quadratic as a squared linear expression. Then, from the equation in standard form, read off the center and radius. For the sphere here, we have

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$
$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$
$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) = -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2$$
$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = -1 + \frac{9}{4} + 4 = \frac{21}{4}.$$

From this standard form, we read that $x_0 = -3/2$, $y_0 = 0$, $z_0 = 2$, and $a = \sqrt{21}/2$. The center is $(-3/2, 0, 2)$. The radius is $\sqrt{21}/2$.

EXAMPLE 5 Here are some geometric interpretations of inequalities and equations involving spheres.

(a) $x^2 + y^2 + z^2 < 4$	The interior of the sphere $x^2 + y^2 + z^2 = 4$.
(b) $x^2 + y^2 + z^2 \leq 4$	The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.
(c) $x^2 + y^2 + z^2 > 4$	The exterior of the sphere $x^2 + y^2 + z^2 = 4$.
(d) $x^2 + y^2 + z^2 = 4, z \leq 0$	The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy-plane (the plane $z = 0$).

Just as polar coordinates give another way to locate points in the xy-plane (Section 11.3), alternative coordinate systems, different from the Cartesian coordinate system developed here, exist for three-dimensional space. We examine two of these coordinate systems in Section 15.7.

Exercises 12.1

Geometric Interpretations of Equations

In Exercises 1–16, give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations.

1. $x = 2, \quad y = 3$
2. $x = -1, \quad z = 0$
3. $y = 0, \quad z = 0$
4. $x = 1, \quad y = 0$
5. $x^2 + y^2 = 4, \quad z = 0$
6. $x^2 + y^2 = 4, \quad z = -2$
7. $x^2 + z^2 = 4, \quad y = 0$
8. $y^2 + z^2 = 1, \quad x = 0$
9. $x^2 + y^2 + z^2 = 1, \quad x = 0$
10. $x^2 + y^2 + z^2 = 25, \quad y = -4$
11. $x^2 + y^2 + (z + 3)^2 = 25, \quad z = 0$
12. $x^2 + (y - 1)^2 + z^2 = 4, \quad y = 0$
13. $x^2 + y^2 = 4, \quad z = y$
14. $x^2 + y^2 + z^2 = 4, \quad y = x$
15. $y = x^2, \quad z = 0$
16. $z = y^2, \quad x = 1$

Geometric Interpretations of Inequalities and Equations

In Exercises 17–24, describe the sets of points in space whose coordinates satisfy the given inequalities or combinations of equations and inequalities.

17. **a.** $x \geq 0, \quad y \geq 0, \quad z = 0$ **b.** $x \geq 0, \quad y \leq 0, \quad z = 0$
18. **a.** $0 \leq x \leq 1$ **b.** $0 \leq x \leq 1, \quad 0 \leq y \leq 1$
    c. $0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1$
19. **a.** $x^2 + y^2 + z^2 \leq 1$ **b.** $x^2 + y^2 + z^2 > 1$
20. **a.** $x^2 + y^2 \leq 1, \quad z = 0$ **b.** $x^2 + y^2 \leq 1, \quad z = 3$
    c. $x^2 + y^2 \leq 1$, no restriction on z
21. **a.** $1 \leq x^2 + y^2 + z^2 \leq 4$
    b. $x^2 + y^2 + z^2 \leq 1, \quad z \geq 0$
22. **a.** $x = y, \quad z = 0$ **b.** $x = y$, no restriction on z
23. **a.** $y \geq x^2, \quad z \geq 0$ **b.** $x \leq y^2, \quad 0 \leq z \leq 2$
24. **a.** $z = 1 - y$, no restriction on x
    b. $z = y^3, \quad x = 2$

In Exercises 25–34, describe the given set with a single equation or with a pair of equations.

25. The plane perpendicular to the
 a. x-axis at $(3, 0, 0)$ b. y-axis at $(0, -1, 0)$
 c. z-axis at $(0, 0, -2)$

26. The plane through the point $(3, -1, 2)$ perpendicular to the
 a. x-axis b. y-axis c. z-axis

27. The plane through the point $(3, -1, 1)$ parallel to the
 a. xy-plane b. yz-plane c. xz-plane

28. The circle of radius 2 centered at $(0, 0, 0)$ and lying in the
 a. xy-plane b. yz-plane c. xz-plane

29. The circle of radius 2 centered at $(0, 2, 0)$ and lying in the
 a. xy-plane b. yz-plane c. plane $y = 2$

30. The circle of radius 1 centered at $(-3, 4, 1)$ and lying in a plane parallel to the
 a. xy-plane b. yz-plane c. xz-plane

31. The line through the point $(1, 3, -1)$ parallel to the
 a. x-axis b. y-axis c. z-axis

32. The set of points in space equidistant from the origin and the point $(0, 2, 0)$

33. The circle in which the plane through the point $(1, 1, 3)$ perpendicular to the z-axis meets the sphere of radius 5 centered at the origin

34. The set of points in space that lie 2 units from the point $(0, 0, 1)$ and, at the same time, 2 units from the point $(0, 0, -1)$

Inequalities to Describe Sets of Points

Write inequalities to describe the sets in Exercises 35–40.

35. The slab bounded by the planes $z = 0$ and $z = 1$ (planes included)

36. The solid cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

37. The half-space consisting of the points on and below the xy-plane

38. The upper hemisphere of the sphere of radius 1 centered at the origin

39. The (a) interior and (b) exterior of the sphere of radius 1 centered at the point $(1, 1, 1)$

40. The closed region bounded by the spheres of radius 1 and radius 2 centered at the origin. (*Closed* means the spheres are to be included. Had we wanted the spheres left out, we would have asked for the *open* region bounded by the spheres. This is analogous to the way we use *closed* and *open* to describe intervals: *closed* means endpoints included, *open* means endpoints left out. Closed sets include boundaries; open sets leave them out.)

