ABSTRACT

Title of dissertation: ASYMPTOTIC PROBLEMS FOR STOCHASTIC PROCESSES WITH REFLECTION AND RELATED PDE's

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Asymptotic problems for stochastic processes with reflection and for related partial differential equations (PDE's) are considered in this thesis. The stochastic processes that we study, depend on a small parameter and are restricted to move in the interior of some domain, while having instantaneous reflection at the boundary of the domain. These stochastic processes are closely related to corresponding PDE problems that depend on a small parameter. We are interested in the behavior of these stochastic processes and of the solutions to the corresponding PDE problems as this small parameter goes to zero.

In particular, we consider two problems that are related to stochastic processes with reflection at the boundary of some domain.

Firstly, we study the Smoluchowski-Kramers approximation for the Langevin equation with reflection. According to the Smoluchowski-Kramers approximation, the solution of the equation $\mu\ddot{q}^{\mu}_t = b(q^{\mu}_t) - \dot{q}^{\mu}_t + \sigma(q^{\mu}_t)\dot{W}_t, q^{\mu}_0 = q, \dot{q}^{\mu}_0 = p$ converges to the solution of the equation $\dot{q}_t = b(q_t) + \sigma(q_t)\dot{W}_t, q_0 = q$ as $\mu \to 0$. We consider here

a similar result for the Langevin process with elastic reflection on the boundary of the half space, i.e. on $\partial\mathbb{R}^n_+ = \{(x^1, \cdots, x^n) \in \mathbb{R}^n : x^1 = 0\}$. After proving that such a process exists and is well defined, we prove that the Langevin process with reflection at $x^1 = 0$ converges in distribution to the diffusion process with reflection on $\partial\mathbb{R}^n_+$. This convergence is the main justification for using a first order equation, instead of a second order one, to describe the motion of a small mass particle that is restricted to move in the interior of some domain and reflects elastically on its boundary.

Secondly, we study the second initial boundary problem in a narrow domain of width $\epsilon \ll 1$, denoted by D^ϵ, for linear second order differential equations with nonlinear boundary conditions. The underlying stochastic process is the Wiener process $(X^\epsilon_t, Y^\epsilon_t)$ in the narrow domain D^ϵ with instantaneous normal reflection at its boundary. Using probabilistic methods we show that the solution of such a problem converges to the solution of a standard reaction-diffusion equation in a domain of reduced dimension as $\epsilon \downarrow 0$. This reduction allows to obtain some results concerning wave front propagation in narrow domains. In particular, we describe conditions leading to jumps of the wave front. This problem is important in applications (e.g., thin waveguides).

ASYMPTOTIC PROBLEMS FOR STOCHASTIC PROCESSES WITH REFLECTION AND RELATED PDE's

by

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Chapter 1

Introduction

1.1 Smoluchowski-Kramers approximation

The motion of a particle of mass μ in a force field $b(q)+\sigma(q)\dot{W}_t$ with a friction (which for brevity is taken to be equal to 1) proportional to velocity is governed by the Newton law

$$\begin{aligned} \mu\ddot{q}_t^\mu &= b(q_t^\mu) - \dot{q}_t^\mu + \sigma(q_t^\mu)\dot{W}_t \qquad (1.1) \\ q_0^\mu &= q \in \mathbb{R}^r \\ \dot{q}_0^\mu &= p \in \mathbb{R}^r \end{aligned}$$

where $b = (b_1, ..., b_r)'$ (the transpose of $(b_1, ..., b_r)$) with $b_j : \mathbb{R}^r \to \mathbb{R}, j = 1, .., r$, $\sigma = [\sigma_{ij}]_{i,j}^r$ with $\sigma_{ij} : \mathbb{R}^r \to \mathbb{R}, i, j = 1, .., r$ have bounded first derivatives and $W_t = (W_t^1, ..., W_t^r)'$ is the standard r-dimensional Wiener process.

The solution to equation (1.1) is also referred as "Physical" Brownian motion that is defined in Langevin's model of Brownian motion. In contrast to the "Mathematical" Brownian motion, which treats the process as a random walk with independent identically distributed steps, the "Physical" Brownian motion allows step dependence. This is clearly an advantage over the "Mathematical" Brownian motion model since a particle moving due to random collisions with other particles (e.g. gas molecules) does not experience independent steps. The reason is that its inertia tends to keep it moving roughly at the same direction as its previous step. The stochastic differential equation (S.D.E.) (1.1) is also called Langevin's equation.

The well-known Smoluchowski-Kramers approximation ([31],[24]) implies that the solution of (1.1) converges in probability as $\mu \to 0$ to the solution of the following first order S.D.E.:

$$\begin{aligned} \dot{q}_t &= b(q_t) + \sigma(q_t)\dot{W}_t \qquad (1.2) \\ q_0 &= q \in \mathbb{R}^r, \end{aligned}$$

In other words, one can prove that for any $\delta, T > 0$ and $q, p \in \mathbb{R}^r$ (see, for example, Lemma 1 in [14]),

$$\lim_{\mu \downarrow 0} P(\max_{0 \le t \le T} |q_t^\mu - q_t| > \delta) = 0. \qquad (1.3)$$

The Smoluchowski-Kramers approximation is the main justification for using a first order equation, instead of a second order one, to describe the motion of a small mass particle.

It is easy to see now that (1.1) can be equivalently written as:

$$\begin{aligned} \dot{q}_t^\mu &= p_t^\mu \\ \mu \dot{p}_t^\mu &= b(q_t^\mu) - p_t^\mu + \sigma(q_t^\mu)\dot{W}_t \qquad (1.4) \\ q_0^\mu &= q \in \mathbb{R}^r, \ \dot{q}_0^\mu = p \in \mathbb{R}^r. \end{aligned}$$

Let us define $\mathbb{R}_+ = \{q^1 \in \mathbb{R} : q^1 \geq 0\}$ and let the configuration space be $D = \mathbb{R}_+ \times \mathbb{R}^{r-1}$. We examine the behavior of the process with elastic reflection on the boundary $\partial D \times \mathbb{R}^r = (\partial \mathbb{R}_+ \times \mathbb{R}^{r-1}) \times \mathbb{R}^r$ of the phase space $D \times \mathbb{R}^r$ that is governed by (1.4), i.e. of the Langevin process with reflection, as $\mu \to 0$.We show that: (a) the Langevin process with reflection is well defined and (b) the first component (the q component) of the Langevin process with reflection at $q^1 = 0$, that is governed by equation (1.4), converges in distribution to the diffusion process with reflection on ∂D that is governed by (1.2). The method is based on properties of the Skorohod reflection problem and on techniques developed in Constantini [4] and [5].

1.2 Reaction diffusion equations with nonlinear boundary conditions in narrow domains

Let $D^\epsilon = \{(x, y) : x \in \mathbb{R}^n, y \in D_x^\epsilon \subset \mathbb{R}^m, D_x^\epsilon = \epsilon D_x\}$ be a narrow domain of width $\epsilon \ll 1$, where D_x is a bounded domain in $\mathbb{R}^m$ with smooth boundary. Consider the following nonlinear problem

$$\begin{aligned} u_t^\epsilon &= \frac{1}{2}\triangle u^\epsilon, && \text{in } (0,T) \times D^\epsilon \qquad (1.5) \\ u^\epsilon(0,x,y) &= f(x), && \text{on } \{0\} \times D^\epsilon \end{aligned}$$

$$\frac{\partial u^\epsilon}{\partial \gamma^\epsilon} = -\epsilon c(x,y,u^\epsilon)u^\epsilon, \qquad \text{on } (0,T)\times\partial D^\epsilon,$$

where γ^ϵ is the inward unit normal to ∂D^ϵ. The functions f and c are sufficiently regular and bounded; f is assumed to be nonnegative.

Equation (1.5) is a semilinear reaction diffusion equation where the reaction takes place on the boundary of the domain. These equations arise naturally in physics, chemical kinetics, combustion theory and biology (e.g. Grieser [19], Kurchment [26], Grindrod [20]). Our goal here is twofold: (i) to study the limit of the solution of (1.5) as $\epsilon \to 0$; and (ii) to study travelling waves of (1.5) as $t \to \infty$ for ϵ small. Hale and Raugel [22] study reaction diffusion equations on narrow domains that have zero (Neyman, Dirichlet or mixed) boundary data and a nonlinear term in the equation. Their treatment is purely analytical. Our approach is mainly probabilistic and makes it possible to consider nonlinear boundary conditions. Of course, one can also consider equation (1.5) with an extra nonlinear term in the equation and our methodology can be applied to this case as well without any difficulties. Furthermore, travelling waves for reaction diffusion equations in, unbounded or bounded but fixed, domains have been studied by several authors and under different assumptions for the nonlinear term (e.g. Evans and Souganidis [7], Freidlin [9], [10], [11], [12], Gärtner [18], Nolen and Xin [27], Hadeler and Rothe [21], Kolmogorov, Petrovskii and Piskunov [25]).

Using a combination of analytical and probabilistic techniques we show that

$$u^\epsilon(t,x,y) \to u(t,x) \text{ as } \epsilon \to 0, \text{ uniformly in any compact sunset of } \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m, \tag{1.6}$$

where u solves the following standard reaction-diffusion equation

$$\begin{aligned} u_t &= \frac{1}{2}\triangle_x u + \frac{1}{2}\nabla(\log V(x))\nabla_x u + \frac{1}{2}\frac{S(x)}{V(x)}c(x,0,u)u, \quad \text{in } (0,T)\times\mathbb{R}^n \qquad (1.7) \\ u(0,x) &= f(x), \qquad \text{on } \{0\}\times\mathbb{R}^n, \end{aligned}$$

where $V(x)$ is the volume of D_x and $S(x)$ is the surface area of ∂D_x. We observe that the effect of the boundary is an *extra first order term* in the limiting equation and the effect of the boundary term is a *nonlinear term* in the limiting equation.

Consider the Wiener process $(X_t^\epsilon, Y_t^\epsilon)$ in D^ϵ with instantaneous normal reflection on the boundary of D^ϵ. Its trajectories can be described by the stochastic differential equations:

$$\begin{aligned} X_t^\epsilon &= x + W_t^1 + \int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon)dL_s^\epsilon \\ Y_t^\epsilon &= y + W_t^2 + \int_0^t \gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)dL_s^\epsilon. \end{aligned} \tag{1.8}$$

Here W_t^1 and W_t^2 are independent Wiener process in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and (x,y) is a point inside D^ϵ. Moreover γ_1^ϵ and γ_2^ϵ are projections of the unit inward normal vector to ∂D^ϵ on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. It is easy to see that

$\lim_{\epsilon\downarrow 0}|\epsilon^{-1}\gamma_1^\epsilon| = \frac{\gamma_1^1}{|\gamma_2^1|}$ and $\lim_{\epsilon\downarrow 0}|\gamma_2^\epsilon| = 1$, where $|\cdot|$ denotes Euclidean length. Furthermore L_t^ϵ is the local time for the process $(X_t^\epsilon, Y_t^\epsilon)$ on ∂D^ϵ, i.e. it is a continuous, non-decreasing process that increases only when $(X_t^\epsilon, Y_t^\epsilon) \in \partial D^\epsilon$ such that the Lebesque measure $\Lambda\{t > 0 : (X_t^\epsilon, Y_t^\epsilon) \in \partial D^\epsilon\} = 0$ (see for instance [23]).

If $(X_t^\epsilon, Y_t^\epsilon)$ is defined by (1.8), then as it can be derived from Theorem 2.5.1 in [9], $u^\epsilon(t, x, y)$ satisfies the following integral equation in the functional space:

$$u^\epsilon(t, x, y) = E_{x,y} f(X_t^\epsilon) \exp[\int_0^t \epsilon c(X_s^\epsilon, Y_s^\epsilon, u^\epsilon(t-s, X_s^\epsilon, Y_s^\epsilon)) dL_s^\epsilon], \tag{1.9}$$

where $E_{x,y}$ denotes expectation and the subscript (x, y) denotes the initial point of $(X_s^\epsilon, Y_s^\epsilon)$.

Let X_t be the solution of the stochastic differential equation

$$X_t = x + W_t^1 + \int_0^t \frac{1}{2}\nabla(\log V(X_s)) ds. \tag{1.10}$$

Then the solution $u(t, x)$ of equation (1.7) satisfies the equality:

$$u(t, x) = E_x f(X_t) \exp[\int_0^t \frac{1}{2}\frac{S(X_s)}{V(X_s)} c(X_s, 0, u(t-s, X_s)) ds]. \tag{1.11}$$

We prove that the component X_t^ϵ of the process $(X_t^\epsilon, Y_t^\epsilon)$ converges in a certain sense to X_t. This together with uniform in $0 < \epsilon < 1$ bounds for $u^\epsilon(t, x, y)$ and its derivatives allow to prove that the solution of (1.9) converges to the solution of (1.11) as $\epsilon \downarrow 0$ uniformly on each compact subset of $[0, \infty) \times \mathbb{R}^{n+m}$.

One can expect that, under certain assumptions on the nonlinear term $c(x, y, u)u$ in (1.5), the solution $u^\epsilon(t, x, y)$ can be approximated by a running-wave-type solution. Corresponding results on the standard reaction diffusion equation (1.7) (see chapter 6 and 7 in Freidlin [9]) allow to describe the asymptotic wavefront motion for (1.5). We will see how the motion of the interface (wavefront) depends on the behavior of the cross-sections D_x of the domain D. In particular, we consider three different cases: (a) K-P-P wave fronts in slowly changing media, (b) wave fronts in slowly changing media and bistable nonlinearity and (c) K-P-P wave fronts in random media. In the case of nonlinear term of K-P-P type the wavefront can have jumps and we can actually characterize the conditions under which the jumps may arise.

1.2.1 Wave front propagation in reaction diffusion equations

Fisher [8] and Kolmogorov, Petrovskii and Piskunov (K-P-P) [25], in 1937, started to consider traveling waves of semilinear reaction diffusion equations of the type (1.7). In particular the equation that they studied is

$$\begin{aligned} u_t &= \frac{D}{2} u_{xx} + c(u)u, \quad \text{in } (0, T) \times \mathbb{R} \qquad (1.12) \\ u(0, x) &= \chi_{x<0}, \qquad\qquad\quad \text{on } \{0\} \times \mathbb{R}, \end{aligned}$$

The nonlinear term $c(u)u$ characterizes the reaction (killing and multiplication of particles) in the absence of diffusion and is of K-P-P type if it is Lipschitz continuous in $u \in \mathbb{R}$ such that $c(u)$ is positive for $u < 1$, negative for $u > 1$ and $\bar{c} = c(0) = \max_{0 \le u \le 1} c(u)$. Reaction diffusion equations that have a K-P-P type nonlinear term are called K-P-P reaction diffusion equations.

It is proved in [25], that for any $h > 0$

$$\begin{aligned} \lim_{t \to \infty} \sup_{x > (\alpha^* + h)t} u(t,x) &= 0 \\ \lim_{t \to \infty} \inf_{x < (\alpha^* - h)t} u(t,x) &= 1 \end{aligned} \tag{1.13}$$

for $\alpha^* = \sqrt{2\bar{c}D}$. The parameter α^* is referred to as the asymptotic speed for problem (1.12) as $t \to \infty$. The asymptotic shape for problem (1.12) as $t \to \infty$ is given by the solution of the following problem

$$\begin{aligned} \frac{D}{2}\theta_{xx}(x) + \alpha^* \theta_x(x) + c(\theta(x))\theta(x) &= 0, x \in \mathbb{R} \\ \lim_{x \to \infty} \theta(x) = 0, \lim_{x \to -\infty} \theta(x) &= 1, \;\; \theta(0) = \frac{1}{2} \end{aligned} \tag{1.14}$$

These results are the first of this type. Freidlin [10] gave the first general result on K-P-P reaction diffusion equations using probabilistic methods, when the diffusion coefficient and the nonlinear term depend on the space variable and are slowly varying in it. By introducing a small parameter $\delta > 0$, via the change of variables $t \to \frac{t}{\delta}$, he separated the study of the asymptotic shape and speed of the traveling wave. He considered the problem

$$\begin{aligned} u^\delta_t(t,x) &= \frac{\delta}{2}\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i x_j}[a^{ij}(x)u^\delta(t,x)] + \sum_{i=1}^n b^i(x)\frac{\partial u^\delta(t,x)}{\partial x_i} + \frac{1}{\delta}c(x,u^\delta(t,x))u^\delta(t,x) \\ u^\delta(0,x) &= f(x)\geq 0, \text{ on } \{0\}\times\mathbb{R}^n, \end{aligned} \tag{1.15}$$

where the nonlinear term is assumed to be of K-P-P type for all $x\in\mathbb{R}^n$ and the $a^{ij}(x)$ functions are bounded with bounded second order derivatives such that $\sum_{i,j=1}^n a^{ij}(x)\lambda_i\lambda_j$ does not degenerate uniformly in $\mathbb{R}^n$.

Remark 1.2.1. *Of course, if we do the time change $t\longrightarrow\frac{t}{\delta}$ in equation (1.7) we get an equation similar to (1.15).*

Let (X^δ_t, P_x) be the Markov diffusion process in $\mathbb{R}$ that corresponds to the operator $L^\delta\cdot = \frac{\delta}{2}\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i x_j}[a^{ij}(x)\cdot] + \sum_{i=1}^n b^i(x)\frac{\partial\cdot}{\partial x_i}$ and let $\sigma(x)$ be a $n\times n$ matrix such that $\sigma(x)\sigma^*(x) = [a^{ij}(x)]$.

Using the Feynman-Kac formula, the solution to problem (1.15) can be represented as

$$u^\delta(t,x) = E_x f(X^\delta_t)\exp[\int_0^t \frac{1}{\delta}c(X^\delta_s, 0, u^\delta(t-s, X^\delta_s))ds]. \tag{1.16}$$

Using the large deviations theory for stochastic differential equations and the representation (1.16), Freidlin [10] studied the limit of u^δ as $\delta\downarrow 0$. In particular, the action functional (see [16] for more details on the action functional and its properties) for the process $X_t^\delta, 0 \le t \le T$, in $\mathcal{C}_{OT}$ as $\delta \downarrow 0$ has the form $\frac{1}{\delta} S_{0T}(\phi)$ where

$$S_{0T}(\phi) = \begin{cases} \frac{1}{2}\int_0^T \sum_{i,j=1}^n a_{ij}(\phi_s)(\dot{\phi}_s^i - b^i(\phi_s))(\dot{\phi}_s^j - b^j(\phi_s))ds, & \text{if } \phi \in \mathcal{C}_{OT} \text{ is} \\ & \text{abs. continuous} \\ +\infty, & \text{for the rest of } \mathcal{C}_{OT}. \end{cases} \tag{1.17}$$

where $[a_{ij}(x)]_{i,j=1}^n$ is the matrix inverse to $[a^{ij}(x)]_{i,j}^n$. Let us further define

$$W(t,x) = \sup\{\int_0^t c(\phi_s)ds - S_{0t}(\phi) : \phi \in \mathcal{C}_{0,t}, \phi_0 = x, \phi_t \in F_o\}. \tag{1.18}$$

where F_o is closure of the support of f and $c(x) = c(x,0) = \max_{0 \le u \le 1} c(x,u)$.

