The Smoluchowski-Kramers approximation for the Langevin equation with reflection

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According to the Smoluchowski-Kramers approximation the solution of the S.D.E.:

$$\begin{aligned} \mu \ddot{q}_t^\mu &= b(q_t^\mu) - \dot{q}_t^\mu + \sigma(q_t^\mu)\dot{W}_t \qquad (1) \\ q_0^\mu &= q \in \mathbb{R}^1 \\ \dot{q}_0^\mu &= p \in \mathbb{R}^1 \end{aligned}$$

converges in probability as $\mu \to 0$ to the solution of the following S.D.E.:

$$\begin{aligned} \dot{q}_t &= b(q_t) + \sigma(q_t)\dot{W}_t \qquad (2) \\ q_0 &= q \in \mathbb{R}^1 \end{aligned}$$

More precisely, one can prove that for any $\delta, T > 0$ and $q, p \in \mathbb{R}^1$,

$$\lim_{\mu \downarrow 0} P(\max_{0 \le t \le T} |q_t^\mu - q_t| > \delta) = 0, \qquad (3)$$

Langevin process with elastic reflection on the boundary

Define the process $(q_t^\mu; p_t^\mu)$ as the solution of the following S.D.E.:

$$\begin{aligned} \dot{q}_t^\mu &= p_t^\mu \\ \mu \dot{p}_t^\mu &= -p_t^\mu + b(q_t^\mu) + \sigma(q_t^\mu)\dot{W}_t \\ q_0^\mu &= q, p_0^\mu = p, \end{aligned} \tag{4}$$

for $t \in [0, \tau_1^\mu)$, where $\tau_1^\mu = inf\{t > 0 : q_t^\mu = 0\}$.

Then define $(q_t^\mu; p_t^\mu)$ for $t \in [\tau_1^\mu, \tau_2^\mu)$ as the solution of (4) with initial conditions $(q_{\tau_1^\mu}^\mu; p_{\tau_1^\mu}^\mu) = (0; -\lim_{t \uparrow \tau_1^\mu} p_t^\mu)$. Here $\tau_2^\mu = inf\{t > \tau_1^\mu : q_t^\mu = 0\}$.

If $0 < \tau_1^\mu < \tau_2^\mu < ... < \tau_k^\mu$ and $(q_t^\mu; p_t^\mu)$ for $t \in [0, \tau_k^\mu)$ are already defined, then define $(q_t^\mu; p_t^\mu)$ for $t \in [\tau_k^\mu, \tau_{k+1})$ as solution of (4) with initial conditions $(q_{\tau_k^\mu}^\mu; p_{\tau_k^\mu}^\mu) = (0; -\lim_{t \uparrow \tau_k^\mu} p_t^\mu)$.

This construction defines the process $(q_t^\mu; p_t^\mu)$ in $\mathbb{R}_+^2$ for all $t \geq 0$. This follows from:

Proposition 1. The strictly increasing sequence of Markov times $\{\tau_k^\mu\}$ converges to $+\infty$ as $k \to +\infty$, i.e.

$$P(\lim_{k \to +\infty} \tau_k^\mu = +\infty) = 1.$$

We will refer to the Langevin process with reflection as l.p.r.$(q_t^\mu; p_t^\mu)$.

Proposition 2. Let $T > 0$. The Markov process l.p.r.$(q_t^\mu; p_t^\mu)$ does not reach the origin, $O = (0,0)$, in finite time T, i.e.

$$P(\exists t \leq T \, s.t. \, \text{l.p.r.}(q_t^\mu; p_t^\mu) = \text{O}) = 0.$$

Consider the following S.D.E. in $\mathbb{R}^2$:

$$\begin{aligned} \dot{q}_t^{\mu} &= p_t^{\mu} \\ \mu \dot{p}_t^{\mu} &= -p_t^{\mu} + \mathrm{sgn}(q_t^{\mu}) b(|q_t^{\mu}|) + \mathrm{sgn}(q_t^{\mu}) \sigma(|q_t^{\mu}|) \dot{W}_t \\ q_0^{\mu} &= q, p_0^{\mu} = p. \end{aligned} \tag{5}$$

Define:

$$\begin{aligned} (\widehat{q}_t^{\mu}; \widehat{p}_t^{\mu}) &= (q_t^{\mu,q}; p_t^{\mu,p}) \text{ for } \tau_{2k}^{\mu} \le t \le \tau_{2k+1}^{\mu,-} \quad (6) \\ (\widehat{q}_t^{\mu}; \widehat{p}_t^{\mu}) &= (q_t^{\mu,-q}; p_t^{\mu,-p}) \text{ for } \tau_{2k+1}^{\mu} \le t \le \tau_{2k+2}^{\mu,-} \end{aligned}$$

(i). $(\widehat{q}_t^{\mu}; \widehat{p}_t^{\mu})$ is a process with reflection on $\partial \mathbb{R}_+^2$.

(ii). $(\widehat{q}_t^{\mu}; \widehat{p}_t^{\mu})$ defined by (6) and l.p.r.$(q_t^{\mu}; p_t^{\mu}) = (|q_t^{\mu}|; \frac{d}{dt}|q_t^{\mu}|)$ coincide.