Distance

In Exercises 41–46, find the distance between points P_1 and P_2.

41. $P_1(1, 1, 1)$, $P_2(3, 3, 0)$

42. $P_1(-1, 1, 5)$, $P_2(2, 5, 0)$

43. $P_1(1, 4, 5)$, $P_2(4, -2, 7)$

44. $P_1(3, 4, 5)$, $P_2(2, 3, 4)$

45. $P_1(0, 0, 0)$, $P_2(2, -2, -2)$

46. $P_1(5, 3, -2)$, $P_2(0, 0, 0)$

Spheres

Find the centers and radii of the spheres in Exercises 47–50.

47. $(x + 2)^2 + y^2 + (z - 2)^2 = 8$

48. $(x - 1)^2 + \left(y + \frac{1}{2}\right)^2 + (z + 3)^2 = 25$

49. $(x - \sqrt{2})^2 + (y - \sqrt{2})^2 + (z + \sqrt{2})^2 = 2$

50. $x^2 + \left(y + \frac{1}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 = \frac{16}{9}$

Find equations for the spheres whose centers and radii are given in Exercises 51–54.

	Center	Radius
51.	$(1, 2, 3)$	$\sqrt{14}$
52.	$(0, -1, 5)$	2
53.	$\left(-1, \frac{1}{2}, -\frac{2}{3}\right)$	$\frac{4}{9}$
54.	$(0, -7, 0)$	7

Find the centers and radii of the spheres in Exercises 55–58.

55. $x^2 + y^2 + z^2 + 4x - 4z = 0$

56. $x^2 + y^2 + z^2 - 6y + 8z = 0$

57. $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$

58. $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9$

Theory and Examples

59. Find a formula for the distance from the point $P(x, y, z)$ to the
 a. x-axis. b. y-axis. c. z-axis.

60. Find a formula for the distance from the point $P(x, y, z)$ to the
 a. xy-plane. b. yz-plane. c. xz-plane.

61. Find the perimeter of the triangle with vertices $A(-1, 2, 1)$, $B(1, -1, 3)$, and $C(3, 4, 5)$.

62. Show that the point $P(3, 1, 2)$ is equidistant from the points $A(2, -1, 3)$ and $B(4, 3, 1)$.

63. Find an equation for the set of all points equidistant from the planes $y = 3$ and $y = -1$.

64. Find an equation for the set of all points equidistant from the point $(0, 0, 2)$ and the xy-plane.

65. Find the point on the sphere $x^2 + (y - 3)^2 + (z + 5)^2 = 4$ nearest
 a. the xy-plane. b. the point $(0, 7, -5)$.

66. Find the point equidistant from the points $(0, 0, 0)$, $(0, 4, 0)$, $(3, 0, 0)$, and $(2, 2, -3)$.

and

$$|\mathbf{F}_2| = \frac{75 \cos 55^\circ}{\sin 55^\circ \cos 40^\circ + \cos 55^\circ \sin 40^\circ}$$

$$= \frac{75 \cos 55^\circ}{\sin (55^\circ + 40^\circ)} \approx 43.18 \text{ N}.$$

The force vectors are then $\mathbf{F}_1 = \langle -33.08, 47.24 \rangle$ and $\mathbf{F}_2 = \langle 33.08, 27.76 \rangle$. ■

Exercises 12.2

Vectors in the Plane

In Exercises 1–8, let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Find the (a) component form and (b) magnitude (length) of the vector.

1. $3\mathbf{u}$
2. $-2\mathbf{v}$
3. $\mathbf{u} + \mathbf{v}$
4. $\mathbf{u} - \mathbf{v}$
5. $2\mathbf{u} - 3\mathbf{v}$
6. $-2\mathbf{u} + 5\mathbf{v}$
7. $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v}$
8. $-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v}$

In Exercises 9–16, find the component form of the vector.

9. The vector $\overrightarrow{PQ}$, where $P = (1, 3)$ and $Q = (2, -1)$
10. The vector $\overrightarrow{OP}$ where O is the origin and P is the midpoint of segment RS, where $R = (2, -1)$ and $S = (-4, 3)$
11. The vector from the point $A = (2, 3)$ to the origin
12. The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, where $A = (1, -1)$, $B = (2, 0)$, $C = (-1, 3)$, and $D = (-2, 2)$
13. The unit vector that makes an angle $\theta = 2\pi/3$ with the positive x-axis
14. The unit vector that makes an angle $\theta = -3\pi/4$ with the positive x-axis
15. The unit vector obtained by rotating the vector $\langle 0, 1 \rangle$ 120° counterclockwise about the origin
16. The unit vector obtained by rotating the vector $\langle 1, 0 \rangle$ 135° counterclockwise about the origin

Vectors in Space

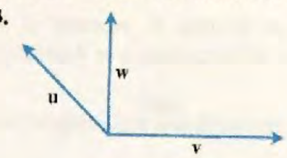
In Exercises 17–22, express each vector in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

17. $\overrightarrow{P_1P_2}$ if P_1 is the point $(5, 7, -1)$ and P_2 is the point $(2, 9, -2)$
18. $\overrightarrow{P_1P_2}$ if P_1 is the point $(1, 2, 0)$ and P_2 is the point $(-3, 0, 5)$
19. $\overrightarrow{AB}$ if A is the point $(-7, -8, 1)$ and B is the point $(-10, 8, 1)$
20. $\overrightarrow{AB}$ if A is the point $(1, 0, 3)$ and B is the point $(-1, 4, 5)$
21. $5\mathbf{u} - \mathbf{v}$ if $\mathbf{u} = \langle 1, 1, -1 \rangle$ and $\mathbf{v} = \langle 2, 0, 3 \rangle$
22. $-2\mathbf{u} + 3\mathbf{v}$ if $\mathbf{u} = \langle -1, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, 1, 1 \rangle$

Geometric Representations

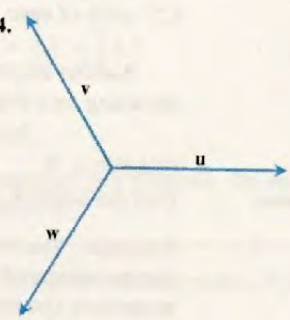
In Exercises 23 and 24, copy vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ head to tail as needed to sketch the indicated vector.