We say that condition (N) is satisfied if for any $t > 0$ and $(t,x) \in \{(t,x) : W(t,x) = 0\}$:

$$W(t,x) = \sup\{\int_0^t c(\phi_s)ds - S_{0t}(\phi) \quad : \quad \phi_0 = x, \phi_t \in F_o, \\ (t-s, \phi_s) \in \{(t,x) : W(t,x) < 0\}\}.$$

Using the representation (1.16) and the properties of the action functional (1.17), Freidlin [10] proved the following theorem.

Theorem 1.2.2. *(Freidlin [10]). Let $u^\delta(t,x)$ be the solution to (1.15). Then, under condition (N) we have:*

$$\lim_{\delta \downarrow 0} u^\delta(t,x) = \begin{cases} 1, & W(t,x) > 0 \\ 0, & W(t,x) < 0. \end{cases} \tag{1.19}$$

The convergence is uniform on every compactum lying in the region $\{(t,x) : t > 0, x \in \mathbb{R}^n, W(t,x) > 0\}$ *and* $\{(t,x) : t > 0, x \in \mathbb{R}^n, W(t,x) < 0\}$ *respectively.*

Hence, the equation $W(t,x) = 0$ defines the position of the interface (wavefront) between areas where u^δ (for $\delta > 0$ small enough) is close to 0 and to 1.

Later on, Evans and Souganidis [7] considered wave front propagation for the solution to equation (1.15) using analytical methods. Using variational methods, they generalized Freidlin's result to the case when condition (N) is not satisfied. Later, Freidlin [11] generalized their results using probabilistic methods.

Without condition (N), the position of the wavefront can be characterized as follows. Instead now of function $W(t,x)$, we consider the function

$$W^*(t,x) = \sup\{\min_{0\le\alpha\le t} \int_0^\alpha c(\phi_s)ds - S_{0\alpha}(\phi) \quad : \quad \phi \in \mathcal{C}_{0,t}(\mathbb{R}^n) \text{ is absolutely continuous},$$
$$\phi_0 = x,\;\; \phi_t \in F_o\}. \tag{1.20}$$

One can prove that $W^*(t,x)$ is Lipschitz continuous and that $W^*(t,x) \le \min\{0, W(t,x)\}$.

Theorem 1.2.3. *(Freidlin [11]). The following statements hold:*

(i). For any compact subset Θ_1 *of the interior of* $\{(t,x) : t > 0, W^*(t,x) = 0\}$,

$$\lim_{\delta\downarrow 0} u^\delta(t,x) = 1 \text{ uniformly in } (t,x) \in \Theta_1.$$

(ii). For any compact subset Θ_2 *of* $\{(t,x) : W^*(t,x) < 0\}$,

$$\lim_{\delta\downarrow 0} u^\delta(t,x) = 0 \text{ uniformly in } (t,x) \in \Theta_2.$$

In our case, we consider wave front propagation for the solution of (1.5) for small $\epsilon > 0$ when $c(x, y, u)$, is of K-P-P type for $y = 0$ and the functions $c(\cdot, 0, u)$, $V(\cdot)$, $S(\cdot)$ and $f(\cdot)$ change slowly in x, i.e. $c(\cdot, 0, u) = c(\delta x, 0, u)$, $V(\cdot) = V(\delta x)$, $S(\cdot) = S(\delta x)$ and $f(\cdot) = f(\delta x)$ for $0 < \delta \ll 1$. Using the aforementioned results we obtain theorems similar to Theorem 1.2.2 and Theorem 1.2.3. In addition, we examine how the motion of the wavefront depends on the behavior of the cross-sections D_x of the domain D. In particular, we prove that the wavefront can have jumps, we specify conditions under which jumps may appear and we characterize the positions at which they may appear.

So far, we have mentioned results only for the traveling waves of K-P-P reaction diffusion equations. Another important class of reaction diffusion equations are those that have bistable nonlinear term. For these equations the nonlinear term satisfies: $c(x, u) > 0$ for $u \in (\mu, 1)$ and $c(x, u) < 0$ for $u \in (0, \mu) \cup (1, \infty)$, where $0 < \mu < 1$. The problem of wave front propagation for a bistable reaction diffusion equation (1.15), with $b^i(x) = 0$ for every i, was considered in detail in Gärtner [18] and it is also presented in section 6.4 of Freidlin [9]. The fact that $c(x, u)$ is negative for small u means that the wave front cannot have jumps. Hence, the propagation of the wave front has a local character. This allows to freeze the coefficients in (1.15) and to reduce the problem to the one-dimensional equation (1.12) with bistable nonlinear term. It is worth noting that the probabilistic approach is, so far, less successful in the bistable case than in the K-P-P case. This is partially related to the fact that in the bistable case, in contrast to the K-P-P case, one cannot separate the asymptotic shape and speed of the wave. The logarithmic asymptotics of (1.16) as $\delta \downarrow 0$ is defined by the trajectories going in the transition area where $\mu < u < 1$ (recall that u is negative outside this interval).

Related to bistable nonlinearities, we consider wave front propagation for the solution of (1.5) for small $\epsilon > 0$ when $c(x, y, u)$, is of bistable type for $y = 0$ and the functions $c(\cdot, 0, u)$, $V(\cdot)$, $S(\cdot)$ and $f(\cdot)$ change slowly in x, as in the K-P-P case. In particular, we consider a specific example and we examine how the asymptotic speed of the wavefront depends on the surface area to volume ratio $\frac{S(x)}{V(x)}$ of the cross-sections D_x of the domain D.

Lastly, one can also consider K-P-P wave fronts in random media. Freidlin, in sections $7.4 - 7.6$ of [9], considers wave front propagation for equations like (1.15) in the case of $x \in \mathbb{R}$, no drift term, constant diffusion coefficient and randomness coming only from the nonlinear part of the equation. He considers the following problem

$$\begin{aligned} u_t(t,x) &= \frac{1}{2}u_{xx}(t,x) + c(x, u(t,x))u(t,x) \\ u(0,x) &= f(x) \geq 0, \text{ on } \{0\} \times \mathbb{R}. \end{aligned} \tag{1.21}$$

The nonlinear function $c(x,u)u$ is assumed to be a random function defined on a complete probability space $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P})$. It is measurable, stationary with respect to x and it satisfies a Lipshitz condition in u with probability one. Moreover, the random function $c(x,u)$ is assumed to be of K-P-P type for all $x \in \mathbb{R}$ with probability one.

Theorem 1.2.4. *(Freidlin [9]). Let $x \in \mathbb{R}$ and $u(t,x)$ satisfy equation (1.21). Under the aforementioned conditions, there exists a unique ν^* such that:*

(i). For all $\nu > \nu^*$,

$$\lim_{t\to\infty} \sup_{x \geq \nu t} u(t,x) = 0, \quad \hat{P} - a.s.$$

(ii). Let us define $\bar{c}_h(x) = \inf_{0<u<h} c(x,u)$ and assume that there is a constant $\kappa > 0$ such that for any $0 < h < 1$ and $x \in \mathbb{R}$,

$$\kappa < \bar{c}_h(x), \quad \hat{P} - a.s.$$

Then for all $\nu \in (0, \nu^*)$,

$$\lim_{t\to\infty} \inf_{0 \leq x \leq \nu t} u(t,x) = 1, \quad \hat{P} - a.s.$$

In 2007, Nolen and Xin [27] considered one dimensional K-P-P reaction-diffusion equations of type (1.21) with random drift and homogeneous in x nonlinear term. They consider the following equation

$$u_t(t,x) \quad = \quad \frac{1}{2}u_{xx}(t,x) + b(x)u_x(t,x) + c(u(t,x))u(t,x)$$

$$u(0,x) \quad = \quad f(x) \in [0,1], \;\; \text{on } \{0\} \times \mathbb{R}. \tag{1.22}$$

where the random drift $b(x, \hat{\omega})$ is measurable, stationary in x and translation in x generates an ergodic transformation of the space $\hat{\Omega}$. Additionally, it satisfies $\hat{E}[b(x, \hat{\omega})] = 0$, $\hat{E}[\sup_{x \in [-2,2]} |b(x, \hat{\omega})|] < \infty$ and it is almost surely locally Lipshitz continuous. Lastly, it satisfies the following mild conditions: there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\begin{aligned}
\limsup_{z \to \infty} \hat{P}[\int_0^z b(x, \hat{\omega}) \geq \alpha_1] &< 1 \\
\limsup_{z \to \infty} \hat{P}[\int_{-z}^0 b(x, \hat{\omega}) \leq \alpha_2] &< 1
\end{aligned}$$

Under the aforementioned conditions, Nolen and Xin [27] prove a theorem for (1.22) that is similar to Theorem 1.2.4.

In our case, we consider wave front propagation for the solution of (1.5) for small $\epsilon > 0$, when $x \in \mathbb{R}$, the boundary ∂D^1 of D^1 is determined by stationary and ergodic random processes on $\mathbb{R}$ and the nonlinear boundary term in (1.5) (for $y = 0$, i.e. $c(x, 0, u)$) is of K-P-P type. The limiting equation (1.7) has random drift and random nonhomogeneous in x nonlinear term. Making use of the results in [9], [27] and of the fact that the operator of the equation (1.7) is self adjoint with respect to an appropriate inner product (it has the form $\frac{1}{2V(x)} \frac{d}{dx}(V(x) \frac{d}{dx})$), we prove a result similar to Theorem 1.2.4.

1.3 Outline of the thesis

The thesis is organized as follows. In Chapter 2, we study the Smoluchowski-Kramers approximation for the Langevin equation with reflection. In particular, in section 2.1, we define the Langevin process with reflection for general diffusion matrix σ with inputs that have bounded first derivatives. In section 2.2 we describe the Skorohod reflection problem and in section 2.3 we consider the limit $\mu \to 0$ when the diffusion matrix is the unit matrix. We note here that the limit when $\mu \to 0$ for a general diffusion matrix as above can be examined similarly.

In Chapter 3, we study the problem of reaction diffusion equations with non-linear boundary conditions in narrow domains. In section 3.2 we consider averaging of integrals in local time. This result allows in section 3.3 to prove convergence of the integral in the right side of the first of equations in (1.8) to the integral term in (1.10) and convergence of exponents in (1.9) and (1.11). Together with a-priori bounds obtained in section 3.3, this implies convergence of $u^{\epsilon}(t, x, y)$ to $u(t, x)$. Some results concerning wavefront propagation are presented in section 3.4. In particular, we consider three different cases: (a) K-P-P wave fronts in slowly changing media, (b) wave fronts in slowly changing media and bistable nonlinearity and (c) K-P-P wave fronts in random media. In the first case, (a), the wave front may have jumps and we can actually characterize the conditions under which the jumps may arise explicitly.

Chapter 2

Smoluchowski-Kramers Approximation for the Langevin Equation with Reflection

2.1 Langevin process with reflection and preliminary results

We begin with the construction of the Langevin process $(q_t^\mu; p_t^\mu)$ in $D\times\mathbb{R}^r$ with elastic reflection on the boundary. Let $b = (b_1, ..., b_r)'$ with $b_j : D \to \mathbb{R}, j = 1, .., r$ and $\sigma = [\sigma_{ij}]$ with $\sigma_{ij} : D \to \mathbb{R}, i, j = 1, .., r$ have bounded first derivatives and σ be non-degenerate. Let $(q, p) \in D \times \mathbb{R}^r$ be the initial point (we assume that $(q^1)^2 + (p^1)^2 \neq 0$). Then $(q_t^\mu; p_t^\mu)$ is the right-continuous Markov process in $D \times \mathbb{R}^r$ defined as follows. Consider the following system of S.D.E.'s:

$$\begin{aligned} \dot{q}_t^{i,\mu} &= p_t^{i,\mu} \\ \mu\dot{p}_t^{i,\mu} &= -p_t^{i,\mu} + b_i(q_t^\mu) + \sum_{j=1}^{r} \sigma_{ij}(q_t^\mu)\dot{W}_t^j \\ q_0^{i,\mu} &= q^i,\ p_0^{i,\mu} = p^i,\ i = 1, ..., r. \end{aligned} \tag{2.1}$$

We define $(q_t^\mu; p_t^\mu)$ to be the solution to (2.1) for $t \in [0, \tau_1^\mu)$, where $\tau_1^\mu = \inf\{t > 0 : q_t^{1,\mu} = 0\}$. Then define $(q_t^\mu; p_t^\mu)$ for $t \in [\tau_1^\mu, \tau_2^\mu)$, where $\tau_2^\mu = \inf\{t > \tau_1^\mu : q_t^\mu = 0\}$, to be the solution of (2.1) with initial conditions

$$(q^{\mu}_{\tau^{\mu}_1}; p^{\mu}_{\tau^{\mu}_1}) = (0, \lim_{t\uparrow\tau^{\mu}_1} q^{2,\mu}_t, ..., \lim_{t\uparrow\tau^{\mu}_1} q^{r,\mu}_t; -\lim_{t\uparrow\tau^{\mu}_1} p^{1,\mu}_t, \lim_{t\uparrow\tau^{\mu}_1} p^{2,\mu}_t, ..., \lim_{t\uparrow\tau^{\mu}_1} p^{r,\mu}_t).$$

If $0 < \tau^{\mu}_1 < \tau^{\mu}_2 < ... < \tau^{\mu}_k$ and $(q^{\mu}_t; p^{\mu}_t)$ for $t \in [0, \tau^{\mu}_k)$ are already defined, then define $(q^{\mu}_t; p^{\mu}_t)$ for $t \in [\tau^{\mu}_k, \tau^{\mu}_{k+1})$ as solution of (2.1) with initial conditions

$$(q^{\mu}_{\tau^{\mu}_k}; p^{\mu}_{\tau^{\mu}_k}) = (0, \lim_{t\uparrow\tau^{\mu}_k} q^{2,\mu}_t, ..., \lim_{t\uparrow\tau^{\mu}_k} q^{r,\mu}_t; -\lim_{t\uparrow\tau^{\mu}_k} p^{1,\mu}_t, \lim_{t\uparrow\tau^{\mu}_k} p^{2,\mu}_t, ..., \lim_{t\uparrow\tau^{\mu}_k} p^{r,\mu}_t)$$

(see Figure 2.1 for an illustration).

This construction defines the process $(q^{\mu}_t; p^{\mu}_t)$ in $D \times \mathbb{R}^r$ for all $t \geq 0$. This follows from Theorem 2.1.4, which states that the process that we constructed above does not have infinitely many jumps in any finite time interval $[0, T]$. Therefore we have the following definition:

Definition 2.1.1. *We call the above recursively constructed process, the Langevin process with elastic reflection on the boundary $\partial D \times \mathbb{R}^r$. This process has jumps on $\partial D \times \mathbb{R}^r$ and is continuous inside $D \times \mathbb{R}^r$.*

We will refer to the Langevin process with reflection as l.p.r.$(q^{\mu}_t; p^{\mu}_t)$. Moreover we will denote by $(q^{\mu,q}_t; p^{\mu,p}_t)$ the trajectories of $(q^{\mu}_t; p^{\mu}_t)$ with initial position (q, p). For easy of notation we also define $-x = (-x^1, x^2, \ldots, x^r)$ and $|x| = (|x^1|, x^2, \ldots, x^r)$ for $x \in \mathbb{R}^r$.

Below we see an illustration of the construction above in the $(q^1 - p^1)$ phase space.

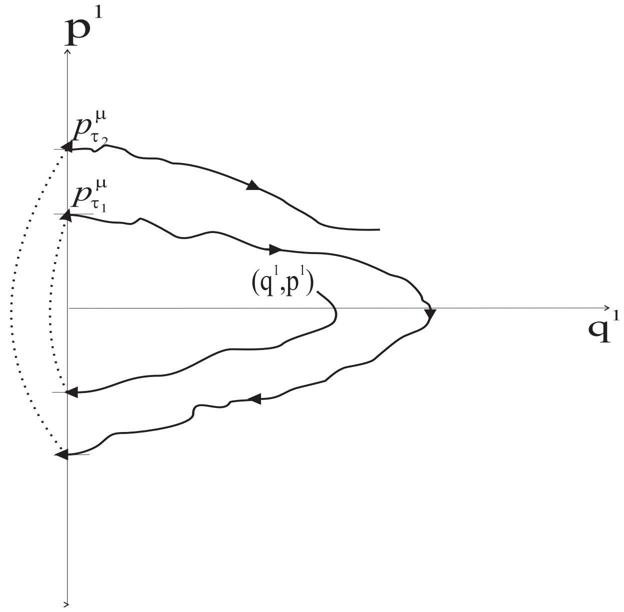

Figure 2.1: Illustration of the Langevin process with reflection in the $(q^1 - p^1)$ phase space

Let us give now another construction of the Langevin process with reflection. Consider the following S.D.E. in $\mathbb{R}^{2r}$:

$$\dot{q}_t^{1,\mu} \quad = \quad p_t^{1,\mu}$$

$$
\begin{aligned}
\mu\dot{p}_t^{1,\mu} &= -p_t^{1,\mu} + \operatorname{sgn}(q_t^{1,\mu})b_1(|q_t^{\mu}|) + \sum_{j=1}^{r} \operatorname{sgn}(q_t^{1,\mu})\sigma_{1j}(|q_t^{\mu}|)\dot{W}_t^j \\
q_0^{1,\mu} &= q^1, p_0^{1,\mu} = p^1,
\end{aligned}
$$

$$
\begin{aligned}
\dot{q}_t^{i,\mu} &= p_t^{i,\mu} \qquad (2.2) \\
\mu\dot{p}_t^{i,\mu} &= -p_t^{i,\mu} + b_i(|q_t^{\mu}|) + \sum_{j=1}^{r} \sigma_{ij}(|q_t^{\mu}|)\dot{W}_t^j \\
q_0^{i,\mu} &= q^i, p_0^{i,\mu} = p^i, i = 2, ..., r,
\end{aligned}
$$

where $\operatorname{sgn}(x)$ takes two values, 1 if $x \geq 0$ and -1 if $x < 0$.

