\mu, W)\}$ and

$\{(q, \Phi, H, X, W)\}$ respectively such that $\{(\widetilde{q}^{\mu}, \widetilde{\Phi}^{\mu}, \widetilde{H}^{\mu}, \widetilde{X}^{\mu}, \widetilde{W}^{\mu})\}$ converges $\widetilde{P}$-almost surely to $\{(\widetilde{q}, \widetilde{\Phi}, \widetilde{H}, \widetilde{X}, \widetilde{W})\}$. Therefore by Theorem 3.2. $(\widetilde{q}, \widetilde{\Phi})$ is a solution to the Skorohod problem for $(D, N, \widetilde{H} + \widetilde{X})$ $\widetilde{P}-$almost surely.

Now by the convergence of $\widetilde{q}^{\mu}$ to $\widetilde{q}$ we get that $\widetilde{X}$ must be given by:

$$\widetilde{X}_t = \int_0^t b(\widetilde{q}_s)ds + \widetilde{W}_t$$

Finally Lemma 4.2 and its proof imply that $\widetilde{H}_t = q_0$.

$\square$

We would like to note here that one could prove the convergence in distribution of the Langevin procces with reflection to the corresponding diffusion process with reflection using the Smoluchowski-Kramers approximation. However the beauty and generality of the results of [3] resulted in using the method that was presented here.

Acknowledgments

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