23.



a. $\mathbf{u} + \mathbf{v}$
b. $\mathbf{u} + \mathbf{v} + \mathbf{w}$
c. $\mathbf{u} - \mathbf{v}$
d. $\mathbf{u} - \mathbf{w}$

24.



a. $\mathbf{u} - \mathbf{v}$
b. $\mathbf{u} - \mathbf{v} + \mathbf{w}$
c. $2\mathbf{u} - \mathbf{v}$
d. $\mathbf{u} + \mathbf{v} + \mathbf{w}$

Length and Direction

In Exercises 25–30, express each vector as a product of its length and direction.

25. $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
26. $9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
27. $5\mathbf{k}$
28. $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$
29. $\frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$
30. $\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}$

31. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

Length	Direction
a. 2	$\mathbf{i}$
b. $\sqrt{3}$	$-\mathbf{k}$
c. $\frac{1}{2}$	$\frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$
d. 7	$\frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$

32. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

Length	Direction
a. 7	$-\mathbf{j}$
b. $\sqrt{2}$	$-\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{k}$
c. $\frac{13}{12}$	$\frac{3}{13}\mathbf{i} - \frac{4}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$
d. $a > 0$	$\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$

33. Find a vector of magnitude 7 in the direction of $\mathbf{v} = 12\mathbf{i} - 5\mathbf{k}$.

34. Find a vector of magnitude 3 in the direction opposite to the direction of $\mathbf{v} = (1/2)\mathbf{i} - (1/2)\mathbf{j} - (1/2)\mathbf{k}$.

Direction and Midpoints

In Exercises 35–38, find

a. the direction of $\overrightarrow{P_1P_2}$ and

b. the midpoint of line segment P_1P_2.

35. $P_1(-1, 1, 5)$ $P_2(2, 5, 0)$

36. $P_1(1, 4, 5)$ $P_2(4, -2, 7)$

37. $P_1(3, 4, 5)$ $P_2(2, 3, 4)$

38. $P_1(0, 0, 0)$ $P_2(2, -2, -2)$

39. If $\overrightarrow{AB} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and B is the point $(5, 1, 3)$, find A.

40. If $\overrightarrow{AB} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$ and A is the point $(-2, -3, 6)$, find B.

Theory and Applications

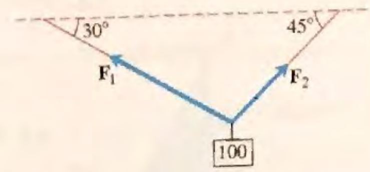
41. **Linear combination** Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$, and $\mathbf{w} = \mathbf{i} - \mathbf{j}$. Find scalars a and b such that $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$.

42. **Linear combination** Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$, and $\mathbf{w} = \mathbf{i} + \mathbf{j}$. Write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is parallel to $\mathbf{w}$. (See Exercise 41.)

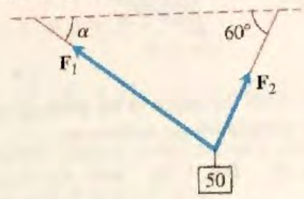
43. **Velocity** An airplane is flying in the direction 25° west of north at 800 km/h. Find the component form of the velocity of the airplane, assuming that the positive x-axis represents due east and the positive y-axis represents due north.

44. (*Continuation of Example 8.*) What speed and direction should the jetliner in Example 8 have in order for the resultant vector to be 500 mph due east?

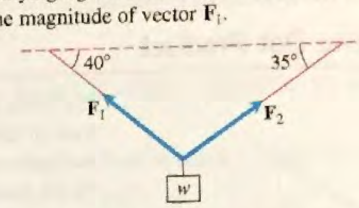
45. Consider a 100-N weight suspended by two wires as shown in the accompanying figure. Find the magnitudes and components of the force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$.



46. Consider a 50-N weight suspended by two wires as shown in the accompanying figure. If the magnitude of vector $\mathbf{F}_1$ is 35 N, find angle α and the magnitude of vector $\mathbf{F}_2$.



47. Consider a w-N weight suspended by two wires as shown in the accompanying figure. If the magnitude of vector $\mathbf{F}_2$ is 100 N, find w and the magnitude of vector $\mathbf{F}_1$.



48. Consider a 25-N weight suspended by two wires as shown in the accompanying figure. If the magnitudes of vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are both 75 N, then angles α and β are equal. Find α.

49. **Location** A bird flies from its nest 5 km in the direction 60° north of east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an xy-coordinate system so that the origin is the bird's nest, the x-axis points east, and the y-axis points north.

a. At what point is the tree located?

b. At what point is the telephone pole?

50. Use similar triangles to find the coordinates of the point Q that divides the segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ into two lengths whose ratio is $p/q = r$.

51. **Medians of a triangle** Suppose that A, B, and C are the corner points of the thin triangular plate of constant density shown here.

a. Find the vector from C to the midpoint M of side AB.

b. Find the vector from C to the point that lies two-thirds of the way from C to M on the median CM.

c. Find the coordinates of the point in which the medians of ΔABC intersect. According to Exercise 19, Section 6.6, this point is the plate's center of mass. (See the accompanying figure.)