Lemma 2.1.2. *Equation (2.2) has a weak solution which is unique in the sense of probability law.*

Proof. The existence follows from the Girsanov's Theorem on the absolute continuous change of measures in the space of trajectories (b and σ are assumed bounded) and the fact that (2.2) with $b = 0$ has a weak solution. The uniqueness follows from Proposition 5.3.10 of [23]. □

Using the processes $(q_t^{\mu,q}; p_t^{\mu,p})$ and $(q_t^{\mu,-q}; p_t^{\mu,-p})$ we can give another construction of the Langevin process with reflection, as follows. Assume that $q^1 > 0$ and $p^1 > 0$, The graphs of p_t^{1,μ,p^1} and $p_t^{1,\mu,-p^1}$ will be exactly symmetric with respect to zero. The same will be true also for the graphs of q_t^{1,μ,q^1} and of $q_t^{1,\mu,-q^1}$. Let

$\tau_0^\mu = 0, \tau_k^\mu = \inf\{t > \tau_{k-1}^\mu : q_t^{1,\mu,q^1} = 0\}$ and $(\widehat{q}_t^\mu; \widehat{p}_t^\mu)$ be a stochastic process, which is defined as follows:

$$
\begin{aligned}
(\widehat{q}_t^\mu; \widehat{p}_t^\mu) &= (q_t^{\mu,q}; p_t^{\mu,p}) \text{ for } \tau_{2k}^\mu \le t \le \tau_{2k+1}^{\mu,-} \\
(\widehat{q}_t^\mu; \widehat{p}_t^\mu) &= (q_t^{\mu,-q}; p_t^{\mu,-p}) \text{ for } \tau_{2k+1}^\mu \le t \le \tau_{2k+2}^{\mu,-}, k = 0, 1, 2, ...
\end{aligned} \tag{2.3}
$$

Process $(\widehat{q}_t^\mu; \widehat{p}_t^\mu)$ is a process with reflection and it can be seen that $(\widehat{q}_t^\mu; \widehat{p}_t^\mu)$, which is the same as $(|q_t^{1,\mu}|, q_t^{2,\mu}, \cdots, q_t^{r,\mu}; \frac{d}{dt}|q_t^{1,\mu}|, \dot{q}_t^{2,\mu}, \cdots, \dot{q}_t^{r,\mu})$, and l.p.r.$(q_t^\mu; p_t^\mu)$ coincide.

In the figures below we give an illustration of the construction of $(\widehat{q}_t^{1,\mu}; \widehat{p}_t^{1,\mu})$. The first figure illustrates with thick continuous and dotted lines $\widehat{q}_t^{1,\mu}$ versus t. The continuous line is q_t^{1,μ,q^1} versus t and the dotted is $q_t^{1,\mu,-q^1}$ versus t. The second figure illustrates with thick continuous and dotted lines $\widehat{p}_t^{1,\mu}$ versus t. The continuous line is p_t^{1,μ,p^1} versus t and the dotted is $p_t^{1,\mu,-p^1}$ versus t.

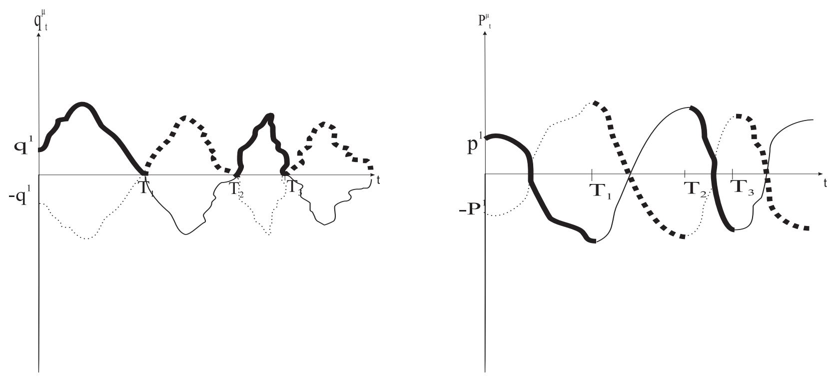

Figure 2.2: A construction of the Langevin process with reflection

Lemma 2.1.3. *Let $T > 0$. The Markov process $(q_t^\mu; p_t^\mu)$ starting at a point (q, p) different from the origin $O = (0, ..., 0; 0, ..., 0)$, that satisfies system (2.2), does not reach the origin O in finite time T, i.e.*

$$P(\exists t \leq T \; s.t. \; (q_t^\mu; p_t^\mu) = O) = 0.$$

Proof. We easily see that it is actually enough to consider only $(q_t^{1,\mu}; p_t^{1,\mu})$. Let $d \ll 1$ be a small number. Define the rectangle $\Delta = \{(q, p) \in \mathbb{R} \times \mathbb{R} : |q| \leq \frac{d^2}{2}, |p| \leq \frac{d}{2}\}$ and suppose that the trajectory starts from some point outside the rectangle Δ, say from $(q, 0) \in \mathbb{R}^2 \setminus \Delta$.

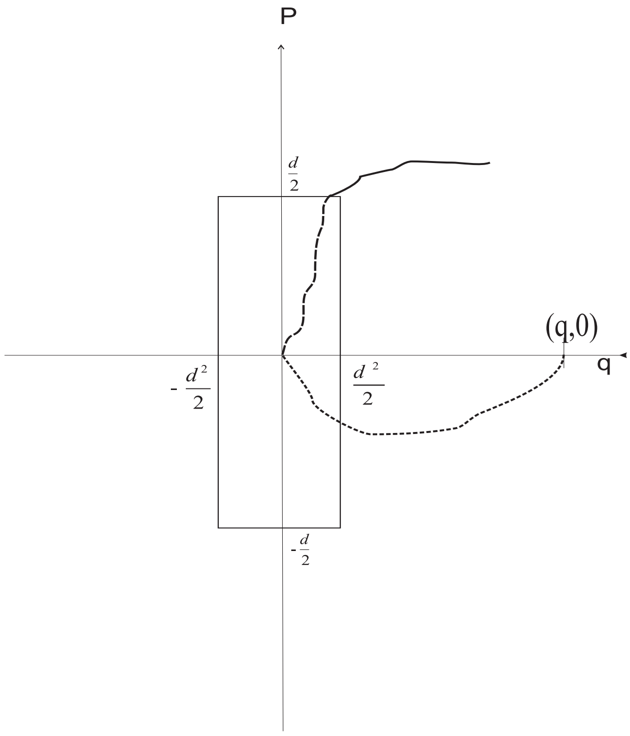

Figure 2.3: The particle does not hit the origin with positive probability

Let also $\chi_\Delta(x)$ denote the indicator function of the set Δ. Then $E^{(q,0)} \int_0^T \chi_\Delta(q_s^1, p_s^1)ds$ is the expected value of the time ,during time $[0, T]$, that the process (q_t^1, p_t^1) with initial point $(q, 0)$ spends inside the rectangle Δ. If $b = 0$ and σ is a matrix with constant entries, (q_t^1, p_t^1) is a Gaussian process. One can write down its density explicitly (see equation (2.2)), which we denote by $\rho(\cdot)$, and obtain the bound

$$E^{(q,0)} \int_0^T \chi_\Delta(q_s^1, p_s^1)ds = \int_\Delta \int_0^T \rho(s, (q, 0), y)dsdy \leq A(T, q)d^3 \tag{2.4}$$

where $A(T,q)$ is a constant that depends on T and q. The general case can be reduced to the case with $b=0$ and σ constant by an absolutely continuous change of measures in the space of trajectories and by a random time change.

We will establish now a lower bound for the quantity $E^{(q,0)}\int_0^T \chi_\Delta(q_s^1,p_s^1)ds$ under the assumption that the process $(q_t^{1,\mu},p_t^{1,\mu})$ will reach $(0,0)$ before time T with positive probability. This will lead to a contradiction.

Again by Girsanov's theorem on the absolute continuity of measures in the space of trajectories it is enough to consider the solution of the following S.D.E:

$$\begin{aligned} \dot{q}_t^1 &= p_t^1 \\ \dot{p}_t^1 &= \frac{1}{\mu}\sum_{j=1}^r \sigma_{1j}(q_t^\mu)\dot{\overline{W}}_t^j \\ q_0^1 &= q^1, p_0^1 = p^1, \end{aligned} \tag{2.5}$$

where $\overline{W}_t^j = \int_0^t \operatorname{sgn}(q_u^{1,\mu})dW_u^j$.

By the self similarity properties of the Wiener process one can find a Wiener process $W_t^{1,*}$ such that $\int_0^t \frac{1}{\mu}\sum_{j=1}^r \sigma_{1j}(q_t^\mu)\dot{\overline{W}}_t^j = W_{\theta(t)}^{1,*}$, where $\theta(t) = \int_0^t \frac{1}{\mu^2}\alpha_{11}(q_s^\mu)ds$ and $\alpha_{11} = \sum_{j,k=1}^r \sigma_{1j}\sigma_{1k}$. So $\int_0^t \frac{1}{\mu}\sum_{j=1}^r \sigma_{1j}(q_t^\mu)\dot{\overline{W}}_t^j$ can be obtained from $W_t^{1,*}$ via a random time change.

By the law of iterated logarithm we get that for all $k\in[0,1]$ there exists a $t_o(k)$ small enough, such that

$$P(t^{\frac{1}{2}+k} \le |W_t^{1,*}| \le t^{\frac{1}{2}-k} \text{ for } t\in[0,t_o(k)]) \ge 1-k.$$

Observe that if $t \in [0, t_o(k)]$ then $\theta(t) \in [0, ct_o(k)]$, where $c = \frac{1}{\mu^2} \sup_{x\in\mathbb{R}} |\alpha_{11}(x)|$. Define also $t_o'(k) = \min\{t_o(k), \frac{t_o(k)}{c}\}$. Then with probability very close to 1, as $k \to 0$, and for all $t \in [0, t_o'(k)]$ it must hold that $|p_t^{1,\mu}| \leq c_1 t^{\frac{1}{2}-k}$ and $q_t^{1,\mu} = \int_0^t p_s^{1,\mu} ds \leq \int_0^t c_1 s^{\frac{1}{2}-k} ds < 2c_1 t^{\frac{3}{2}-k}$, for a constant c_1.

Let τ be the first time, after the time that the Markov process reached the origin, that it exits from the rectangle Δ, i.e. $\tau = \inf\{t > 0 : (q_t^1, p_t^1) \in \mathbb{R}^2 \setminus \Delta\}$. Then it follows that

$$E^{(q,0)} \int_0^T \chi_\Delta(q_s^1, p_s^1) ds > E\{\tau\} \times P(\exists t \leq T \; s.t. \; (q_t^{1,\mu}; p_t^{1,\mu}) = (0,0)) \tag{2.6}$$

Define $\tau_q = \inf\{t > 0 : |q_t^{1,\mu}| > \frac{d^2}{2}\}$ and $\tau_p = \inf\{t > 0 : |p_t^{1,\mu}| > \frac{d}{2}\}$. By the above bounds for $q_t^{1,\mu}$ and $p_t^{1,\mu}$ we get that $\tau_q > c_q d^{\frac{4}{3}}$ and $\tau_p > c_p d^2$, where c_q, c_p are some constants independent of d. So the trajectory exits the rectangle faster in the direction of p than in the direction of q and the exit time is of order d^2. Therefore, by this and by (2.4), we have that

$$Bd^2 < E^{(q,0)} \int_0^T \chi_D(q_s^1, p_s^1) ds \leq Ad^3, \tag{2.7}$$

which cannot hold for constants A and B and small enough d. So we have a contradiction and hence it is true that $P(\exists t \leq T \; s.t. \; (q_t^{1,\mu}; p_t^{1\mu}) = (0,0)) = 0$. $\square$

Theorem 2.1.4. *We have the following two statements:*

(i). Let $T > 0$. The Markov process l.p.r.$(q_t^\mu; p_t^\mu)$ (with arbitrary b) does not reach

the origin $O = (0, ..., 0; 0, ..., 0)$ *in finite time* T*, namely*

$$P(\exists t \leq T\ s.t.\ l.p.r.(q_t^\mu; p_t^\mu) = O) = 0.$$

(ii). The sequence of Markov times $\{\tau_k^\mu\}$ *converges to* $+\infty$ *as* $k \to +\infty$*, i.e.*

$$P(\lim_{k \to +\infty} \tau_k^\mu = +\infty) = 1$$

Proof. The Langevin process with reflection, l.p.r.$(q_t^\mu; p_t^\mu)$, coincides at any time t either with $(q_t^{\mu,q}; p_t^{\mu,p})$ or with $(q_t^{\mu,-q}; p_t^{\mu,-p})$. Therefore we have that:

$$\begin{aligned} P(\exists t \leq T\ s.t.\ \text{l.p.r.}(q_t^\mu; p_t^\mu) = \text{O}) &\leq P(\exists t \leq T\ s.t.\ (q_t^{\mu,q}; p_t^{\mu,p}) = \text{O}) \\ &+ P(\exists t \leq T\ s.t.\ (q_t^{\mu,-q}; p_t^{\mu,-p}) = \text{O}). \end{aligned}$$

Hence Lemma 2.1.3 implies that

$$P(\exists t \leq T\ s.t.\ \text{l.p.r.}(q_t^\mu; p_t^\mu) = \text{O}) = 0.$$

Part (ii) is an easy consequence of part (i). It is easy to see that $\{\tau_k^\mu\}$ is an unbounded, strictly increasing sequence of Markov times. Indeed, if on the contrary we assume that there exists a N such that $\tau_k^\mu \leq N$ for all k with positive probability, then the trajectories of l.p.r.$(q_t^\mu; p_t^\mu)$ will have limit points. The only possible limit point however is the origin $(0, ..., 0; 0, ..., 0)$. But by part (i) the probability that within any time T the trajectory will reach the origin is 0. So $\{\tau_k^\mu\}$ is an unbounded strictly increasing sequence of Markov times. Therefore we have that $P(\lim_{k\to+\infty} \tau_k^{\mu} = +\infty) = 1$.

$\square$

Therefore the Langevin process with reflection has only finitely many jumps in any time interval $[0,T]$ with probability 1. Hence our definition for the Langevin process with reflection is correct.

2.2 The Skorohod reflection problem

The convergence of the Langevin process with reflection that will be presented in section 2.3 relies on results about solutions of the Skorohod reflection problem, proven in [5] and [34].

Let us first recall that $D = \mathbb{R}_+ \times \mathbb{R}^{r-1}$, $\partial D = \partial\mathbb{R}_+ \times \mathbb{R}^{r-1}$ and let $N(q)$ be the set of inward normals at $q \in \partial D$. Denote also by $\mathbb{D}(\mathbb{R}_+, D)$ the space of *cadlág* (right continuous with left limits) functions with values in D, endowed with the Skorohod topology and by $\mathbb{B}.\mathbb{V}.(\mathbb{R}_+, D)$ the set of *cadlág* functions with bounded variation and values in D.

Definition 2.2.1. *Let w be a function in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^r)$ such that $w(0) \in D$. We say that the pair (q, l) with $q \in \mathbb{D}(\mathbb{R}_+, D)$, $l \in \mathbb{B}.\mathbb{V}.(\mathbb{R}_+, \mathbb{R}^r)$ is a solution to the Skorohod* *problem for (D, N, w) if*

$$q_t = w_t + l_t$$

$$l_t = \int_0^t \nu(s) d|l|_s, \nu(s) \in N(q_s), d|l| - a.e.$$

$$d|l|(t : q_t \in D) = 0,$$

where $|l|$ denotes the total variation of l and is called the local time of the solution.

The following theorem characterizes the continuity properties of solutions of the Skorohod reflection problem.

Theorem 2.2.2. *Let W be a compact subset of $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^r)$ in the Skorohod topology such that $w(0) \in D$ for every $w \in W$. Moreover let $\mathcal{Q}$ be the set of $(q, l, |l|, w) \in \mathbb{D}(\mathbb{R}_+, D) \times \mathbb{B}.\mathbb{V}.(\mathbb{R}_+, \mathbb{R}^r) \times \mathbb{B}.\mathbb{V}.(\mathbb{R}_+, \mathbb{R}_+) \times \mathbb{D}(\mathbb{R}_+, \mathbb{R}^r)$ such that (q, l) is the solution to the Skorohod problem for (D, N, w) for some $w \in W$ and q is continuous. The set D is convex and so $\mathcal{Q}$ is a relatively compact subset of $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{3r+1})$ in the Skorohod topology and for every accumulation point of $(q, l, |l|, w)$ in $\mathcal{Q}$ we have that (q, l) is a solution to the Skorohod problem for (D, N, w).*

Proof. This is a special case of Theorem 3.2 in [4]. □

2.3 Convergence of the Langevin process with reflection

In this section we consider the limit of l.p.r.(q_t^μ) as $\mu \to 0$ when the diffusion matrix is the unit matrix. Below we will assume that $t \le T$, where T ia s positive real number.

Consider the stochastic process $(q_t^\mu; p_t^\mu)$ in $D \times \mathbb{R}^r$, which satisfies the following system of S.D.E.'s:

$$\begin{aligned} \dot{q}_t^\mu &= p_t^\mu \\ \mu \dot{p}_t^\mu &= -p_t^\mu + b(q_t^\mu) + \dot{W}_t + \nu(q_t^\mu) \cdot \dot{\Psi}_t^\mu \\ q_0^\mu &= q_0, p_0^\mu = p_0, \end{aligned} \tag{2.8}$$

where $q_t^\mu = (q_t^{1,\mu}, \cdots, q_t^{r,\mu})'$, $p_t^\mu = (p_t^{1,\mu}, \cdots, p_t^{r,\mu})'$, $W_t = (W_t^1, \cdots, W_t^r)'$, $\nu(q)$ denotes the unit inward normal to D at $q \in \partial D$, $b(q) = (b_1(q), ..., b_r(q))'$ and $\Psi_t^\mu = \mu \sum_{s \le t} (-2p_{s-}^\mu \cdot \nu(q_s^\mu)) \cdot \chi_{\partial D}(q_s^\mu)$. It is easy to see that (2.8) is pathwise equivalent to the Langevin process with reflection in $D \times \mathbb{R}^r$ of Definition 2.1.1 and so it admits a unique weak solution.

We will follow the method introduced in [4]. The main idea is to represent q^μ as the first component of a solution to the Skorohod problem for $(D, N, H^\mu + X^\mu)$, where $H^\mu + X^\mu$ is a semimartingale. The family $\{H^\mu + X^\mu\}$ turns out to be tight and this enables us to use Theorem 2.2.2 to conclude that the family $\{q^\mu\}$ is tight as well.