Then Proposition 2. follows from:

Lemma 3. Let $T > 0$. The process $(q_t^\mu; p_t^\mu)$ does not reach the origin, $O = (0,0)$, in finite time T, i.e.

$$P(\exists t \leq T\, s.t.\, (q_t^\mu; p_t^\mu) = \mathrm{O}) = 0.$$

Proof. Let $d \ll 1$ be a small number. Define the rectangle $\Delta = \{(q,p) \in \mathbb{R}^1 \times \mathbb{R}^1 : |q| \leq \frac{d^2}{2}, |p| \leq \frac{d}{2}\}$ and suppose that the trajectory starts from some point outside the rectangle Δ, say from $(q,0) \in \mathbb{R}^2 \setminus \Delta$.

If we assume that the process $(q_t^\mu; p_t^\mu)$ will reach $(0,0)$ before time T with positive probability, one can show that

$$Bd^2 < E^{(q,0)} \int_0^T \chi_\Delta(q_s^\mu; p_s^\mu) ds \leq Ad^3, \qquad (7)$$

which cannot hold for constants A and B and small enough d.

Convergence of the Langevin process with reflection

Consider the following S.D.E. in $\mathbb{R}^1$:

$$\begin{aligned} \dot{q}_t &= \text{sgn}(q_t) b(|q_t|) + \text{sgn}(q_t)\sigma(|q_t|)\dot{W}_t \\ q_0 &= q, \end{aligned} \tag{8}$$

Theorem 4. For the time interval $[0, T]$

$$\text{l.p.r.}(q_\cdot^\mu) \rightarrow |q_\cdot|, \text{ weakly as } \mu \rightarrow 0. \tag{9}$$

Proof. Consider first the following S.D.E.s in $\mathbb{R}^2$ and $\mathbb{R}^1$ respectively:

$$\begin{aligned} \dot{\widetilde{q}}_t^\mu &= \widetilde{p}_t^\mu \\ \mu\dot{\widetilde{p}}_t^\mu &= -\widetilde{p}_t^\mu + \sigma(|\widetilde{q}_t^\mu|)\dot{\widetilde{W}}_t \\ \widetilde{q}_0^\mu &= q, \widetilde{p}_0^\mu = p, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \dot{\widetilde{q}}_t &= \sigma(|\widetilde{q}_t|)\dot{\widetilde{W}}_t \\ \widetilde{q}_0 &= q, \end{aligned} \tag{11}$$

where $\widetilde{W}_t$ is the standard one-dimensional Wiener process.

Lemma 5. The following hold:

(i). For every $\delta > 0$ we have that

$$E\int_0^T \chi_{\{|\widetilde{q}_s|\leq\delta\}} ds \leq c\delta,$$

where c is a constant.

(ii). $\widetilde{q}_t^\mu \to \widetilde{q}_t$ uniformly in $[0, T]$ in probability.

Consider now the following S.D.E.'s:

$$\begin{aligned}\dot{\overline{q}}_t^\mu &= \overline{p}_t^\mu \\ \mu\dot{\overline{p}}_t^\mu &= -\overline{p}_t^\mu + \mathrm{sgn}(\overline{q}_t^\mu)b(|\overline{q}_t^\mu|) + \sigma(|\overline{q}_t^\mu|)\dot{\widetilde{W}}_t \\ \overline{q}_0^\mu &= q, \overline{p}_0^\mu = p \end{aligned} \quad (12)$$

and

$$\begin{aligned}\dot{\overline{q}}_t &= \mathrm{sgn}(\overline{q}_t)b(|\overline{q}_t|) + \sigma(|\overline{q}_t|)\dot{\widetilde{W}}_t \\ \overline{q}_0 &= q \end{aligned} \quad (13)$$

Then by the Lemma above one can show that:

Lemma 6. For the time interval $[0, T]$, $\overline{q}_\cdot^\mu \to \overline{q}_\cdot$, weakly as $\mu \to 0$.

Consider lastly the solution of the following S.D.E.'s:

$$
\begin{aligned}
\dot{q}_t^\mu &= p_t^\mu \\
\mu \dot{p}_t^\mu &= -p_t^\mu + \mathrm{sgn}(q_t^\mu) b(|q_t^\mu|) + \mathrm{sgn}(q_t^\mu)\sigma(|q_t^\mu|)\dot{W}_t \\
q_0^\mu &= q, p_0^\mu = p \qquad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{q}_t &= \mathrm{sgn}(q_t) b(|q_t|) + \mathrm{sgn}(q_t)\sigma(|q_t|)\dot{W}_t \\
q_0 &= q, \qquad (15)
\end{aligned}
$$

Now by the observation that $\widetilde{W}_t^\mu = \int_0^t \mathrm{sqn}(q_s^\mu) dWs$ and $\widetilde{W}_t = \int_0^t \mathrm{sqn}(q_s) dWs$ are again Wiener processes and by Lemma 6, it follows that:

Theorem 7. For the time interval $[0, T]$, $|q_\cdot^\mu| \to |q_\cdot|$, weakly as $\mu \to 0$, or otherwise that

$$
\text{l.p.r.}(q_\cdot^\mu) \to |q_\cdot|, \text{ weakly as } \mu \to 0. \qquad (16)
$$