Exercises 12.3

Dot Product and Projections

In Exercises 1–8, find

a. $\mathbf{v}\cdot\mathbf{u}$, $|\mathbf{v}|$, $|\mathbf{u}|$

b. the cosine of the angle between $\mathbf{v}$ and $\mathbf{u}$

c. the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$

d. the vector $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

1. $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \sqrt{5}\mathbf{k}$, $\mathbf{u} = -2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$
2. $\mathbf{v} = (3/5)\mathbf{i} + (4/5)\mathbf{k}$, $\mathbf{u} = 5\mathbf{i} + 12\mathbf{j}$
3. $\mathbf{v} = 10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}$, $\mathbf{u} = 3\mathbf{j} + 4\mathbf{k}$
4. $\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$, $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
5. $\mathbf{v} = 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
6. $\mathbf{v} = -\mathbf{i} + \mathbf{j}$, $\mathbf{u} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k}$
7. $\mathbf{v} = 5\mathbf{i} + \mathbf{j}$, $\mathbf{u} = 2\mathbf{i} + \sqrt{17}\mathbf{j}$
8. $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right\rangle$

Angle Between Vectors

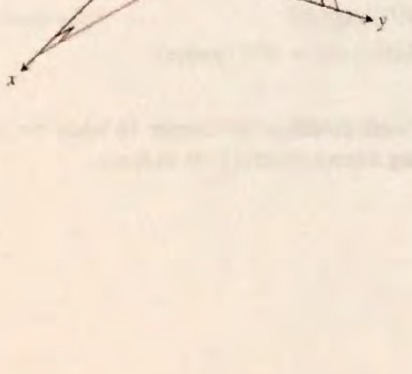
T Find the angles between the vectors in Exercises 9–12 to the nearest hundredth of a radian.

9. $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$
10. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{k}$
11. $\mathbf{u} = \sqrt{3}\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
12. $\mathbf{u} = \mathbf{i} + \sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$
13. **Triangle** Find the measures of the angles of the triangle whose vertices are $A = (-1, 0)$, $B = (2, 1)$, and $C = (1, -2)$.
14. **Rectangle** Find the measures of the angles between the diagonals of the rectangle whose vertices are $A = (1, 0)$, $B = (0, 3)$, $C = (3, 4)$, and $D = (4, 1)$.
15. **Direction angles and direction cosines** The *direction angles* α, β, and γ of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are defined as follows:

α is the angle between $\mathbf{v}$ and the positive x-axis $(0 \le \alpha \le \pi)$

β is the angle between $\mathbf{v}$ and the positive y-axis $(0 \le \beta \le \pi)$

γ is the angle between $\mathbf{v}$ and the positive z-axis $(0 \le \gamma \le \pi)$.



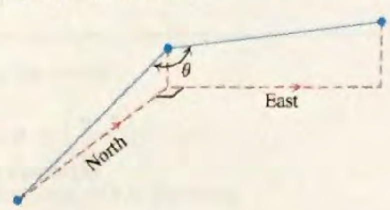
a. Show that

$$\cos\alpha = \frac{a}{|\mathbf{v}|}, \quad \cos\beta = \frac{b}{|\mathbf{v}|}, \quad \cos\gamma = \frac{c}{|\mathbf{v}|},$$

and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the *direction cosines* of $\mathbf{v}$.

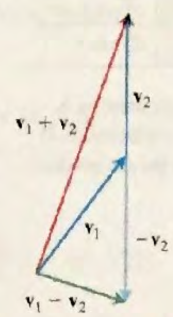
b. **Unit vectors are built from direction cosines** Show that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a unit vector, then a, b, and c are the direction cosines of $\mathbf{v}$.

16. **Water main construction** A water main is to be constructed with a 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.

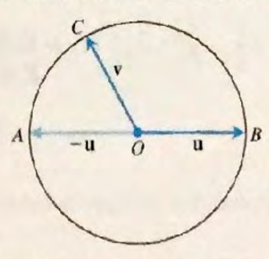


Theory and Examples

17. **Sums and differences** In the accompanying figure, it looks as if $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$ are orthogonal. Is this mere coincidence, or are there circumstances under which we may expect the sum of two vectors to be orthogonal to their difference? Give reasons for your answer.



18. **Orthogonality on a circle** Suppose that AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B. Show that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

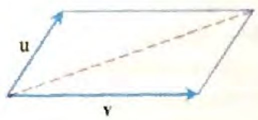


19. **Diagonals of a rhombus** Show that the diagonals of a rhombus (parallelogram with sides of equal length) are perpendicular.

20. **Perpendicular diagonals** Show that squares are the only rectangles with perpendicular diagonals.

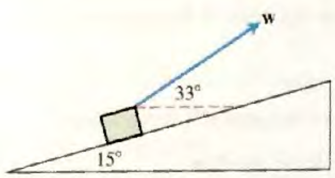
21. **When parallelograms are rectangles** Prove that a parallelogram is a rectangle if and only if its diagonals are equal in length. (This fact is often exploited by carpenters.)

22. **Diagonal of parallelogram** Show that the indicated diagonal of the parallelogram determined by vectors **u** and **v** bisects the angle between **u** and **v** if $|\mathbf{u}| = |\mathbf{v}|$.



23. **Projectile motion** A gun with muzzle velocity of 1200 ft/sec is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

24. **Inclined plane** Suppose that a box is being towed up an inclined plane as shown in the figure. Find the force **w** needed to make the component of the force parallel to the inclined plane equal to 2.5 lb.



25. a. **Cauchy-Schwartz inequality** Since $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, show that the inequality $|\mathbf{u}\cdot\mathbf{v}| \le |\mathbf{u}||\mathbf{v}|$ holds for any vectors **u** and **v**.

    b. Under what circumstances, if any, does $|\mathbf{u}\cdot\mathbf{v}|$ equal $|\mathbf{u}||\mathbf{v}|$? Give reasons for your answer.

26. **Dot multiplication is positive definite** Show that dot multiplication of vectors is *positive definite*; that is, show that $\mathbf{u}\cdot\mathbf{u} \ge 0$ for every vector **u** and that $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

27. **Orthogonal unit vectors** If $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal unit vectors and $\mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2$, find $\mathbf{v}\cdot\mathbf{u}_1$.