We can suppose that there is a unique underlying complete probability space $(\Omega, \mathbb{F}, P)$. Let $\widehat{\mathbb{F}}$ denote the the $\sigma-$algebra of $\mathbb{F}$ of sets with $P-$ measure 0 or 1 and define the filtration

$$\mathbb{F}_t^\mu = \widehat{\mathbb{F}} \cup \sigma((q_s^\mu; p_s^\mu), s \leq t).$$

Lemma 2.3.1. *For every μ the pair of stochastic processes $(q_\cdot^\mu, L_\cdot^\mu)$, where*

$$L_t^\mu = \int_0^t \nu(q_s^\mu) d\Psi_t^\mu \tag{2.9}$$

is an almost surely solution to the Skorohod reflection problem for $(D, N, H^\mu + X^\mu)$, where

$$\begin{aligned} H_t^\mu &= q_0 + \mu p_0 - \mu p_t^\mu \\ X_t^\mu &= \int_0^t b(q_s^\mu) ds + W_t \end{aligned} \tag{2.10}$$

Proof. Consider the integral form of (2.8). Taking into account that $\int_0^t p_s^\mu ds = q_t^\mu - q_0$ and solving for q_t^μ we see that:

$$q_t^\mu = H_t^\mu + X_t^\mu + L_t^\mu$$

Then (q^μ, L^μ) verifies Definition 2.2.1 with probability 1. $\square$

Lemma 2.3.2. *For every $T > 0$ we have that $\lim_{\mu \to 0} E[\sup_{t \leq T} |\mu p_t^\mu|^2] = 0$.*

Proof. Assume first that $b = 0$. Consider equations (2.8) and apply the It$\widehat{o}$ formula for semimartingales to the function $f(q,p) = |p|^2$ for every pair of times s, t such that $0 \leq s \leq t \leq T$. Doing that we get

$$|p_t^\mu|^2 \;=\; |p_s^\mu|^2 - \frac{2}{\mu}\int_s^t |p_u^\mu|^2 du + \frac{2}{\mu}\int_s^t p_u^\mu \cdot dW_u + \frac{1}{\mu^2} r(t-s) \tag{2.11}$$

It is interesting to observe that the local time Ψ_t^μ does not appear above. This comes from the fact that under elastic reflection $|p_t^\mu|^2 = |p_{t-}^\mu|^2$ for every $t > 0$.

Consider now a constant $c > 0$ and functions $x, g \in \mathbb{D}([0,T], \mathbb{R})$ with $g(0) = 0$ such that:

$$x_t \leq x_s - c\int_s^t x_u du + g_t - g_s,\; 0 \leq s \leq t \leq T \tag{2.12}$$

Then one can easily see that

$$x_t \leq e^{-ct}(x_0 + g_t) + c\int_0^t e^{-c(t-u)}(g_t - g_u)du,\; 0 \leq t \leq T \tag{2.13}$$

By taking expected value to (2.11) and applying (2.13) with $c = \frac{2}{\mu}$, $g_t = \frac{1}{\mu^2} rt$ and $x_t = |p_t^\mu|^2$, we get

$$E|p_t^\mu|^2 \;\leq\; e^{-\frac{2}{\mu}t}(|p|^2 + \frac{1}{\mu^2} rt) + \frac{2}{\mu^3}\int_0^t e^{-\frac{2}{\mu}(t-u)} r(t-u)du$$

$$= e^{-\frac{2}{\mu}t}|p|^2 + \frac{r}{\mu^2}(\frac{\mu}{2} - \frac{\mu}{2}e^{-\frac{2t}{\mu}}) \tag{2.14}$$

This implies the statement of the Lemma for $b = 0$. The general case can be reduced to the case with $b = 0$ by an absolutely continuous change of measures in the space of trajectories. □

The following two theorems are restatements of Theorems 3.8.6 and 3.10.2 respectively of [6].

Theorem 2.3.3. *Let $\{Y^n\}$ be a family of processes with sample paths in $\mathbb{D}(\mathbb{R}_+, D)$. Assuming that for every $\epsilon > 0$ and rational $t \geq 0$ there exist a compact set $\Gamma(\epsilon, t) \subset D$ such that $\liminf_n P(Y^n(t) \in \Gamma(\epsilon, t)) \geq 1 - \epsilon$, then the following are equivalent*

(i). $\{Y^n\}$ is relatively compact.

(ii). For each $T > 0$, there exists $\beta > 0$ and a family of nonnegative random variables $\{\gamma^n(\delta), 0 < \delta < 1\}$ satisfying

$$E(|Y^n(t + u) - Y^n(t)|^\beta \, |\mathbb{F}_t^n) \leq E(\gamma^n(\delta)|\mathbb{F}_t^n),$$

for $t \in [0, T]$ and $u \in [0, \delta]$ and in addition $\lim_{\delta \to 0} \limsup_n E(\gamma^n(\delta)) = 0$.

Theorem 2.3.4. *Let $\{Y^n\}$ and Y be processes with sample paths in $\mathbb{D}(\mathbb{R}_+, D)$ such that Y_n converges in distribution to Y. Then Y is almost surely continuous if and only if $\int_0^\infty e^{-u}[\sup_{0 \leq t \leq u} |Y^n(t) - Y^n(t-)| \wedge 1]du \Rightarrow 0$.*

The following lemma shows that the family $\{H^\mu + X^\mu\}$ is tight in the Skorohod topology.

Lemma 2.3.5. *The family $\{H^\mu + X^\mu\}$ defined in (2.10) is relatively compact and all of its accumulation points are continuous.*

Proof. It is easily seen that $\{X^\mu\}$ is relatively compact and that all of its accumulation points are continuous.

Now Lemma 2.3.2 suggests that:

$$\lim_{\mu\to 0} E[\sup_{t\le T} |H_t^\mu|^2] \le c \tag{2.15}$$

$$\lim_{\mu\to 0} E[\sup_{|t-s|\le\delta} |H_t^\mu - H_s^\mu|] \le c_1\delta, \tag{2.16}$$

where c, c_1 are positive constants independent of μ.

Chebychev's inequality and (2.15) imply that

$$\liminf_{n\to\infty} P(|H^{1/n}(t)| \le \lambda) \ge 1 - \frac{c}{\lambda^2}.$$

Therefore by this and (2.16), Theorem 2.3.3 gives us that $\{H^\mu\}$ is relatively compact. Lastly (2.16) and Theorem 2.3.4 implies that all its accumulation points are continuous. $\square$

Theorem 2.3.6. *The family $\{(q^\mu, L^\mu, \Psi^\mu, H^\mu, X^\mu)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{4r+1})$.*

Proof. It follows from Lemma 2.3.5 and 2.2.2 □

Now that tightness has been established we will proceed with the identification of the stochastic differential equation with reflection that describes the behavior of q^μ as $\mu \to 0$.

Consider the following S.D.E. with reflection:

$$q_t = q_0 + \int_0^t b(q_s)ds + W_t + L_t \tag{2.17}$$

where $L_t = \int_0^t \nu(q_s)d|L|_s, \nu(s) \in N(q_s)$ and $d|L|(\{t : q_t \in D\}) = 0$. It is known that (2.17) has a unique weak solution (q, L) ([1]).

Theorem 2.3.7. *The family $\{(q^\mu, L^\mu)\}$ converges in distribution to the unique solution (q, L) of (2.17).*

Proof. By Theorem 2.3.6 we have that the five-tuple $\{(q^\mu, L^\mu, H^\mu, X^\mu, W)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{5r})$. Hence it (or a subsequence) converges in distribution to a stochastic process $\{(q, L, H, X, W)\}$. By the Skorohod representation theorem, one can find a probability space $(\widetilde{\Omega}, \widetilde{\mathbb{F}}, \widetilde{P})$ and realizations $\{(\widetilde{q}^\mu, \widetilde{L}^\mu, \widetilde{H}^\mu, \widetilde{W}^\mu)\}$ and $\{(\widetilde{q}, \widetilde{L}, \widetilde{H}, \widetilde{X}, \widetilde{W})\}$ of $\{(q^\mu, L^\mu, H^\mu, X^\mu, W)\}$ and $\{(q, L, H, X, W)\}$ respectively such that $\{(\widetilde{q}^\mu, \widetilde{L}^\mu, \widetilde{H}^\mu, \widetilde{X}^\mu, \widetilde{W}^\mu)\}$ converges $\widetilde{P}$-almost surely to $\{(\widetilde{q}, \widetilde{L}, \widetilde{H}, \widetilde{X}, \widetilde{W})\}$. Therefore by Theorem 2.2.2 $(\widetilde{q}, \widetilde{L})$ is a solution to the Skorohod problem for $(D, N, \widetilde{H} + \widetilde{X})$ $\widetilde{P}-$almost surely.

Now by the convergence of $\widetilde{q}^{\mu}$ to $\widetilde{q}$ we get that $\widetilde{X}$ must be given by:

$$\widetilde{X}_t = \int_0^t b(\widetilde{q}_s)ds + \widetilde{W}_t$$

Finally, Lemma 2.3.2 and its proof imply that $\widetilde{H}_t = q_0$. $\square$

Chapter 3

Reaction Diffusion Equations with Nonlinear Boundary Conditions in Narrow Domains

3.1 Introduction

For each $x \in \mathbb{R}^n$, let D_x be a bounded domain in $\mathbb{R}^m$ with a smooth boundary ∂D_x. Assume, for brevity, that D_x is homeomorphic to a ball in $\mathbb{R}^m$ and contains $0 \in \mathbb{R}^m$. Consider the domain $D = \{(x, y) : x \in \mathbb{R}^n, y \in D_x\} \subset \mathbb{R}^{n+m}$. Assume that the boundary ∂D of D is smooth enough and denote by $\gamma(x, y)$ the inward unit normal to ∂D. Assume that $\gamma(x, y)$ is not parallel to the subspace $\mathbb{R}^n \subset \mathbb{R}^{n+m}$ for any $(x, y) \in \partial D$.

Denote by $D^\epsilon, 0 < \epsilon << 1$, the domain in $\mathbb{R}^{n+m}$ obtained from D by contraction: $D^\epsilon = \{(x, y) : x \in \mathbb{R}^n, y\epsilon^{-1} \in D_x\}$. If $n = 1$, D^ϵ is a narrow tube (or a strip for $m = 1$) for $0 < \epsilon << 1$. If $n > 1$, then D^ϵ is a thin layer.

Consider the problem:

$$
\begin{aligned}
u_t^\epsilon &= \frac{1}{2}\triangle u^\epsilon, && \text{in } (0, T) \times D^\epsilon && (3.1)\\
u^\epsilon(0, x, y) &= f(x), && \text{on } \{0\} \times D^\epsilon
\end{aligned}
$$

$$\frac{\partial u^\epsilon}{\partial \gamma^\epsilon} = -\epsilon c(x,y,u^\epsilon)u^\epsilon, \qquad \text{on } (0,T)\times\partial D^\epsilon,$$

where γ^ϵ is the inward unit normal to ∂D^ϵ. The functions f and c are sufficiently regular and bounded; f is assumed to be nonnegative. We study the behavior of solution of problem (3.1) as $\epsilon \downarrow 0$. Using probabilistic methods, we prove that $u^\epsilon(t,x,y)$ converges as $\epsilon \downarrow 0$ to the solution $u(t,x)$ of the problem:

$$\begin{aligned} u_t &= \frac{1}{2}\triangle_x u + \frac{1}{2}\nabla(\log V(x))\nabla_x u + \frac{1}{2}\frac{S(x)}{V(x)}c(x,0,u)u, \quad \text{in } (0,T)\times\mathbb{R}^n \\ u(0,x) &= f(x), \qquad \text{on } \{0\}\times\mathbb{R}^n. \end{aligned} \tag{3.2}$$

Here $V(x)$ is the volume of D_x and $S(x)$ is the surface area of ∂D_x. One can expect that, under certain assumptions on the nonlinear term $c(x,y,u)u$ in (3.1), the solution $u^\epsilon(t,x,y)$ can be approximated by a running-wave-type solution. Corresponding results on the standard reaction diffusion equation (3.2) (see chapter 6 and 7 in [9]) allow to describe the asymptotic wavefront motion for (3.1). We see how the motion of the interface (wavefront) depends on the behavior of the cross-sections D_x of the domain D. In particular, using the results of [9] (chapter 6) we prove that in the case of the nonlinear term of K-P-P type the wavefront can have jumps.

Consider the Wiener process $(X_t^\epsilon, Y_t^\epsilon)$ in D^ϵ with instantaneous normal reflection on the boundary of D^ϵ. Its trajectories can be described by the stochastic differential equations:

$$\begin{aligned} X_t^\epsilon &= x + W_t^1 + \int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon) dL_s^\epsilon \\ Y_t^\epsilon &= y + W_t^2 + \int_0^t \gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon) dL_s^\epsilon. \end{aligned} \tag{3.3}$$

Here W_t^1 and W_t^2 are independent Wiener process in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and (x, y) is a point inside D^ϵ. Moreover γ_1^ϵ and γ_2^ϵ are projections of the unit inward normal vector to ∂D^ϵ on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. It is easy to see that $\lim_{\epsilon\downarrow 0} |\epsilon^{-1}\gamma_1^\epsilon| = \frac{\gamma_1^1}{|\gamma_2^1|}$ and $\lim_{\epsilon\downarrow 0} |\gamma_2^\epsilon| = 1$, where $|\cdot|$ denotes Euclidean length. Furthermore L_t^ϵ is the local time for the process $(X_t^\epsilon, Y_t^\epsilon)$ on ∂D^ϵ, i.e. it is a continuous, non-decreasing process that increases only when $(X_t^\epsilon, Y_t^\epsilon) \in \partial D^\epsilon$ such that the Lebesque measure $\Lambda\{t > 0 : (X_t^\epsilon, Y_t^\epsilon) \in \partial D^\epsilon\} = 0$ (see for instance [23]).

If $(X_t^\epsilon, Y_t^\epsilon)$ is defined by (3.3), then as it can be derived from Theorem 2.5.1 in [9], $u^\epsilon(t, x, y)$ satisfies the following integral equation in the functional space:

$$u^\epsilon(t, x, y) = E_{x,y} f(X_t^\epsilon) \exp[\int_0^t \epsilon c(X_s^\epsilon, Y_s^\epsilon, u^\epsilon(t - s, X_s^\epsilon, Y_s^\epsilon)) dL_s^\epsilon], \tag{3.4}$$

where $E_{x,y}$ denotes expectation and the subscript (x, y) denotes the initial point of $(X_s^\epsilon, Y_s^\epsilon)$. Equation (3.4) has a unique solution if, say, $c(x, y, u)$ has a bounded derivative in u.

Let X_t be the solution of the stochastic differential equation

$$X_t = x + W_t^1 + \int_0^t \frac{1}{2} \nabla(\log V(X_s)) ds. \tag{3.5}$$

Then the solution $u(t,x)$ of equation (3.2) satisfies the equality:

$$u(t,x) = E_x f(X_t) \exp[\int_0^t \frac{1}{2} \frac{S(X_s)}{V(X_s)} c(X_s, 0, u(t-s, X_s)) ds]. \qquad (3.6)$$

We prove that the component X_t^ϵ of the process $(X_t^\epsilon, Y_t^\epsilon)$ converges in a certain sense to X_t. This together with uniform in $0 < \epsilon < 1$ bounds for $u^\epsilon(t,x,y)$ and its derivatives allow to prove that the solution of (3.4) converges to the solution of (3.6) as $\epsilon \downarrow 0$ uniformly on each compact subset of $[0, \infty) \times \mathbb{R}^{n+m}$.

3.2 Averaging of integrals in local time

Let $H(x,y)$ be a smooth and bounded function. We want to consider the limiting behavior as $\epsilon \downarrow 0$ of expressions like $\int_0^t \epsilon H(X_s^\epsilon, Y_s^\epsilon/\epsilon) dL_s^\epsilon$ (see Lemma 3.2.1 below). We will assume that the unit inward normal $\gamma(x,y)$ to ∂D and the function $H(x,y)$ are both three times differentiable in x and y.

Lemma 3.2.1. *Define* $Q(x) = \frac{1}{V(x)} \int_{\partial D_x} H(x,y) dS_x$*, where* dS_x *is the surface element on* ∂D_x*. Then for every* $T > 0$ *and small enough* ϵ*, there exists a constant* K *independent of* ϵ *such that:*

(i). $\sup_{0 \le t \le T} E|\int_0^t \frac{1}{2} Q(X_s^\epsilon) ds - \int_0^t \epsilon H(X_s^\epsilon, Y_s^\epsilon/\epsilon) |\gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)| dL_s^\epsilon|^2 \le K\epsilon^2.$

(ii). For every $\delta > 0$ *we have*

$$P\{\sup_{0 \le t \le T} |\int_0^t \frac{1}{2} Q(X_s^\epsilon) ds - \int_0^t \epsilon H(X_s^\epsilon, Y_s^\epsilon/\epsilon) |\gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)| dL_s^\epsilon| > \delta\} \le K \frac{\epsilon^2}{\delta^2}.$$

The proof of Lemma 3.2.1 relies on the following lemma, which we prove first.

Lemma 3.2.2. *For every $T > 0$ and small enough ϵ, there exists a constant K_1 independent of ϵ such that:*

$$E|\epsilon L_T^\epsilon|^2 \leq K_1$$

Proof. Consider the auxiliary problem

$$\begin{aligned} \triangle_y v(x,y) &= Q(x), \;\; y \in D_x \subset \mathbb{R}^m \\ \frac{\partial_y v(x,y)}{\partial n(x,y)} &= -1, \;\; y \in \partial D_x, \end{aligned} \tag{3.7}$$

where $n(x,y) = \frac{\gamma_2^1(x,y)}{|\gamma_2^1(x,y)|}$ and $x \in \mathbb{R}^n$ is a parameter. Let

$$Q(x) = \frac{S(x)}{V(x)}, \tag{3.8}$$

where $S(x)$ is the surface area of D_x and $V(x)$ is the volume of D_x. As it can be derived from [2], a smooth in x and y solution $v(x,y)$ of problem (3.7) exists and is bounded together with its first and second derivatives. So we can apply Itô formula to the function $\epsilon v(x, y/\epsilon)$, and get:

$$\begin{aligned} \epsilon^2 v(X_t^\epsilon, Y_t^\epsilon/\epsilon) &= \epsilon^2 v(x, y/\epsilon) + \int_0^t \epsilon^2 \frac{1}{2} \triangle_x v(X_s^\epsilon, Y_s^\epsilon/\epsilon) ds + \int_0^t \frac{1}{2} \triangle_y v(X_s^\epsilon, Y_s^\epsilon/\epsilon) ds \\ &+ \int_0^t \epsilon^2 (\nabla_x v(X_s^\epsilon, Y_s^\epsilon/\epsilon), dW_s^1) + \int_0^t \epsilon (\nabla_y v(X_s^\epsilon, Y_s^\epsilon/\epsilon), dW_s^2) \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \epsilon^2 (\nabla_x v(X_s^\epsilon, Y_s^\epsilon/\epsilon), \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon)) dL_s^\epsilon \\
&+ \int_0^t \epsilon (\nabla_y v(X_s^\epsilon, Y_s^\epsilon/\epsilon), \gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)) dL_s^\epsilon
\end{aligned} \tag{3.9}
$$

Recalling now that $\lim_{\epsilon\downarrow 0} |\epsilon^{-1}\gamma_1^\epsilon| = \frac{\gamma_1^1}{|\gamma_2^1|}$ and $\lim_{\epsilon\downarrow 0} |\gamma_2^\epsilon| = 1$ and that v satisfies (3.7) one easily concludes that there is an $\epsilon_0 = \epsilon_0(\| |\nabla_x v| \|, \gamma_1^1) > 0$ such that for every $\epsilon < \epsilon_0$:

$$
\begin{aligned}
E|\epsilon L_T^\epsilon|^2 &\leq C[\epsilon^4(2\|v^2\| + \|\frac{1}{2}\triangle_x v\|^2 T^2 + \||\nabla_x v|^2\| T) + \\
&+ \epsilon^2 \| |\nabla_y v|^2 \| T + \|\frac{1}{2} Q\|^2 T],
\end{aligned}
$$

where for any function g, $\|g\| = \sup_z |g(z)|$. Here, we also used the fact that the local time is increasing function of t. This proves the statement of the lemma. $\square$

Proof. Proof of Lemma 3.2.1

We consider the auxiliary problem

$$
\begin{aligned}
\triangle_y v(x,y) &= Q(x), \ \ y \in D_x \subset \mathbb{R}^m \\
\frac{\partial_y v(x,y)}{\partial n(x,y)} &= -H(x,y), \ \ y \in \partial D_x,
\end{aligned} \tag{3.10}
$$

where $n(x,y) = \frac{\gamma_2^1(x,y)}{|\gamma_2^1(x,y)|}$ and $x \in \mathbb{R}^n$ is a parameter.