28. **Cancellation in dot products** In real-number multiplication, if $uv_1 = uv_2$ and $u \ne 0$, we can cancel the u and conclude that $v_1 = v_2$. Does the same rule hold for the dot product? That is, if $\mathbf{u}\cdot\mathbf{v}_1 = \mathbf{u}\cdot\mathbf{v}_2$ and $\mathbf{u} \ne \mathbf{0}$, can you conclude that $\mathbf{v}_1 = \mathbf{v}_2$? Give reasons for your answer.

29. Using the definition of the projection of **u** onto **v**, show by direct calculation that $(\mathbf{u} - \operatorname{proj}_{\mathbf{v}}\mathbf{u})\cdot\operatorname{proj}_{\mathbf{v}}\mathbf{u} = 0$.

30. A force $\mathbf{F} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ is applied to a spacecraft with velocity vector $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$. Express **F** as a sum of a vector parallel to **v** and a vector orthogonal to **v**.

Equations for Lines in the Plane

31. **Line perpendicular to a vector** Show that $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by = c$ by establishing that the slope of the vector **v** is the negative reciprocal of the slope of the given line.

32. **Line parallel to a vector** Show that the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is parallel to the line $bx - ay = c$ by establishing that the slope of the line segment representing **v** is the same as the slope of the given line.

In Exercises 33–36, use the result of Exercise 31 to find an equation for the line through P perpendicular to **v**. Then sketch the line. Include **v** in your sketch *as a vector starting at the origin.*

33. $P(2, 1)$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$
34. $P(-1, 2)$, $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$
35. $P(-2, -7)$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$
36. $P(11, 10)$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

In Exercises 37–40, use the result of Exercise 32 to find an equation for the line through P parallel to **v**. Then sketch the line. Include **v** in your sketch *as a vector starting at the origin.*

37. $P(-2, 1)$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$
38. $P(0, -2)$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$
39. $P(1, 2)$, $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$
40. $P(1, 3)$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

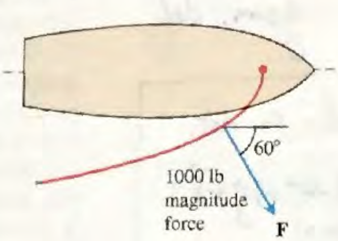
Work

41. **Work along a line** Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude 5 N) in moving an object along the line from the origin to the point (1, 1) (distance in meters).

42. **Locomotive** The Union Pacific's *Big Boy* locomotive could pull 6000-ton trains with a tractive effort (pull) of 602,148 N (135,375 lb). At this level of effort, about how much work did *Big Boy* do on the (approximately straight) 605-km journey from San Francisco to Los Angeles?

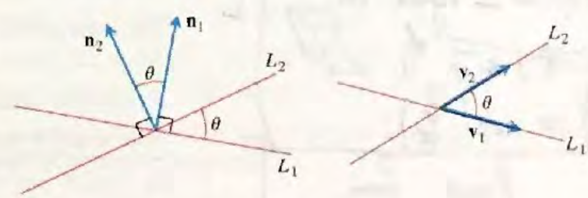
43. **Inclined plane** How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200-N force at an angle of 30° from the horizontal?

44. **Sailboat** The wind passing over a boat's sail exerted a 1000-lb magnitude force **F** as shown here. How much work did the wind perform in moving the boat forward 1 mi? Answer in foot-pounds.



Angles Between Lines in the Plane

The **acute angle between intersecting lines** that do not cross at right angles is the same as the angle determined by vectors normal to the lines or by the vectors parallel to the lines.



and

$$|\mathbf{F}_2| = \frac{75 \cos 55°}{\sin 55° \cos 40° + \cos 55° \sin 40°}$$

$$= \frac{75 \cos 55°}{\sin (55° + 40°)} \approx 43.18 \text{ N}.$$

The force vectors are then $\mathbf{F}_1 = \langle -33.08, 47.24 \rangle$ and $\mathbf{F}_2 = \langle 33.08, 27.76 \rangle$. ■

Exercises 12.2

Vectors in the Plane

In Exercises 1–8, let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Find the (a) component form and (b) magnitude (length) of the vector.

1. $3\mathbf{u}$
2. $-2\mathbf{v}$
3. $\mathbf{u} + \mathbf{v}$
4. $\mathbf{u} - \mathbf{v}$
5. $2\mathbf{u} - 3\mathbf{v}$
6. $-2\mathbf{u} + 5\mathbf{v}$
7. $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v}$
8. $-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v}$

In Exercises 9–16, find the component form of the vector.

9. The vector $\overrightarrow{PQ}$, where $P = (1, 3)$ and $Q = (2, -1)$
10. The vector $\overrightarrow{OP}$ where O is the origin and P is the midpoint of segment RS, where $R = (2, -1)$ and $S = (-4, 3)$
11. The vector from the point $A = (2, 3)$ to the origin
12. The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, where $A = (1, -1)$, $B = (2, 0)$, $C = (-1, 3)$, and $D = (-2, 2)$
13. The unit vector that makes an angle $\theta = 2\pi/3$ with the positive x-axis
14. The unit vector that makes an angle $\theta = -3\pi/4$ with the positive x-axis
15. The unit vector obtained by rotating the vector $\langle 0, 1 \rangle$ 120° counterclockwise about the origin
16. The unit vector obtained by rotating the vector $\langle 1, 0 \rangle$ 135° counterclockwise about the origin

Vectors in Space

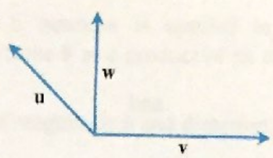
In Exercises 17–22, express each vector in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

17. $\overrightarrow{P_1P_2}$ if P_1 is the point $(5, 7, -1)$ and P_2 is the point $(2, 9, -2)$
18. $\overrightarrow{P_1P_2}$ if P_1 is the point $(1, 2, 0)$ and P_2 is the point $(-3, 0, 5)$
19. $\overrightarrow{AB}$ if A is the point $(-7, -8, 1)$ and B is the point $(-10, 8, 1)$
20. $\overrightarrow{AB}$ if A is the point $(1, 0, 3)$ and B is the point $(-1, 4, 5)$
21. $5\mathbf{u} - \mathbf{v}$ if $\mathbf{u} = \langle 1, 1, -1 \rangle$ and $\mathbf{v} = \langle 2, 0, 3 \rangle$
22. $-2\mathbf{u} + 3\mathbf{v}$ if $\mathbf{u} = \langle -1, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, 1, 1 \rangle$

Geometric Representations

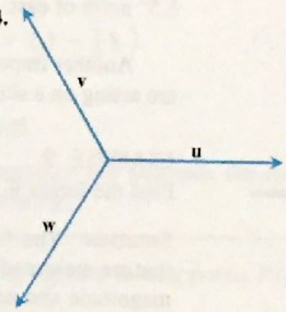
In Exercises 23 and 24, copy vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ head to tail as needed to sketch the indicated vector.

23.



a. $\mathbf{u} + \mathbf{v}$
b. $\mathbf{u} + \mathbf{v} + \mathbf{w}$
c. $\mathbf{u} - \mathbf{v}$
d. $\mathbf{u} - \mathbf{w}$

24.



a. $\mathbf{u} - \mathbf{v}$
b. $\mathbf{u} - \mathbf{v} + \mathbf{w}$
c. $2\mathbf{u} - \mathbf{v}$
d. $\mathbf{u} + \mathbf{v} + \mathbf{w}$

Length and Direction

In Exercises 25–30, express each vector as a product of its length and direction.

25. $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
26. $9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
27. $5\mathbf{k}$
28. $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$
29. $\frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$
30. $\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}$

31. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

Length	Direction
a. 2	$\mathbf{i}$
b. $\sqrt{3}$	$-\mathbf{k}$
c. $\frac{1}{2}$	$\frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$
d. 7	$\frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$

32. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

Length	Direction
a. 7	$-\mathbf{j}$
b. $\sqrt{2}$	$-\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{k}$
c. $\frac{13}{12}$	$\frac{3}{13}\mathbf{i} - \frac{4}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$
d. $a > 0$	$\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$

33. Find a vector of magnitude 7 in the direction of $\mathbf{v} = 12\mathbf{i} - 5\mathbf{k}$.

34. Find a vector of magnitude 3 in the direction opposite to the direction of $\mathbf{v} = (1/2)\mathbf{i} - (1/2)\mathbf{j} - (1/2)\mathbf{k}$.

Direction and Midpoints

In Exercises 35–38, find

a. the direction of $\overrightarrow{P_1P_2}$ and

b. the midpoint of line segment P_1P_2.

35. $P_1(-1, 1, 5)$ $P_2(2, 5, 0)$

36. $P_1(1, 4, 5)$ $P_2(4, -2, 7)$

37. $P_1(3, 4, 5)$ $P_2(2, 3, 4)$

38. $P_1(0, 0, 0)$ $P_2(2, -2, -2)$

39. If $\overrightarrow{AB} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and B is the point $(5, 1, 3)$, find A.

40. If $\overrightarrow{AB} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$ and A is the point $(-2, -3, 6)$, find B.

Theory and Applications

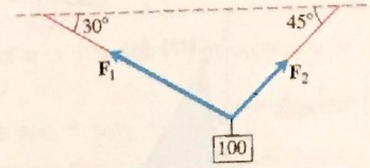
41. **Linear combination** Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$, and $\mathbf{w} = \mathbf{i} - \mathbf{j}$. Find scalars a and b such that $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$.

42. **Linear combination** Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$, and $\mathbf{w} = \mathbf{i} + \mathbf{j}$. Write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is parallel to $\mathbf{w}$. (See Exercise 41.)

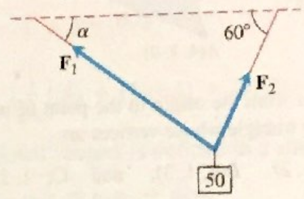
43. **Velocity** An airplane is flying in the direction 25° west of north at 800 km/h. Find the component form of the velocity of the airplane, assuming that the positive x-axis represents due east and the positive y-axis represents due north.

44. (*Continuation of Example 8.*) What speed and direction should the jetliner in Example 8 have in order for the resultant vector to be 500 mph due east?

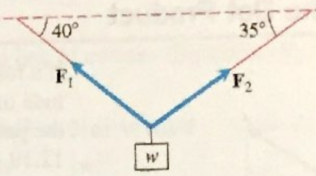
45. Consider a 100-N weight suspended by two wires as shown in the accompanying figure. Find the magnitudes and components of the force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$.



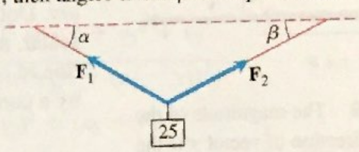
46. Consider a 50-N weight suspended by two wires as shown in the accompanying figure. If the magnitude of vector $\mathbf{F}_1$ is 35 N, find angle α and the magnitude of vector $\mathbf{F}_2$.



47. Consider a w-N weight suspended by two wires as shown in the accompanying figure. If the magnitude of vector $\mathbf{F}_2$ is 100 N, find w and the magnitude of vector $\mathbf{F}_1$.



48. Consider a 25-N weight suspended by two wires as shown in the accompanying figure. If the magnitudes of vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are both 75 N, then angles α and β are equal. Find α.



49. **Location** A bird flies from its nest 5 km in the direction 60° north of east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an xy-coordinate system so that the origin is the bird's nest, the x-axis points east, and the y-axis points north.

a. At what point is the tree located?

b. At what point is the telephone pole?

50. Use similar triangles to find the coordinates of the point Q that divides the segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ into two lengths whose ratio is $p/q = r$.