The necessary and sufficient condition for the existence of a solution for (3.10) is that

$$
Q(x) = \frac{1}{V(x)} \int_{\partial D_x} H(x,y) dS_x, \tag{3.11}
$$

where dS_x is the surface element on ∂D_x and $V(x) = \text{volume}(D_x)$.

Applying Itô formula to the function $\epsilon v(x, y/\epsilon)$ and using the bounds obtained in Lemma 3.2.2 we get the following inequalities:

$$\begin{aligned}
&\sup_{0\le t\le T} E|\int_0^t \frac{1}{2}Q(X_s^\epsilon)ds - \int_0^t \epsilon H(X_s^\epsilon, Y_s^\epsilon/\epsilon)|\gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)|dL_s^\epsilon|^2 \le \\
\le\ &\epsilon^4 C(2\|v^2\| + \|\frac{1}{2}\triangle_x v\|^2 T^2 + \||\nabla_x v|^2\|T + \||\nabla_x v|\|^2 K_1) + \epsilon^2 C\||\nabla_y v|^2\|T,
\end{aligned}$$

which proves statement (i) of the lemma.

For part (ii) one makes use of the Doob maximal inequalities (see [23], page 14):

$$\begin{aligned}
E[\sup_{0\le t\le T}|\int_0^t (\nabla_x v(X_s^\epsilon, Y_s^\epsilon/\epsilon), dW_s^1)|]^2 &\le 4\parallel |\nabla_x v|^2 \parallel T \\
E[\sup_{0\le t\le T}|\int_0^t (\nabla_y v(X_s^\epsilon, Y_s^\epsilon/\epsilon), dW_s^2)|]^2 &\le 4\parallel |\nabla_y v|^2 \parallel T
\end{aligned}$$

Then, following the procedure that proved part (i) we get that there is an $\epsilon_0 > 0$ such that for every $\epsilon < \epsilon_0$:

$$\begin{aligned}
&E[\sup_{0\le t\le T}|\int_0^t \frac{1}{2}Q(X_s^\epsilon)ds - \int_0^t \epsilon H(X_s^\epsilon, Y_s^\epsilon/\epsilon)|\gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)|dL_s^\epsilon|]^2 \le \\
\le\ &\epsilon^4 C(4\|v^2\| + \|\frac{1}{2}\triangle_x v\|^2 T^2 + 4\||\nabla_x v|^2\|T + \||\nabla_x v|\|^2 K_1) + \epsilon^2 C\||\nabla_y v|^2\|T,
\end{aligned}$$

which together with Chebyshev inequality proves statement (ii) of the lemma. $\square$

3.3 Limit of u^ϵ.

In this section we consider the limit as $\epsilon \to 0$ of the solution u^ϵ to problem (3.1). The result is given in Theorem 3.3.4. The proof will proceed as follows. First, (in Proposition 3.3.2) we write down an integral equation in the space of trajectories for the solution of (3.1). Then, in Lemma 3.3.3 we consider the mean square limit as $\epsilon \to 0$ of the underlying stochastic process with instantaneous normal reflection on the boundary of D^ϵ (see (3.3)). Lastly, an important ingredient to the proof are the a-priori bounds for u^ϵ and its derivatives. These a-priori bounds are independent of ϵ, their derivation is standard and are given for completeness in Proposition 3.3.7.

We assume that the initial function $f(x)$ of problem (3.1) is bounded, non-negative and can have finite number of simple discontinuities. The function $c(x, y, u)$ is assumed to be uniformly bounded in all arguments, continuous in x,y, Lipschitz continuous in u and that there exist constants $M, N > 0$ such that $c(\cdot, \cdot, u) < -M$ for $u > N$.

In addition we assume that the boundary of D^1 satisfies $\partial D^1 \in \mathcal{C}^{3+a}(\mathbb{R}^m)$, where $a \in (0, 1)$.

Remark 3.3.1. *For the existence of a classical solution to (3.1) one actually needs only to assume $\partial D^1 \in \mathcal{C}^{2+a}(\mathbb{R}^m)$. The assumption $\partial D^1 \in \mathcal{C}^{3+a}(\mathbb{R}^m)$ is being done* *solely for the purpose of Lemma 3.3.3 and Theorem 3.3.4.*

Let $(X^\epsilon, Y^\epsilon, L^\epsilon)$ in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1_+$ satisfy (3.3). Then we have:

Proposition 3.3.2. *Problem (3.1) has a unique classical solution in* $[0, T) \times D^\epsilon$ *which satisfies:*

$$u^\epsilon(t, x, y) = E_{x,y} f(X_t^\epsilon) \exp[\int_0^t \epsilon c(X_s^\epsilon, Y_s^\epsilon, u^\epsilon(t - s, X_s^\epsilon, Y_s^\epsilon)) dL_s^\epsilon]. \tag{3.12}$$

Proof. Under our assumptions, the uniqueness and existence of a classical solution to (3.1) follows from Theorem 7.5.13 of [17]). The equation (3.12) follows from Theorem 2.5.1 of [9]. □

In order now to consider the limit as $\epsilon \to 0$ of (3.12), we need first to examine the asymptotic behavior of X_t^ϵ as $\epsilon \to 0$.

We will prove that X_t^ϵ converges as $\epsilon \downarrow 0$ to X_t, where X_t is the solution to

$$X_t = x + W_t^1 + \int_0^t \frac{1}{2} \nabla(\log V(X_s)) ds, \tag{3.13}$$

where $V(x) = \text{volume}(D_x)$. Hence, we see that as $\epsilon \downarrow 0$, the effect of the reflection on the boundary is an extra drift term. A sketch of the proof for the above result is given in chapter 7 of [13]. More details are given here.

Lemma 3.3.3. *For any $T > 0$ we have*

$$\sup_{0\le t\le T} E_x|X_t^\epsilon - X_t|^2 \to 0 \ \ as\ \ \epsilon \to 0. \tag{3.14}$$

Proof. It is not difficult to see that $\gamma_1^\epsilon(x,y) = \epsilon\frac{\gamma_1^1(x,y)}{|\gamma_2^1(x,y)|}|\gamma_2^\epsilon(x,y)|$. Moreover, a straightforward application of the divergence theorem gives

$$\int_{\partial D_x} \frac{\gamma_1^1(x,y)}{|\gamma_2^1(x,y)|} dS_x = \nabla V(x). \tag{3.15}$$

Then, Lemma 3.2.1 with $H(x,y) = \frac{\gamma_1^1(x,y)}{|\gamma_2^1(x,y)|}$ and $Q(x) = \nabla \log V(x)$ implies that for small enough ϵ there exists a constant K independent of ϵ such that

$$\sup_{0\le t\le T} E|\int_0^t \frac{1}{2}\nabla \log(V(X_s^\epsilon))ds - \int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon)dL_s^\epsilon|^2 \le \epsilon^2 K. \tag{3.16}$$

Now we write

$$\begin{aligned} X_t^\epsilon - X_t &= \int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon)dL_s^\epsilon - \int_0^t \frac{1}{2}\nabla \log(V(X_s))ds \\ &= [\int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon)dL_s^\epsilon - \int_0^t \frac{1}{2}\nabla \log(V(X_s^\epsilon))ds] \\ &+ [\int_0^t \frac{1}{2}\nabla \log(V(X_s^\epsilon))ds - \int_0^t \frac{1}{2}\nabla \log(V(X_s))ds] \end{aligned} \tag{3.17}$$

Then Gronwall Lemma and (3.16) give:

$$\sup_{0\le t\le T} E_x|X_t^\epsilon - X_t|^2 \to 0 \ \text{ as } \ \epsilon \to 0, \tag{3.18}$$

which is the statement of the lemma. □

Consider now the solution, u, to the equation

$$u(t,x) = E_x f(X_t) \exp[\int_0^t \overline{c}(X_s, u(t-s, X_s))ds], \tag{3.19}$$

where

$$\overline{c}(x, u(t,x)) = \frac{1}{2}\frac{S(x)}{V(x)} c(x, 0, u(t,x)). \tag{3.20}$$

For notational convenience we will also denote $\overline{c}(t,x) = \overline{c}(x, u(t,x))$.

Since $\overline{c}(x,u)$ is Lipschitz continuous in u, the solution of (3.19) exists and is unique. Our assumptions on the functions f, c and the boundary ∂D_x, imply that the solution u to (3.19) is actually the classical solution of the following parabolic problem:

$$\begin{aligned} &u_t = \frac{1}{2}\triangle_x u + \frac{1}{2}\nabla(\log V(x)) \cdot \nabla_x u + \frac{1}{2}\frac{S(x)}{V(x)} c(x, 0, u(t,x))u, \text{ in } (0,T) \times \mathbb{R}^n \\ &u(0,x) = f(x), \text{ on } \{0\} \times \mathbb{R}^n. \end{aligned} \tag{3.21}$$

Theorem 3.3.4. *Under our assumptions, we have that*

$$u^\epsilon(t,x,y) \to u(t,x) \text{ as } \epsilon \to 0, \textit{ uniformly in any compact sunset of } \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m,$$

where $u^\epsilon(t,x,y)$, $u(t,x)$ are the solutions to (3.1) and (3.21) respectively.

Proof. By Proposition 3.3.7 and the well known theorem of Ascoli-Arzela we get that there exists a subsequence of $\{u^\epsilon\}$ (which for convenience we will denote again by $\{u^\epsilon\}$) and a function u, such that:

$$u^\epsilon \to u \text{ as } \epsilon \to 0, \text{ uniformly in compacts.}$$

We will prove that u actually satisfies (3.19) which then implies that u satisfies (3.21). Fix t and x and consider the solution $v(y) = v^{\epsilon,t,x}(y)$ to the elliptic boundary value problem:

$$\begin{aligned}
\triangle_y v(y) &= \overline{c}^\epsilon(t,x), \;\; y \in D_x \subset \mathbb{R}^m \\
\frac{\partial_y v(y)}{\partial n(x,y)} &= -\frac{1}{|\gamma_2^\epsilon(x,\epsilon y)|} c(x,\epsilon y, u^\epsilon(t,x,\epsilon y)), \;\; y \in \partial D_x.
\end{aligned} \tag{3.22}$$

Problem (3.22) is solvable if

$$\overline{c}^\epsilon(t,x) = \frac{1}{V(x)} \int_{\partial D_x} \frac{1}{|\gamma_2^\epsilon(x,\epsilon y)|} c(x,\epsilon y, u^\epsilon(t,x,\epsilon y)) dS_x. \tag{3.23}$$

Proceeding similarly now to Lemma 3.2.1 and recalling that v satisfies (3.22), we see that there is a constant $K^\epsilon = K(\|\nabla_x v^\epsilon\|, \|\nabla_y v^\epsilon\|, \|\triangle_x v^\epsilon\|, \|\triangle_y v^\epsilon\|, \|v_t^\epsilon\|, \|\gamma_1^1\|, T)$ such that:

$$\begin{aligned}
&\sup_{0\le t\le T} E\left|\int_0^t \frac{1}{2}\overline{c}^\epsilon(t-s,X_s^\epsilon)ds - \int_0^t \epsilon c(X_s^\epsilon, Y_s^\epsilon, u^\epsilon(t-s,X_s^\epsilon,Y_s^\epsilon))dL_s^\epsilon\right|^2 \\
\le\;& \epsilon^2 K^\epsilon (1 + \sup_{0\le t\le T} E[\epsilon L_t^\epsilon]^2)
\end{aligned} \tag{3.24}$$

We observe that K^ϵ depends on ϵ only through functions that are uniformly bounded in ϵ (Proposition 3.3.7). This observation and Lemma 3.2.2 imply that as $\epsilon \to 0$:

$$\sup_{0\le t\le T} E\left|\int_0^t \frac{1}{2}\overline{c}^\epsilon(t-s,X_s^\epsilon)ds - \int_0^t \epsilon c(X_s^\epsilon, Y_s^\epsilon, u^\epsilon(t-s,X_s^\epsilon,Y_s^\epsilon))dL_s^\epsilon\right|^2 \to 0. \tag{3.25}$$

Moreover the Lebesque dominated convergence Theorem, Lemma 3.3.3, the compactness of the family $\{u^\epsilon\}$ and (3.23), imply that as $\epsilon \to 0$:

$$\sup_{0\le t\le T} E|\int_0^t \frac{1}{2}\overline{c}^\epsilon(t-s, X_s^\epsilon)ds - \int_0^t \overline{c}(t-s, X_s)ds|^2 \to 0, \tag{3.26}$$

where $\overline{c}^\epsilon, \overline{c}$ and X_t are given by (3.23), (3.20) and (3.13) respectively.

Now let $u^\epsilon(t,x,y), u(t,x)$ be the solutions to (3.12) and (3.19) respectively. Taking into account relations (3.25), (3.26), the weak convergence of X_t^ϵ to X_t as $\epsilon \to 0$ (which is implied by Lemma 3.3.3) and Proposition 3.3.2 we get the statement of the Theorem. $\square$

We conclude this section with the a-priori bounds for the Hölder norm of the solution and for the sup-norm of the solution, the first and the second derivatives of the solution of (3.1). These bounds will be uniform in ϵ. The method follows closely [17].

Let us first introduce some notation.

We write $U_T^\epsilon = [0,T) \times D^\epsilon$, $\overline{U}_T^\epsilon = [0,T) \times \overline{D}^\epsilon$, $\partial U_T^\epsilon = [0,T) \times \partial D^\epsilon$ and $V_T^\epsilon = (0,T) \times D^\epsilon$, where $\overline{D}^\epsilon = D^\epsilon \cup \partial D^\epsilon$.

For $0 < a < 1$, $T > 0$ and for any function g we write:

$$\begin{aligned} \| g \|_{U_T^\epsilon} &= \sup_{(t,z)\in U_T^\epsilon} |g(t,z)| \\ \|H^a g\|_{U_T^\epsilon} &= \sup_{(t,z),(t',z')\in U_T^\epsilon} \frac{|g(t,z)-g(t',z')|}{|t-t'|^{a/2}+|z-z'|^a} \end{aligned}$$

$$\begin{aligned}
\|g\|_{U_T^\epsilon,a} &= \|g\|_{U_T^\epsilon} + \|H^a g\|_{U_T^\epsilon} \\
\|g\|_{D^\epsilon,T,1+a} &= \|g\|_{U_T^\epsilon,a} + \|g_t\|_{U_T^\epsilon} + \|Dg\|_{(0,T)\times D^\epsilon} \\
\overline{\|g\|}_{D^\epsilon,T,1+a} &= \|g\|_{\overline{D}^\epsilon,T,1+a}
\end{aligned}$$

Moreover for notational convenience we will write $z = (x, y)$.

Lemma 3.3.5. *Under our assumptions there exists a constant C_1, independent of $\epsilon > 0$, such that*

$$0 \leq u^\epsilon \leq C_1 \text{ in } \overline{U}_T^\epsilon.$$

Proof. Lemma 3.3.5 can be proven using equation (3.12). Here we give an analytic proof of the claim. For any fixed $b > 0$ we define the function

$$w^\epsilon = (u^\epsilon - b)^+ = \max\{u^\epsilon - b, 0\}.$$

It is easy to show that

$$w_t^\epsilon \leq \frac{1}{2}\triangle w^\epsilon \qquad \text{on } (0,T) \times D^\epsilon.$$

in the weak sense. Let us choose now $b = \max\{N, \|f\|\}$, where N is such that if $u > N$ then $c(\cdot,\cdot,u) < -M$ for some $M > 0$. Then

$$w^\epsilon(0, x, y) = 0.$$

Let us now assume that w^ϵ attains a maximum positive value on the boundary ∂V_T^ϵ at the point (t_o, x_o, y_o). Since u^ϵ is continuous up to the boundary, there exists a connected set Δ such that $(t_o, x_o, y_o) \in \Delta$, $\Delta \subset \partial V_T^\epsilon$ and $w^\epsilon > 0$ on Δ, i.e. $u^\epsilon > b$ on Δ. Since (t_o, x_o, y_o) is a maximum for w^ϵ and γ^ϵ is the inward normal derivative we get that $\frac{\partial w^\epsilon}{\partial \gamma^\epsilon} \leq 0$ at (t_o, x_o, y_o). But on Δ we have that $\frac{\partial w^\epsilon}{\partial \gamma^\epsilon} = \frac{\partial u^\epsilon}{\partial \gamma^\epsilon} = -\epsilon c(x, y, u^\epsilon)u^\epsilon$. Taking into account the particular choice of b and that $c(\cdot, \cdot, u) < -M$ for $u > N$, we get that $-\epsilon c(x, y, u^\epsilon)u^\epsilon > 0$ at (t_o, x_o, y_o). Thus, we have a contradiction and so maximum principle implies that

$$w^\epsilon = 0 \Longrightarrow u^\epsilon < b \text{ in } \overline{U}_T^\epsilon.$$

Lastly maximum principle again implies that $u^\epsilon \geq 0$. $\square$

Let us consider the following linear parabolic pde:

$$\begin{aligned} v_t^\epsilon &= \frac{1}{2}\triangle v^\epsilon, && \text{in } (0,T) \times D^\epsilon \\ v^\epsilon(0,x,y) &= f(x), && \text{on } \{0\} \times D^\epsilon \\ \frac{\partial v^\epsilon}{\partial \gamma^\epsilon} &= -\epsilon c(x,y)v^\epsilon, && \text{on } (0,T) \times \partial D^\epsilon, \end{aligned} \tag{3.27}$$

where f, c are bounded smooth functions. Under the standard hypotheses problem (3.27) has a unique classical solution (Theorem 5.3.2 in [17]).