51. **Medians of a triangle** Suppose that A, B, and C are the corner points of the thin triangular plate of constant density shown here.

a. Find the vector from C to the midpoint M of side AB.

b. Find the vector from C to the point that lies two-thirds of the way from C to M on the median CM.

c. Find the coordinates of the point in which the medians of ΔABC intersect. According to Exercise 19, Section 6.6, this point is the plate's center of mass. (See the accompanying figure.)

Exercises 12.3

Dot Product and Projections

In Exercises 1–8, find

a. $\mathbf{v} \cdot \mathbf{u}$, $|\mathbf{v}|$, $|\mathbf{u}|$

b. the cosine of the angle between $\mathbf{v}$ and $\mathbf{u}$

c. the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$

d. the vector $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

1. $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \sqrt{5}\mathbf{k}$, $\mathbf{u} = -2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$
2. $\mathbf{v} = (3/5)\mathbf{i} + (4/5)\mathbf{k}$, $\mathbf{u} = 5\mathbf{i} + 12\mathbf{j}$
3. $\mathbf{v} = 10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}$, $\mathbf{u} = 3\mathbf{j} + 4\mathbf{k}$
4. $\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$, $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
5. $\mathbf{v} = 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
6. $\mathbf{v} = -\mathbf{i} + \mathbf{j}$, $\mathbf{u} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k}$
7. $\mathbf{v} = 5\mathbf{i} + \mathbf{j}$, $\mathbf{u} = 2\mathbf{i} + \sqrt{17}\mathbf{j}$
8. $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right\rangle$

Angle Between Vectors

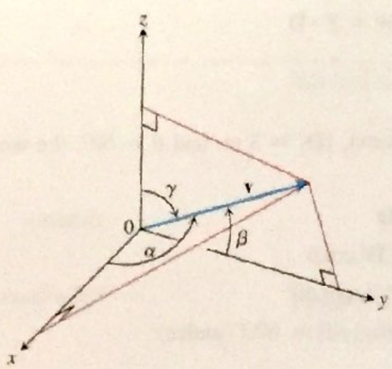
T Find the angles between the vectors in Exercises 9–12 to the nearest hundredth of a radian.

9. $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$
10. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{k}$
11. $\mathbf{u} = \sqrt{3}\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
12. $\mathbf{u} = \mathbf{i} + \sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$
13. **Triangle** Find the measures of the angles of the triangle whose vertices are $A = (-1, 0)$, $B = (2, 1)$, and $C = (1, -2)$.
14. **Rectangle** Find the measures of the angles between the diagonals of the rectangle whose vertices are $A = (1, 0)$, $B = (0, 3)$, $C = (3, 4)$, and $D = (4, 1)$.
15. **Direction angles and direction cosines** The *direction angles* α, β, and γ of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are defined as follows:

α is the angle between $\mathbf{v}$ and the positive x-axis $(0 \le \alpha \le \pi)$

β is the angle between $\mathbf{v}$ and the positive y-axis $(0 \le \beta \le \pi)$

γ is the angle between $\mathbf{v}$ and the positive z-axis $(0 \le \gamma \le \pi)$.



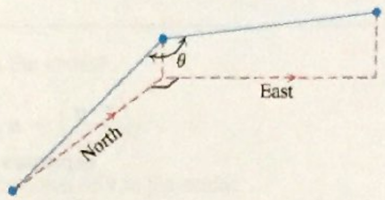
a. Show that

$$\cos\alpha = \frac{a}{|\mathbf{v}|}, \qquad \cos\beta = \frac{b}{|\mathbf{v}|}, \qquad \cos\gamma = \frac{c}{|\mathbf{v}|},$$

and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the *direction cosines* of $\mathbf{v}$.

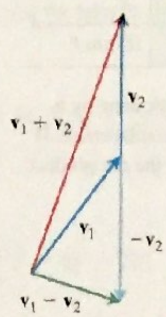
b. **Unit vectors are built from direction cosines** Show that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a unit vector, then a, b, and c are the direction cosines of $\mathbf{v}$.

16. **Water main construction** A water main is to be constructed with a 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.

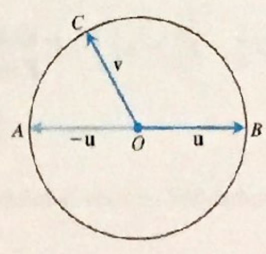


Theory and Examples

17. **Sums and differences** In the accompanying figure, it looks as if $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$ are orthogonal. Is this mere coincidence, or are there circumstances under which we may expect the sum of two vectors to be orthogonal to their difference? Give reasons for your answer.



18. **Orthogonality on a circle** Suppose that AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B. Show that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

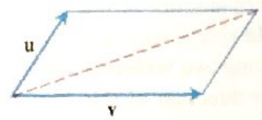


19. **Diagonals of a rhombus** Show that the diagonals of a rhombus (parallelogram with sides of equal length) are perpendicular.

20. **Perpendicular diagonals** Show that squares are the only rectangles with perpendicular diagonals.

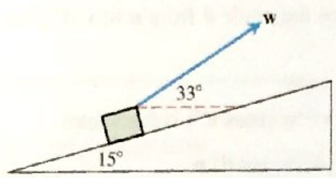
21. **When parallelograms are rectangles** Prove that a parallelogram is a rectangle if and only if its diagonals are equal in length. (This fact is often exploited by carpenters.)

22. **Diagonal of parallelogram** Show that the indicated diagonal of the parallelogram determined by vectors **u** and **v** bisects the angle between **u** and **v** if $|\mathbf{u}| = |\mathbf{v}|$.



23. **Projectile motion** A gun with muzzle velocity of 1200 ft/sec is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

24. **Inclined plane** Suppose that a box is being towed up an inclined plane as shown in the figure. Find the force **w** needed to make the component of the force parallel to the inclined plane equal to 2.5 lb.