Lemma 3.3.6. *There is a constant C, independent of ϵ, and an open set $I \subset (0,1)$*

such that for any $a \in I$:

$$\overline{\|v^\epsilon\|}_{D^\epsilon,T,1+a} + \|D^2 v^\epsilon\|_{V_T^\epsilon} \leq C. \tag{3.28}$$

Proof. We will give just a sketch of the proof, since the analysis follows [28], [29], [30] and [17]. The calculations are lengthy but standard.

We solve the second initial-boundary value problem (3.27) by reducing it to an integral equation, i.e. we write:

$$v^\epsilon(t,z) = \int_0^t \int_{\partial D^\epsilon} \Gamma^\epsilon(t,z,\tau,\xi)\phi^\epsilon(\tau,\xi) d\partial D_\xi^\epsilon d\tau + \int_{D^\epsilon} \Gamma^\epsilon(t,z,0,\xi) f(\xi) d\xi, \tag{3.29}$$

where $\Gamma^\epsilon(t,z,\tau,\xi) = (2\sqrt{\pi})^{-n-m}(t-\tau)^{-\frac{n+m}{2}} \exp[-\frac{\sum_{i=1}^{n+m}(z_i-\xi_i)^2}{4(t-\tau)}]$ is the fundamental solution to the heat equation and $\phi(t,z)$ is the solution to a Voltera type integral equation:

$$\begin{aligned} \phi^\epsilon(t,z) &= 2\int_0^t \int_{\partial D^\epsilon} [\frac{\partial \Gamma^\epsilon(t,z,\tau,\xi)}{\partial \gamma^\epsilon} + \epsilon c(z) \Gamma^\epsilon(t,z,\tau,\xi)]\phi^\epsilon(\tau,\xi) d\partial D_\xi^\epsilon d\tau \\ &+ 2[\int_{D^\epsilon} \frac{\partial \Gamma^\epsilon(t,z,0,\xi)}{\partial \gamma^\epsilon} f(\xi) d\xi + \epsilon c(z) \int_{D^\epsilon} \Gamma^\epsilon(t,z,0,\xi) f(\xi) d\xi] \end{aligned} \tag{3.30}$$

Let us now define

$$\begin{aligned} F^\epsilon(t,z) &= \int_{D^\epsilon} \frac{\partial \Gamma^\epsilon(t,z,0,\xi)}{\partial \gamma^\epsilon} f(\xi) d\xi + \epsilon c(z) \int_{D^\epsilon} \Gamma^\epsilon(t,z,0,\xi) f(\xi) d\xi \\ M_1(t,z,\tau,\xi) &= \frac{\partial \Gamma^\epsilon(t,z,\tau,\xi)}{\partial \gamma^\epsilon} + \epsilon c(z) \Gamma^\epsilon(t,z,\tau,\xi) \\ M_{\nu+1}(t,z,\tau,\xi) &= \int_0^t \int_{\partial D^\epsilon} M_1(t,z,t',z') M_\nu(t',z',\tau,\xi) d\partial D_{z'}^\epsilon dt' \end{aligned}$$

It can be shown (see [17]) that there is a Hölder continuous (in space variables) and bounded (with bound and Hölder coefficient independent of ϵ) solution ϕ^ϵ for (3.30), expressed in the form:

$$\phi^\epsilon(t,x) = 2F^\epsilon(t,z) + 2\sum_{\nu=1}^{\infty}\int_0^t \int_{\partial D^\epsilon} M_\nu(t,z,\tau,\xi)F^\epsilon(\tau,\xi)d\partial D^\epsilon_\xi d\tau \tag{3.31}$$

Using the boundedness and the Hölder continuity of (3.31) and (3.29), one can show (see [28], [29], [30] and [17]) that there is a constant C, independent of ϵ, such that

$$\overline{\|v^\epsilon\|}_{D^\epsilon,T,1+a} + \|D^2 v^\epsilon\|_{V_T^\epsilon} \leq C.$$

□

Now, we are ready to prove the result for the a-priori bounds:

Proposition 3.3.7. *There is a constant C, independent of ϵ, and an open set $I \subset (0,1)$ such that for any $b > a \in I$ (a is the constant from Lemma 3.6.):*

$$\overline{\|u^\epsilon\|}_{D^\epsilon,T,1+b} + \|D^2 u^\epsilon\|_{V_T^\epsilon} \leq C. \tag{3.32}$$

where u^ϵ is a classical solution to (3.1).

Proof. We use Schauder's fixed point Theorem. Let us first define for convenience $\overline{\|\cdot\|}_{2+a} = \overline{\|\cdot\|}_{D^\epsilon,T,1+a} + \|D^2\cdot\|_{V_T^\epsilon}$.

Let $\mathcal{C}^{2+a}$ be the Banach space of all functions $u^\epsilon(t,z)$ that are continuous in $\overline{U}_T^\epsilon$ with norm $\overline{\|u^\epsilon\|}_{2+a}$.

For any $C > 0$, let $\mathcal{C}_C^{2+a}$ be the set $\{u^\epsilon : u^\epsilon \in \mathcal{C}^{2+a}, \overline{\|u^\epsilon\|}_{2+a} \leq C\}$.

For every $u^\epsilon \in \mathcal{C}_C^{2+a}$ define $w^\epsilon = Tu^\epsilon$ to be the solution to the following problem:

$$\begin{aligned}
w_t^\epsilon &= \frac{1}{2}\triangle w^\epsilon, && \text{in } (0,T) \times D^\epsilon \\
w^\epsilon(0,z) &= f(x), && \text{on } \{0\} \times D^\epsilon \\
\frac{\partial w^\epsilon}{\partial \gamma^\epsilon} &= -\epsilon c(z, u^\epsilon) w^\epsilon, && \text{on } (0,T) \times \partial D^\epsilon,
\end{aligned} \tag{3.33}$$

Then, similarly as in Lemma 3.3.6, one can write:

$$w^\epsilon(t,z) = \int_0^t \int_{\partial D^\epsilon} \Gamma^\epsilon(t,z,\tau,\xi)\phi^\epsilon(\tau,\xi) d\partial D_\xi^\epsilon d\tau + \int_{D^\epsilon} \Gamma^\epsilon(t,z,0,\xi) f(\xi) d\xi, \tag{3.34}$$

where $\phi^\epsilon(t,z)$ satisfies:

$$\begin{aligned}
\phi^\epsilon(t,z) &= 2\int_0^t \int_{\partial D^\epsilon} [\frac{\partial \Gamma^\epsilon(t,z,\tau,\xi)}{\partial \gamma^\epsilon} + \epsilon c(z,u^\epsilon)\Gamma^\epsilon(t,z,\tau,\xi)]\phi^\epsilon(\tau,\xi) d\partial D_\xi^\epsilon d\tau \\
&+ 2[\int_{D^\epsilon} \frac{\partial \Gamma^\epsilon(t,z,0,\xi)}{\partial \gamma^\epsilon} f(\xi) d\xi + \epsilon c(z,u^\epsilon) \int_{D^\epsilon} \Gamma^\epsilon(t,z,0,\xi) f(\xi) d\xi]
\end{aligned} \tag{3.35}$$

We shall prove that T has a fixed point.

Since u^ϵ and c are bounded functions, one can show, in the same way as in the proof of Lemma 3.3.6, that the function $\phi^\epsilon(t,z)$ that satisfies (3.35) is bounded and Hölder continuous (in space variables) with bound and Hölder constant independent of ϵ.

Using this result and representation (3.34) one can conclude (Lemma 3.3.6) that there is a constant C such that

$$\overline{\|w^\epsilon\|}_{2+a} \leq C.$$

So T maps $\mathcal{C}_C^{2+a}$ into itself for an appropriately chosen constant C.

Now let $\{u_n^\epsilon\}$ be a sequence of functions that belong to $\mathcal{C}_C^{2+a}$ and $w_n^\epsilon, \phi_n^\epsilon$ be defined by (3.34) and (3.35) when $u^\epsilon = u_n^\epsilon$. Assume that $\overline{\|u_n^\epsilon - u^\epsilon\|}_{2+a} \to 0$ as $n \to \infty$. We need to show that $\overline{\|w_n^\epsilon - w^\epsilon\|}_{2+a} \to 0$ as $n \to \infty$.

The continuity of the function $c(z, u)$ in u-variables imply that $\|\phi_n^\epsilon - \phi^\epsilon\|_{U_T^\epsilon} \to 0$ as $n \to \infty$. This and (3.34) give us $\overline{\|w_n^\epsilon - w^\epsilon\|}_{2+a} \to 0$.

Therefore, T is a continuous map.

Next, we need to show that T maps $\mathcal{C}_C^{2+a}$ into a compact subset of $\mathcal{C}_C^{2+a}$. This is an easy consequence of Theorem 7.1.1 of [17], which states that for $0 < a < b < 1$, the bounded subsets of $\mathcal{C}^{2+b}$ are pre-compact subsets of $\mathcal{C}^{2+a}$.

Lastly, $\mathcal{C}_C^{2+b}$ is a closed convex set of the Banach space $\mathcal{C}^{2+b}$.

Therefore, by Schauder's Fixed Point Theorem we get that T has a fixed point, i.e. there exists a u^ϵ such that $u^\epsilon = Tu^\epsilon$ and actually

$$u^\epsilon = Tu^\epsilon \in \mathcal{C}_C^{2+b}.$$

$\square$

3.4 Some results on wave front propagation

In this section we will see some applications of Theorem 3.3.4 to the question of wave front propagation in narrow domains. As we mentioned in the introduction, corresponding results on the standard reaction diffusion equation (3.2) (see chapter 6 and 7 in [9], [18] and [27]) allow to describe the asymptotic wavefront motion for (3.1).

We will focus on two different cases. In subsection 4.1 we consider the case where the functions $c(\cdot, 0, u)$, $V(\cdot)$, $S(\cdot)$ and $f(\cdot)$ change slowly in x, i.e. $c(\cdot, 0, u) = c(\delta x, 0, u)$, $V(\cdot) = V(\delta x)$, $S(\cdot) = S(\delta x)$ and $f(\cdot) = f(\delta x)$ for $0 < \delta \ll 1$. We first assume that the nonlinear boundary term in (3.1), $c(x, y, u)$, is of K-P-P type for $y = 0$, i.e. $c(x, 0, u)$ is positive for $u < 1$, negative for $u > 1$ and $c(x) = c(x, 0, 0) = \max_{0 \le u \le 1} c(x, 0, u)$. We will see how the motion of the wavefront depends on the behavior of the cross-sections D_x of the domain D. In particular, using the results of [9] (chapter 6) we will see that in the case of the nonlinear term of K-P-P type and for $x \in \mathbb{R}$ the wavefront can have jumps. Actually, the jumps of the wavefront appear at positions where the tube becomes thinner. The results are given in Theorem 3.4.1, Theorem 3.4.4 and Theorem 3.4.6. Then we briefly discuss the bistable case, i.e. when $c(x, 0, u) > 0$ for $u \in (\mu, 1)$ and $c(x, 0, u) < 0$ for $u \in (0, \mu) \cup (1, \infty)$, where $0 < \mu < 1$. In this case, we consider a specific example and we see how the asymptotic speed of the wavefront depends on the surface area to volume ratio $\frac{S(x)}{V(x)}$. In subsection 4.2, we return to the K-P-P case, but now we consider front propagation when $x \in \mathbb{R}$ and the boundary ∂D^1 of D^1 is determined by stationary random processes on $\mathbb{R}$ on some probability space $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P})$. The conclusion is in Theorem 3.4.13.

We will denote by $\overline{c}(x, u) := \frac{1}{2}\frac{S(x)}{V(x)}c(x, 0, u(t, x))$ the nonlinear term in (3.2). Obviously, the type of $\overline{c}(x, u)$ (K-P-P or bistable) is determined by $c(x, 0, u)$ and vice-versa.

3.4.1 Wave fronts in slowly changing media

Let us assume that the functions $c(\cdot, u)$, $V(\cdot)$, $S(\cdot)$ and $f(\cdot)$ change slowly in x, i.e. $c(\cdot, u) = c(\delta x, u)$, $V(\cdot) = V(\delta x)$, $S(\cdot) = S(\delta x)$ and $f(\cdot) = f(\delta x)$ for $0 < \delta \ll 1$.

We start with the case where the nonlinear term $\overline{c}(x, u)$ of (3.2) is of K-P-P type. We additionally assume that the closure of the support of f, F_o, coincides with the closure of its interior. Lastly, we take for brevity $x \in \mathbb{R}^1$ and $\overline{c}(x) = \overline{c}(x, 0) = \frac{1}{2}\frac{S(x)}{V(x)}c(x, 0, 0)$ (recall that $c(x, 0, 0) = \sup_{0 \le u \le 1} c(x, 0, u)$) to be an increasing function.

Let $\phi : [0,T] \to \mathbb{R}^1$ and introduce the functional

$$R_{0,T}(\phi) = \begin{cases} \int_0^T [\overline{c}(\phi_s) - \frac{1}{2}|\dot{\phi}_s|^2] ds, & \phi \text{ is absolutely continuous} \\ +\infty, & \text{for the rest of } \mathcal{C}_{0,T}. \end{cases} \tag{3.36}$$

Put

$$W(t,x) = \sup\{R_{0,t}(\phi) : \phi \in \mathcal{C}_{0,t}(\mathbb{R}^1), \phi_0 = x, \phi_t \in F_o\}. \tag{3.37}$$

We say that condition (N) is satisfied if for any $t > 0$ and $(t,x) \in \{(t,x) : W(t,x) = 0\}$:

$$W(t,x) = \sup\{R_{0,t}(\phi) : \phi_0 = x, \phi_t \in F_o, (t-s, \phi_s) \in \{(t,x) : W(t,x) < 0\}\}.$$

As it is mentioned in chapter 10 of [16], condition (N) is fulfilled for the smooth and increasing function $\overline{c}(x)$. Moreover as we shall see in Theorem 3.4.1, $W(t,x)$ determines the motion of the wave front for u^ϵ for small enough $\epsilon > 0$.

Let us consider $u(t,x)$, the solution to equation (3.2), for $n = 1$. If we set $u^\delta(t,x) = u(t/\delta, x/\delta)$, then u^δ is the solution to the following parabolic problem:

$$\begin{aligned} u_t^\delta &= \frac{\delta}{2} u_{xx}^\delta + \frac{\delta}{2} \frac{V_x(x)}{V(x)} u_x^\delta + \frac{1}{\delta} \overline{c}(x, u^\delta(t,x)) u^\delta, \text{ in } (0,\infty) \times \mathbb{R}^1 \\ u^\delta(0,x) &= f(x) \geq 0, \text{ on } \{0\} \times \mathbb{R}^1. \end{aligned} \tag{3.38}$$

Under the assumptions above, as we have mentioned in section 1.2.1, Theorem 1.2.2 holds. So, $W(t,x)$ determines the motion of the wave front for $u^\delta(t,x)$ under condition (N).

Let us consider now equation (3.1) for $n = 1$, $c(\cdot, 0, u) = c(\delta x, 0, u)$, $f(\cdot) = f(\delta x)$ in a slowly changing in x narrow domain $D^{\epsilon,\delta}$, so that $V(\cdot) = V(\delta x)$, $S(\cdot) = S(\delta x)$. Let us define $u^{\epsilon,\delta}(t, x, y) = u^{\epsilon}(t/\delta, x/\delta, y)$. Under the assumptions above, Theorems 3.3.4 and 1.2.2 imply that $W(t, x)$ will determine the motion of the wave front in this case too, as follows:

Theorem 3.4.1. *The following statement holds:*

$$\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} u^{\epsilon,\delta}(t, x, y) = \begin{cases} 1, & W(t, x) > 0 \\ \\ 0, & W(t, x) < 0. \end{cases} \tag{3.39}$$

So the equation $W(t, x) = 0$ defines the position of the interface (wavefront) between areas where $u^{\epsilon,\delta}$ (for $\epsilon > 0$ and $\delta > 0$ small enough) is close to 0 and to 1. Actually, as we shall see below the wavefront may have jumps. It is known (see chapter 6 in [9]), that because of the dependance of $\bar{c}(x)$ on x, the wave front of u^{δ} may have jumps and new sources may be "igniting" ahead of the front. We will give sufficient conditions that guarantee such jumps for a class of smooth and increasing functions $\bar{c}(x)$. Hence Theorem 3.4.1 implies that one can predict appearances of new sources and jumps of the wave front of $u^{\epsilon,\delta}$ for $\epsilon > 0$ and $\delta > 0$ small enough.

Let $t^* = t^*(x, \bar{c}(\cdot))$ be such that $W(t^*, x) = 0$. Such a $t^*(x, \bar{c}(\cdot))$ is defined in a unique way.

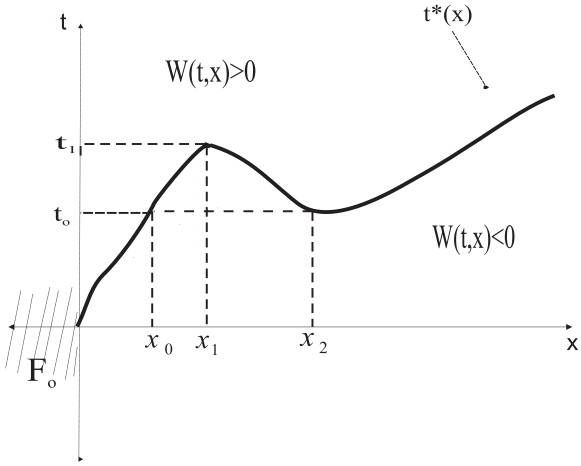

Figure 3.1: A wave front that jumps from x_0 to x_2 at time t_0.

We have the following proposition (see chapter 6 in [9] for more details):

Proposition 3.4.2. *Let* $t^*(x)$ *be as in Figure 3.1 and* $F_o = \{x \in \mathbb{R}^1, x < 0\}$*. Then the wavefront jumps from* x_o *to* x_2 *at time* t_o *(see Figure 3.1), i.e.:*

(i). If $t \leq t_0$ *then* $\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} u^{\epsilon,\delta}(t,x,y) = 1$ *for a connected set:*

$F_t = \{x \in \mathbb{R}^1 : W(t,x) > 0$ *and* $x < x_0\}$.

(ii). If $t_0 < t < t_1$ *then the set where* $\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} u^{\epsilon,\delta}(t,x,y) = 1$ *consists of two*

connected components:

$F_t = \{x \in \mathbb{R}^1 : W(t,x) > 0 \text{ and } x < x_1\} \cup \{x \in \mathbb{R}^1 : W(t,x) > 0 \text{ and } x > x_1\}$.

The set $\{x \in \mathbb{R}^1 : W(t,x) > 0 \text{ and } x < x_1\}$ *is at a positive distance from the set* $\{x \in \mathbb{R}^1 : W(t,x) > 0 \text{ and } x > x_1\}$ *for* $t_0 < t < t_1$.

(iii). If $t \geq t_1$ *then* $\lim_{\delta\downarrow 0} \lim_{\epsilon\downarrow 0} u^{\epsilon,\delta}(t,x,y) = 1$ *for a connected set:*

$F_t = \{x \in \mathbb{R}^1 : W(t,x) > 0\}$.

Based now on comparison results (Lemma 3.4.3) we give sufficient conditions that guarantee jumps of the wavefront. In particular, we prove (Theorem 3.4.4) that if $\bar{c}(x)$ is a rapidly increasing smooth function, then $t^* = t^*(x, \bar{c}(\cdot))$ such that $W(t^*, x) = 0$ is as in Figure 3.1.

The functional $R_{0,T}(\phi)$ defined in (3.36) and the function $W(t,x)$ defined in (3.37) depend also on $\bar{c}$. Hence, we write sometimes $R_{0,T}(\phi, \bar{c}(\cdot))$ and $W(t, x, \bar{c}(\cdot))$ in order to emphasize this dependence.

We have the following comparison result:

Lemma 3.4.3. *(i). Let A be a positive number. Then* $t^*(x, Ac(\cdot)) = \frac{1}{\sqrt{A}} t^*(x, c(\cdot))$.

(ii). Let a be a positive number and define $c_a(x) = c(ax)$. *Then* $t^*(x, c_a(\cdot)) = \frac{1}{a} t^*(ax, c(\cdot))$.

(iii). Let c_1, c_2 be two functions such that $c_1(x) < c_2(x)$ for every $x \in \mathbb{R}^1$. Then

$$t^*(x, c_1(\cdot)) > t^*(x, c_2(\cdot)).$$

Proof. Let us write $t_A^* = t^*(x, Ac(\cdot))$ and let ϕ^A be the extremal so that $W(t_A^*, x, Ac(\cdot)) = R_{0,t_A^*}(\phi^A, Ac(\cdot)) = 0$. Such an extremal satisfies the following Euler-Lagrance equation:

$$\begin{aligned} \ddot{\phi}^A(s) &= -Ac'(\phi^A(s)) \\ \phi^A(0) &= x \\ \phi^A(t_A^*) &= 0. \end{aligned} \tag{3.40}$$

Let us define now the function $\phi(s) = \phi^A(s/\sqrt{A})$. We claim that the function $\phi(s)$ is the extremal so that $W(\sqrt{A}t_A^*, x, c(\cdot)) = R_{0,\sqrt{A}t_A^*}(\phi, c(\cdot)) = 0$. Indeed, it is easy to see that the definition of ϕ and the fact that $R_{0,t_A^*}(\phi^A, Ac(\cdot)) = 0$ imply that $R_{0,\sqrt{A}t_A^*}(\phi, c(\cdot)) = 0$. Moreover, ϕ satisfy an Euler-Lagrange equation of the form (3.40) with $Ac(x)$ and t_A^* replaced by $c(x)$ and $\sqrt{A}t_A^*$ respectively. This proves the claim, which implies part (i) of the lemma.

Part (ii) of the lemma can be proven in a similar way. We define $t_a^* = t^*(x, c_a(\cdot))$ and let ϕ^a to be the extremal so that $W(t_a^*, x, c_a(\cdot)) = R_{0,t_a^*}(\phi^a, c_a(\cdot)) = 0$. Then similarly as it is done in part (i), one should consider the function $\phi(s)$ that is defined by $\phi(s) = a\phi^a(s/a)$.

We prove now part (iii) of the lemma. Let us define $t_1^* = t^*(x, c_1(\cdot))$ and $t_2^* = t^*(x, c_2(\cdot))$. Moreover let ϕ^1 be the extremal so that $W(t_1^*, x, c_1(\cdot)) = R_{0,t_1^*}(\phi^1, c_1(\cdot)) = 0$. Since $c_1(x) < c_2(x)$ we have

$$0 = R_{0,t_1^*}(\phi^1, c_1(\cdot)) < R_{0,t_1^*}(\phi^1, c_2(\cdot)). \tag{3.41}$$

Furthermore, it is easy to see that $W(t, x)$ is an increasing function of t.

Let us assume now that $t_1^* \leq t_2^*$. This assumption and the fact that $W(t_2^*, x, c_2(\cdot)) = 0$ imply that $W(t_1^*, x, c_2(\cdot)) \leq 0$. By recalling the definition of function W, one easily concludes that:

$$R_{0,t_1^*}(\phi^1, c_2(\cdot)) \leq 0. \tag{3.42}$$

However inequality (3.42) contradicts (3.41). Therefore $t^*(x, c_1(\cdot)) > t^*(x, c_2(\cdot))$. $\square$

In section 6.2 of [9], it is proven that if $\bar{c}(x)$, instead of the smooth function $\frac{1}{2}\frac{S(x)}{V(x)}c(x, 0, 0)$, is a piecewise constant function, denoted by $d(x)$, such that

$$d(x) = \begin{cases} d_1, & x < x_2 \\ \\ d_2, & x \geq x_2. \end{cases} \tag{3.43}$$

with $d_2 > 2d_1 > 0$, then the function $t^* = t^*(x, d(\cdot))$ such that $W(t^*, x, d(\cdot)) = 0$ is not monotone, as in Figure 3.1. More specifically the curves connecting the point $(0, 0)$ with (x_1, t_1) and (x_1, t_1) with (x_2, t_0) are line segments and for $x > x_2$, $t^* = t^*(x, d(\cdot))$ is the solution to

$$\sup_t \{d_2(t^* - t) + d_1 t - \frac{(x - x_2)^2}{2(t^* - t)} - \frac{x_2^2}{2t}\} = 0.$$

Moreover in this case

$$t_0 = x_2 \frac{\sqrt{2(d_2 - d_1)}}{d_2} \tag{3.44}$$

$$t_1 = \frac{1}{2\sqrt{2d_1}}(x_2 + \sqrt{2d_1}t_0) \tag{3.45}$$

We will write $t_0 = t_0(d)$ and $t_1 = t_1(d)$ to emphasize the dependence of t_0 and t_1 on the function $d(x)$.

With the help of the result above and Lemma 3.4.3 we give sufficient conditions that guarantee jumps of the wavefront of $u^\delta(t,x)$ (and by Theorem 3.4.1 of $u^\epsilon(t,x,y)$ for $\epsilon > 0$ and $\delta > 0$ small enough) for a class of smooth and increasing functions $\bar{c}$.

Let us define the set

$$\Delta = \{(d_1, d_2) \in \mathbb{R}^1_+ \times \mathbb{R}^1_+ : d_2 > 2d_1\}. \tag{3.46}$$

Theorem 3.4.4. *Let $d(x)$ be the step function defined in (3.43) such that $(d_1, d_2) \in \Delta$. Consider real numbers A and a such that*

(i). $a, A > 1$.

(ii). $a\sqrt{A} < \frac{1}{2}[1 + \frac{d_2}{2\sqrt{d_1(d_2-d_1)}}]$.

Then for any smoothly increasing function $\bar{c}(x)$ such that

$$d(x) < \bar{c}(x) < Ad(ax) \tag{3.47}$$

the wavefront corresponding to $\bar{c}$ *has jumps. In particular, the excitation reaches the region* $\{x > \frac{x_1}{a} + \delta\}$ *before it reaches the point* $\frac{x_1}{a}$*, where* δ *is a small enough positive number and* $\frac{x_1}{a}$ *is as in Figures 3.2 and 3.3.*

Proof. Let us define $\bar{d}(x) = Ad(ax)$. Since $a, A > 1$, the function d(x) is shifted vertically upwards and horizontally to the left. So we get that $d(x) < \bar{d}(x)$ (see Figure 3.2).

Parts (i) and (ii) of Lemma 3.4.3 imply that $t^*(x, \bar{d}(\cdot)) = \frac{1}{a\sqrt{A}} t^*(ax, d(\cdot))$. This and part (iii) of Lemma 3.4.3 give that if $\bar{c}$ satisfies (3.47), then $t^*(x, \bar{c}(\cdot))$ will satisfy (see Figure 3.3):

$$\frac{1}{a\sqrt{A}} t^*(ax, d(\cdot)) < t^*(x, \bar{c}(\cdot)) < t^*(x, d(\cdot)). \tag{3.48}$$

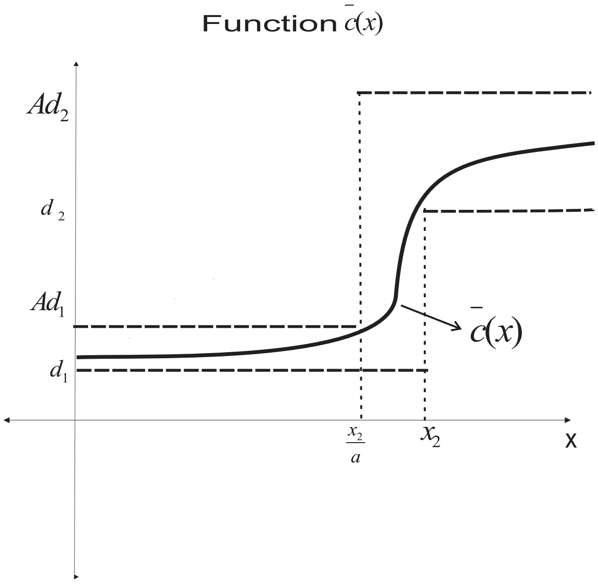

Figure 3.2: $d(x) < \bar{c}(x) < Ad(ax)$

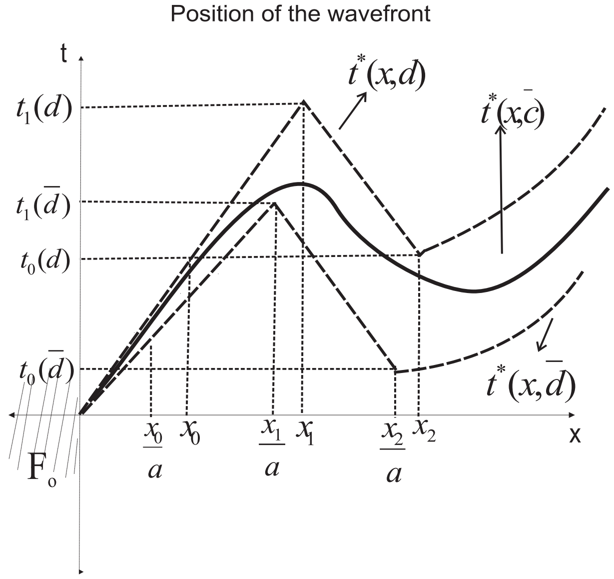

Figure 3.3: $t^*(x, \bar{d}) < t^*(x, \bar{c}) < t^*(x, d)$

We know that $t^*(x, d(\cdot))$ and $t^*(x, \bar{d}(\cdot))$ are not monotone (recall that d and $\bar{d}$ are piecewise constant functions). We will show that $t^*(x, \bar{c}(\cdot))$ is also not monotone (i.e it is as in Figure 3.1). Let us assume that

$$t_1(\bar{d}) > t_0(d), \tag{3.49}$$

where $t_0(d)$ is as in (3.44) and $t_1(\bar{d})$ is defined similarly to $t_1(d)$ in (3.45) with d_1, d_2, x_2 replaced by $Ad_1, Ad_2, \frac{x_2}{a}$ respectively. In particular (3.49) holds if condition

(ii) above holds, i.e. if $a\sqrt{A} < \frac{1}{2}[1+\frac{d_2}{2\sqrt{d_1(d_2-d_1)}}]$. Moreover, it is easy to see that $d_2 > 2\sqrt{d_1(d_2-d_1)}$ is true for any $d_2 > d_1 > 0$. This implies that $\frac{1}{2}[1+\frac{d_2}{2\sqrt{d_1(d_2-d_1)}}] > 1$, which has to be true since $a, A > 1$.

Inequality (3.49) can be equivalently written as $t^*(\frac{x_1}{a}, \bar{d}(\cdot)) > t(x_2, d(\cdot))$. By this and (3.48) we immediately get that

$$t^*(x_2, \bar{c}(\cdot)) < t^*(\frac{x_1}{a}, \bar{c}(\cdot)) \tag{3.50}$$

which, since $\frac{x_1}{a} < x_1 < x_2$, implies that $t^*(x, \bar{c}(\cdot))$ is as in Figure 3.1 and so new sources are igniting ahead of the wavefront.

In Figures 3.2 and 3.3 we see an illustration of the construction. □

Example. An example of a function $\bar{c}(x)$ that satisfies the requirements of Theorem 3.4.4 is

$$\bar{c}(x) = \frac{Ad_2\mu + d_1 e^{-\lambda(x-k)}}{\mu + e^{-\lambda(x-k)}}, \tag{3.51}$$

where $(d_1, d_2) \in \Delta$, a, A satisfy assumptions (i) and (ii) of Theorem 3.4.4, $k \in (\frac{x_2}{a}, x_2)$ and the constants μ and λ are chosen so that $\bar{c}(\frac{x_2}{a}) < Ad_1$ and $\bar{c}(x_2) > d_2$.

In particular now if $\bar{c}(x) = \frac{1}{2}\frac{S(x)}{V(x)}$, i.e. $c(x,0,0) = 1$, is an increasing smooth function that satisfies the requirements of Theorem 3.4.4, then the jump of the wavefront of $u^{\epsilon,\delta}(t,x,y)$, for $\epsilon > 0$ and $\delta > 0$ small enough, occurs when $\frac{S(x)}{V(x)}$ increases rapidly. This implies, at least when the tube D^1 retains its shape as

x increases, that the jumps of the wave front occur at places where the tube D^1 becomes thinner, i.e. when $V(x)$ decreases significantly.

Remark 3.4.5. *Similar results hold for layers as well, i.e. for $x \in \mathbb{R}^n$ with $n > 1$.*

Using the results in [11] one can consider the limiting behavior as $\delta, \epsilon \downarrow 0$ of $u^{\epsilon,\delta}(t,x,y)$ when condition (N) is not fulfilled. We briefly discuss the result for the general case $x \in \mathbb{R}^n$.

Instead now of function $W(t,x)$ defined by (3.37), we consider the function

$$
\begin{aligned}
W^*(t,x) = \sup\{\min_{0\le s\le t} R_{0,s}(\phi) \quad : \quad & \phi \in \mathcal{C}_{0,t}(\mathbb{R}^n) \text{ is absolutely continuous,} \\
& \phi_0 = x, \;\; \phi_t \in F_o\}. \qquad (3.52)
\end{aligned}
$$

As it is mentioned in section 1.2.1, $W^*(t,x)$ is Lipschitz continuous and $W^*(t,x) \le \min\{0, W(t,x)\}$.

Then, Theorem 1.2.3 and Theorem 3.3.4 imply that $W^*(t,x)$ determines the motion of the wave front as follows:

Theorem 3.4.6. *The following statements hold:*

(i). For any compact subset Θ_1 of the interior of $\{(t,x) : t > 0, W^*(t,x) = 0\}$,

$$\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} u^{\epsilon,\delta}(t,x,y) = 1 \text{ uniformly in } (t,x) \in \Theta_1.$$

(ii). For any compact subset Θ_2 *of* $\{(t,x) : W^*(t,x) < 0\}$,

$$\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} u^{\epsilon,\delta}(t,x,y) = 0 \text{ \textit{uniformly in} } (t,x)\in\Theta_2.$$

We conclude this subsection with the case that the nonlinear term $\overline{c}(x,u)$ of (3.2) is of bistable type, i.e. $\overline{c}(x,u) > 0$ for $u \in (\mu,1)$, $\overline{c}(x,u) < 0$ for $u \in (0,\mu)\cup(1,\infty)$, where $0<\mu<1$. This problem was considered in [18] and it was also presented in section 6.4 of [9].

Here, we restrict the analysis to a concrete example that allows to give an exact formula for the asymptotic speed of the wavefront of $u^{\epsilon,\delta}$ for $\epsilon>0$ and $\delta>0$ small enough. As we will see, the asymptotic speed of the wavefront is proportional to the square root of the surface area to volume ratio $\sqrt{\frac{S(x)}{V(x)}}$.

To be specific, let $x\in\mathbb{R}^n$, $c(x,0,u) = (u-\mu)(1-u)$, $0<\mu<\frac{1}{2}$ and assume that the function $u^\delta(t,x)$ (compare with (3.38)) is the solution to

$$\begin{aligned} u^\delta_t &= \frac{\delta}{2V(x)}\operatorname{div}(V(x)\nabla_x u^\delta) + \frac{1}{\delta}\frac{1}{2}\frac{S(x)}{V(x)}(u^\delta-\mu)(1-u^\delta)u^\delta, \quad \text{in } (0,\infty)\times\mathbb{R}^n \\ u^\delta(0,x) &= f(x), \quad \text{on } \{0\}\times\mathbb{R}^n. \end{aligned} \tag{3.53}$$

Consider a point $x\in\mathbb{R}^n$ to be excited at time t, if $u^\delta(t,x)$ (the solution to (3.53)) is close to 1 and non-excited if $u^\delta(t,x)$ is close to 0. Then, the Corollary of Theorem 4.1 of [18] gives us that for small $\delta>0$ the region $\{x\in\mathbb{R}^n : f(x)>\mu\}$ becomes excited and the region $\{x \in \mathbb{R}^n : f(x) < \mu\}$ becomes non-excited after a short starting phase. Now, let $u^{\epsilon,\delta}(t,x,y) = u^{\epsilon}(t/\delta, x/\delta, y)$, where $u^{\epsilon}(t,x,y)$ is the solution to (3.1). Theorem 3.3.4 implies that the same conclusions hold for $u^{\epsilon,\delta}(t,x,y)$ for $\epsilon > 0$ and $\delta > 0$ small enough.

To compute the asymptotic propagation speed of excitation at $x \in \mathbb{R}^n$, let us consider the equation for the wave profile:

$$\frac{1}{2}v''_{\xi\xi}(\xi) + a(x)v'_{\xi}(\xi) + \frac{1}{2}\frac{S(x)}{V(x)}(v(\xi) - \mu)(1 - v(\xi))v(\xi) = 0, \;\; \xi \in \mathbb{R} \quad (3.54)$$

$$\lim_{\xi \to -\infty} v(\xi) = 1, \;\; \lim_{\xi \to \infty} v(\xi) = 0.$$

As it can be verified by direct substitution, equation (3.54) is solvable if $a(x)$ is given by the formula

$$a(x) = \sqrt{\frac{1}{2}\frac{S(x)}{V(x)}}(\frac{1}{2} - \mu). \quad (3.55)$$

Moreover, in our case, (3.55) is also the asymptotic propagation speed of excitation at $x \in \mathbb{R}^n$ and it is independent of direction.

Lastly, it is known that as the size of D_x increases (without changing shape), the surface area to volume ratio $\frac{S(x)}{V(x)}$ decreases. In the case $x \in \mathbb{R}$, this fact, equation (3.55) and Theorem 3.3.4 imply that the wavefront of $u^{\epsilon,\delta}$ (for $\epsilon > 0$ and $\delta > 0$ small enough) slows down when the tube becomes thicker. A similar result holds for layers.

3.4.2 K-P-P fronts in random media

In this subsection we consider wave front propagation for the solution of (3.1) for small $\epsilon > 0$, when $x \in \mathbb{R}$, the boundary ∂D^1 of D^1 is determined by stationary and ergodic random processes on $\mathbb{R}$ and the nonlinear boundary term in (3.1) (for $y = 0$, i.e. $c(x, 0, u)$) is of K-P-P type. As we did in subsection 3.4.1, we first consider (Theorem 3.4.12) wavefront propagation for the solution of (3.2) and then with the aid of Theorem 3.3.4 we consider (Theorem 3.4.13) wavefront propagation for the solution of (3.1) for small enough $\epsilon > 0$. As we will see, the cross sections D_x of D affect the speed of the wavefront through the surface to volume ratio $\frac{S(x)}{V(x)}$.

In sections $7.4 - 7.6$ of [9] wave front propagation for equations like (3.2) is considered in the case where there is no drift term and the randomness comes only from the nonlinear part of the equation. Moreover, in [27] the authors considered the case of reaction-diffusion equations of type (3.2) with a random drift and homogeneous in x nonlinear term. In the case considered here, both the drift and the nonlinear term are random. In [9], pp. 524-525, the author remarks that one could use the procedure developed in sections $7.4 - 7.6$ of [9] to study wavefronts in one-dimensional uniformly bounded random drift with random nonlinear term. We will see that one can prove Theorem 3.4.12, which is analogous to Theorem 7.6.1 in [9], by following the proof of Theorem 7.6.1 in [9]. We make use of the results in [27] and of the fact that the operator of the equation (3.2) is self adjoint with respect to an appropriate inner product (it has the form $\frac{1}{2V(x)}\frac{d}{dx}(V(x)\frac{d}{dx})$). Actually the latter simplifies the analysis significantly. Instead of repeating the proof of [9] here, we only outline the differences.

Let us first list our assumptions. Consider a probability space $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P})$. We assume that the random field $V(x, \hat{\omega})$ (namely the volume) is three times continuously differentiable, i.e. $V \in \mathcal{C}^3(\mathbb{R})$, with $\hat{P}$ probability one. Suppose that $\Theta(x) = (\frac{d}{dx}(\log V(x)), \frac{S(x)}{V(x)})$ is a random vector function on $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P})$ and that it is measurable, stationary in x and translation in x generates an ergodic transformation of the space $\hat{\Omega}$. Moreover, the function $\frac{d}{dx}(\log V(x))$ is assumed bounded, with zero mean (i.e. $\hat{E}[\frac{d}{dx}(\log V(x))] = 0$). We additionally assume (for the purposes of Lemma 3.4.7 and 3.4.9) that there is a set of nonzero $\hat{P}$ probability on which

$$\lim_{z \to \infty} \int_0^z [V(x, \hat{\omega})]^{-1} dx = +\infty. \tag{3.56}$$

If condition (3.56) holds on a set of nonzero measure then, by the ergodicity assumption, it must hold with $\hat{P}$ probability one.

As far as the non-linear term $\overline{c}(x, u, \hat{\omega})u = \frac{1}{2}\frac{S(x,\hat{\omega})}{V(x,\hat{\omega})}c(x, 0, u)u$ is concerned, in addition to the stationarity and ergodicity assumptions, we also make the following assumptions. For all $x \in \mathbb{R}$, c is of K.P.P type, i.e. $c(x, 0, u)$ is positive for $u < 1$, negative for $u > 1$, continuous in u for $u \geq 0$ and $c(x) = c(x, 0, 0) = \sup_{0<u} c(x, 0, u)$.

Moreover with $\hat{P}$ probability one, the function $\overline{c}(x,u,\hat{\omega})u$ satisfies a Lipschitz condition of the form

$$|\overline{c}(x,u_1,\hat{\omega})u_1 - \overline{c}(x,u_2,\hat{\omega})u_2| \le \frac{1}{2}\frac{S(x,\hat{\omega})}{V(x,\hat{\omega})}\zeta(x)|u_1-u_2|, \text{ for } x,u_1,u_2 \in \mathbb{R},$$

such that for all $t \ge 0$ and $x \in \mathbb{R}$,

$$E_x \exp\{\int_0^t \frac{1}{2}\frac{S(X_s)}{V(X_s)}\zeta(X_s)\} < \infty, \ \ \hat{P}\text{-a.s.},$$

where (X_t, P_x) is a diffusion process with random generator $L = \frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx}(\log V(x,\hat{\omega}))\frac{d}{dx}$.

The initial function $f(x)$ is assumed to be nonnegative, bounded and non-random. Moreover we assume that there exist an $\eta > 0$ such that f vanishes on the interval $[\eta, +\infty)$, $\int_{\mathbb{R}} f(x)dx > 0$ and $\int_{\mathbb{R}} \max(f(x)-1,0)dx < \infty$.

Let now $\mu(z)$ be the function defined by the equality

$$\mu(z) = \hat{E}[\ln E_1 \chi_{\tau_0<\infty} \exp\{\int_0^{\tau_0} [\overline{c}(X_s)+z]ds\}], \ \ z \in \mathbb{R}, \tag{3.57}$$

where $\overline{c}(x) = \frac{1}{2}\frac{S(x)}{V(x)}c(x,0,0)$ and τ_0 is the first hitting time of the process X_t to the point 0.

Since $\overline{c}(x) \ge 0$ we have that $\mu(z) = \infty$ for $z > 0$. Therefore, a non-positive number $\bar{g}_\mu$ exists such that $\mu(z) < \infty$ for $z < \bar{g}_\mu$ and $\mu(z) = \infty$ for $z > \bar{g}_\mu$.

For τ_0 one has the following lemma:

Lemma 3.4.7. *Condition (3.56) implies that* $P_1(\tau_0 < \infty) = 1, \quad \hat{P} - a.s..$

Proof. Let us define τ_y^x to be the first time the process X_t^x hits the point $x \geq y$, i.e.

$$\tau_y^x = \inf\{t > 0 : X_t^x \leq y\}.$$

Consider the function $w(x; L) = P(\tau_0^x < \tau_L^x)$ for $x \in [0, L]$. This function solves the equation

$$\begin{aligned} w_{xx}(x; L) + \frac{d}{dx}(\log V(x))w_x(x; L) &= 0 \\ w(0; L) = 1, w(L; L) &= 0 \end{aligned}$$

The solution to this equation reads as follows

$$w(x; L) = 1 - \frac{\int_0^x \frac{1}{V(z)}dz}{\int_0^L \frac{1}{V(z)}dz}.$$

The latter and condition (3.56) implies that for any fixed x

$$\lim_{L\to\infty} w(x; L) = 1.$$

This proves that $P_1(\tau_0 < \infty) = 1, \quad \hat{P} - a.s..$ ☐

Lemma 3.4.8. *Under the assumptions imposed above, function* $\mu(z)$ *has the following properties:*

(i). For all $z \in \mathbb{R}$, $\mu(z) = \lim_{t\to\infty} \frac{1}{t} \ln E_t \chi_{\tau_0<\infty} \exp\{\int_0^{\tau_0} [\bar{c}(X_s) + z]ds\}$.

(ii). Function $\mu(z)$ is convex, lower semicontinuous and monotonically non-decreasing in z. Moreover $\mu(z)$ is continuously differentiable and the derivative $\mu'(z)$ is positive and monotonically increasing for $z < \bar{g}_\mu$, where $\bar{g}_\mu$ is the non-positive number mentioned before Lemma 3.4.7. This number is the discontinuity point of $\mu(z)$.

(iii). $\mu(z) \leq 0$ for $z \leq \bar{g}_\mu$ and $\mu(z) = \infty$ for $z > \bar{g}_\mu$ where $\bar{g}_\mu \leq 0$.

Proof. Property (i) can be proven as Proposition 1 of [27]. Property (ii) follows similarly as Theorem 7.5.1(ii) of [9].

We prove property (iii) using the methodology of Theorem 7.5.1(iii) of [9]. Here one uses the fact that the operator of (3.2) has the form $\frac{1}{2V(x)}\frac{d}{dx}(V(x)\frac{d}{dx})$, i.e. it is self adjoint.

Under our assumptions the function $\bar{c}(x)$ is bounded. For any $z < \bar{g}_\mu$ the function

$$q(x; z) = E_x \exp \int_0^{\tau_0} [\bar{c}(X_s) + z] ds, x \geq 0,$$

is finite. Furthermore, it satisfies the equation

$$\frac{1}{2}\frac{d^2 q(x; z)}{dx^2} + \frac{1}{2}\frac{d \log V(x)}{dx}\frac{dq(x; z)}{dx} + [\bar{c}(x) + z] q(x; z) = 0. \tag{3.58}$$

Function $q(x; z)$ is continuously differentiable in x, and the first derivative is absolutely continuous. Equation (3.58) holds almost everywhere with respect to the

Lebesque measure.

Let us now define $\phi(x;z) = \log q(x;z)$. This function solves the equation

$$\frac{1}{2}\frac{d^2\phi(x;z)}{dx^2} + \frac{1}{2}(\frac{d\phi(x;z)}{dx})^2 + \frac{1}{2}\frac{d\log V(x)}{dx}\frac{d\phi(x;z)}{dx} + [\bar{c}(x) + z] = 0. \qquad (3.59)$$

One may differentiate equation (3.59) with respect to z and obtain that the function $\psi(x;z) = \frac{d\phi(x;z)}{dz}$ satisfies the equation

$$\frac{1}{2}\frac{d^2\psi(x;z)}{dx^2} + [\frac{d\phi(x;z)}{dx} + \frac{1}{2}\frac{d\log V(x)}{dx}]\frac{d\psi(x;z)}{dx} + 1 = 0. \qquad (3.60)$$

Recalling that $\phi(x;z) = \log q(x;z)$ we can rewrite (3.60) as

$$\frac{1}{2}\frac{d^2\psi(x;z)}{dx^2} + \frac{1}{2}\frac{d\log[u^2(x;z)V(x)]}{dx}\frac{d\psi(x;z)}{dx} + 1 = 0. \qquad (3.61)$$

The general solution of the latter is

$$\psi(x;z) = \psi(z;0) + \int_0^x [\frac{d\psi(z;0)}{dx} - 2\int_0^y [u^2(w;z)V(w)]dw]\frac{dy}{[u^2(y;z)V(y)]} \qquad (3.62)$$

Let us now assume that $\mu(z) > 0$ for $z < \bar{g}_\mu$. Then, by (i), $q(x;z)$ converges exponentially to $+\infty$ as $x \to \infty$. Then, equation 3.62 implies that the function $\psi(x;z)$ is bounded by above as $x \to \infty$. However, this cannot hold. Indeed, by (i) and Lemma 3.4.7

$$\lim_{x\to\infty} \frac{\phi(z;x)}{x} = \mu(z).$$

Since the functions $\mu(z)$ and $\phi(x;z)$ are convex and differentiable in z, for $z < \bar{g}_\mu$, the last relation can be differentiated with respect to z. We get

$$\lim_{x\to\infty} \frac{\psi(x;z)}{x} = \mu'(z).$$

Since, by (ii), $\mu'(z) > 0$, we have that $\lim_{x\to\infty} \psi(x; z) = \infty$, which contradicts the above. Hence, we conclude that $\mu(z) \leq 0$ for $z \leq \bar{g}_\mu$. $\square$

We also observe that $\mu(z) \geq \mu_o(z)$ where $\mu_o(z) = \hat{E}[\ln E_1(\chi_{\tau_0<\infty} e^{z\tau_0})]$. As it has been proven in Lemma 2.2 of [27], function $\mu_o(z)$ has properties (i)-(iii) of Lemma 3.4.8 as well (for $\bar{c}(x) = 0$). In addition, the following lemma holds, which is a restatement of Proposition 2 of [27].

Lemma 3.4.9. *Condition (3.56) and $\hat{E}[\frac{d}{dx}(\log V(x, \hat{\omega}))] = 0$ imply that the discontinuity point of $\mu_o(z)$ is $\bar{g}_{\mu_o} = 0$.*

We will assume that $-\infty < \bar{g}_\mu < 0$ (by Lemma 3.4.8(iii) or Lemma 3.4.9 we already know that $\bar{g}_\mu \leq 0$) and we define $I(y) = \sup_{z\leq\bar{g}_\mu}[yz - \mu(z)]$ for $y \in \mathbb{R}$.

Lemma 3.4.8 and the fact that $\mu(z) \geq \mu_o(z)$ imply that the arguments in the beginning of section 7.6 of [9] carry out here as well. In particular, we have that

1. $I(y) = +\infty$ for $y \leq 0$.
2. $I(y) \to +\infty$ as $y \downarrow 0$.
3. $I(y)$ is finite and strictly decreasing for $y > 0$.
4. $I(y) \to -\infty$ as $y \to +\infty$.

Therefore, we conclude that there is a unique $\nu^* > 0$ such that $I(\frac{1}{\nu^*}) = 0$ and $\nu^* = \inf_{z \leq \bar{g}_\mu} \frac{z}{\mu(z)}$.

Remark 3.4.10. *We would like to emphasize that the existence and uniqueness of a positive ν^* follows mainly from properties (i)-(iii) of $\mu(z)$ (Lemma 3.4.8). In particular property (iii) was proven using the fact that the operator of (3.2) is self adjoint.*

Similarly, as Theorem 7.6.1 in [9] was proven, one can prove Theorem 3.4.12 below.

Note that by following the proof of Theorem 7.6.1 in [9], one needs to estimate certain probabilities for τ_0 and X_t. For this purpose we have the following lemma:

Lemma 3.4.11. *Let δ be a positive number and $U_\delta(0) = \{x : |x| \leqslant \delta\}$. Then*

(i). $\inf_{x \in U_\delta(0)} P_x\{\tau_0 \leqslant 1\} > 0$, *$\hat{P}$-a.s.*

(ii). $\inf_{x \in U_\delta(0), s \in (0,1]} P_x\{X_s \in U_\delta(0)\} > 0$, *$\hat{P}$-a.s.*

(iii). For $a > 0$ and $\eta > \delta > 0$ we have

$$\inf_{x \in U_\delta(-a)} P_x\{\tau_{-\eta-a} > 1, X_1 \in U_\delta(0)\} > 0,\ \hat{P}\text{-a.s.}$$

(iv). For any $\nu \in \mathbb{R}$ and $\eta > 0$ we have

$$\lim_{t \to \infty} \sup_{|x| \leq |\nu| t} P_x\{\sup_{s \in [0,t]} |X_s - x| \geq \eta t\} = 0,\ \hat{P}\text{-a.s.}$$

Proof. The proof of all statements follows from the corresponding statements for W_t^1 in place of X_t (see for example section 7.5 of [9]) and by the Girsanov's theorem on the absolute continuous change of measures in the space of trajectories. Property (iv) is Lemma 4.2 of [27]. $\square$

Therefore, we have the following Theorem:

Theorem 3.4.12. *Let $x \in \mathbb{R}$ and $u(t,x)$ satisfy equation (3.2). Under our assumptions we have:*

(i). For all $\nu > \nu^*$,

$$\lim_{t\to\infty} \sup_{x \geq \nu t} u(t,x) = 0, \;\; \hat{P} - a.s.$$

(ii). Let us define $\bar{c}_h(x) = \frac{1}{2}\frac{S(x)}{V(x)} \inf_{0<u<h} c(x,0,u)$ and assume that there is a constant $\kappa > 0$ such that for any $0 < h < 1$ and $x \in \mathbb{R}$,

$$\kappa < \bar{c}_h(x), \;\; \hat{P} - a.s.$$

Then for all $\nu \in (0, \nu^*)$,

$$\lim_{t\to\infty} \inf_{0 \leq x \leq \nu t} u(t,x) = 1, \;\; \hat{P} - a.s.$$

Finally, Theorem 3.3.4 and Theorem 3.4.12 imply:

Theorem 3.4.13. *Let $(x, y) \in \mathbb{R} \times \mathbb{R}^m$ and $u^\epsilon(t, x, y)$ satisfy equation (3.1). Under our assumptions we have:*

(i). For all $\nu > \nu^*$,

$$\lim_{t \to \infty} \sup_{x \geq \nu t} \lim_{\epsilon \to 0} u^\epsilon(t, x, y) = 0, \quad \hat{P} - a.s.$$

(ii). Let us define $\bar{c}_h(x) = \frac{1}{2} \frac{S(x)}{V(x)} \inf_{0<u<h} c(x, 0, u)$ and assume that there is a constant $\kappa > 0$ such that for any $0 < h < 1$ and $x \in \mathbb{R}$,

$$\kappa < \bar{c}_h(x), \quad \hat{P} - a.s.$$

Then for all $\nu \in (0, \nu^*)$,

$$\lim_{t \to \infty} \inf_{0 \leq x \leq \nu t} \lim_{\epsilon \to 0} u^\epsilon(t, x, y) = 1, \quad \hat{P} - a.s.$$

Remark 3.4.14. *Theorem 3.4.12 was proven in ([9]) with the assumption in part (ii) replaced by the assumption that for any $0 < h < 1$ and $\nu \in \mathbb{R}$,*

$$\limsup_{t \to \infty} \frac{1}{t} \ln E_{\nu t} \exp\{-\int_0^t \bar{c}_h(X_s) ds\} < 0, \quad \hat{P} - a.s, \tag{3.63}$$

which is however difficult to verify. Obviously, the assumption made in part (ii) of Theorems 3.4.12 and 3.4.13 implies (3.63).

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