25. a. **Cauchy-Schwartz inequality** Since $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$, show that the inequality $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}||\mathbf{v}|$ holds for any vectors **u** and **v**.

    b. Under what circumstances, if any, does $|\mathbf{u} \cdot \mathbf{v}|$ equal $|\mathbf{u}||\mathbf{v}|$? Give reasons for your answer.

26. **Dot multiplication is positive definite** Show that dot multiplication of vectors is *positive definite*; that is, show that $\mathbf{u} \cdot \mathbf{u} \ge 0$ for every vector **u** and that $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

27. **Orthogonal unit vectors** If $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal unit vectors and $\mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2$, find $\mathbf{v} \cdot \mathbf{u}_1$.

28. **Cancellation in dot products** In real-number multiplication, if $uv_1 = uv_2$ and $u \ne 0$, we can cancel the u and conclude that $v_1 = v_2$. Does the same rule hold for the dot product? That is, if $\mathbf{u} \cdot \mathbf{v}_1 = \mathbf{u} \cdot \mathbf{v}_2$ and $\mathbf{u} \ne \mathbf{0}$, can you conclude that $\mathbf{v}_1 = \mathbf{v}_2$? Give reasons for your answer.

29. Using the definition of the projection of **u** onto **v**, show by direct calculation that $(\mathbf{u} - \text{proj}_\mathbf{v}\,\mathbf{u}) \cdot \text{proj}_\mathbf{v}\,\mathbf{u} = 0$.

30. A force $\mathbf{F} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ is applied to a spacecraft with velocity vector $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$. Express **F** as a sum of a vector parallel to **v** and a vector orthogonal to **v**.

Equations for Lines in the Plane

31. **Line perpendicular to a vector** Show that $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by = c$ by establishing that the slope of the vector **v** is the negative reciprocal of the slope of the given line.

32. **Line parallel to a vector** Show that the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is parallel to the line $bx - ay = c$ by establishing that the slope of the line segment representing **v** is the same as the slope of the given line.

In Exercises 33–36, use the result of Exercise 31 to find an equation for the line through P perpendicular to **v**. Then sketch the line. Include **v** in your sketch *as a vector starting at the origin.*

33. $P(2, 1)$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$
34. $P(-1, 2)$, $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$
35. $P(-2, -7)$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$
36. $P(11, 10)$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

In Exercises 37–40, use the result of Exercise 32 to find an equation for the line through P parallel to **v**. Then sketch the line. Include **v** in your sketch *as a vector starting at the origin.*

37. $P(-2, 1)$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$
38. $P(0, -2)$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$
39. $P(1, 2)$, $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$
40. $P(1, 3)$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

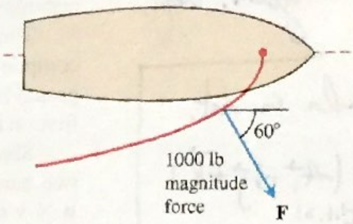
Work

41. **Work along a line** Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude 5 N) in moving an object along the line from the origin to the point (1, 1) (distance in meters).

42. **Locomotive** The Union Pacific's *Big Boy* locomotive could pull 6000-ton trains with a tractive effort (pull) of 602,148 N (135,375 lb). At this level of effort, about how much work did *Big Boy* do on the (approximately straight) 605-km journey from San Francisco to Los Angeles?

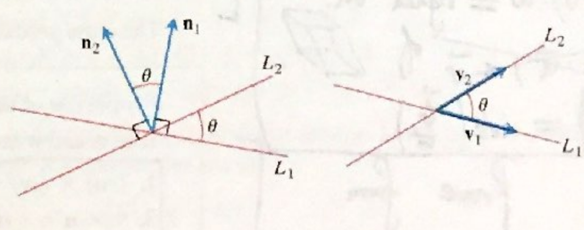
43. **Inclined plane** How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200-N force at an angle of 30° from the horizontal?

44. **Sailboat** The wind passing over a boat's sail exerted a 1000-lb magnitude force **F** as shown here. How much work did the wind perform in moving the boat forward 1 mi? Answer in foot-pounds.



Angles Between Lines in the Plane

The **acute angle between intersecting lines** that do not cross at right angles is the same as the angle determined by vectors normal to the lines or by the vectors parallel to the lines.



parallelogram. The number $|\mathbf{w}||\cos\theta|$ is the parallelepiped's height. Because of this geometry, $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ is also called the **box product** of **u**, **v**, and **w**.

By treating the planes of **v** and **w** and of **w** and **u** as the base planes of the parallelepiped determined by **u**, **v**, and **w**, we see that

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}.$$

The dot and cross may be interchanged in a triple scalar product without altering is value.

Since the dot product is commutative, we also have

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}).$$

The triple scalar product can be evaluated as a determinant:

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} &= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3\end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3\end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2\end{vmatrix}\mathbf{k}\right]\cdot\mathbf{w} \\ &= w_1\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3\end{vmatrix} - w_2\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3\end{vmatrix} + w_3\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2\end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3\end{vmatrix}.\end{aligned}$$

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3\end{vmatrix}$$

EXAMPLE 6 Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution Using the rule for calculating a 3×3 determinant, we find

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4\end{vmatrix} = (1)\begin{vmatrix} 0 & 3 \\ 7 & -4\end{vmatrix} - (2)\begin{vmatrix} -2 & 3 \\ 0 & -4\end{vmatrix} + (-1)\begin{vmatrix} -2 & 0 \\ 0 & 7\end{vmatrix} = -23.$$

The volume is $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| = 23$ units cubed. ■

Exercises 12.4

Cross Product Calculations

In Exercises 1–8, find the length and direction (when defined) of $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$.

1. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$
2. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j}$
3. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
4. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{0}$
5. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = -3\mathbf{j}$
6. $\mathbf{u} = \mathbf{i}\times\mathbf{j}$, $\mathbf{v} = \mathbf{j}\times\mathbf{k}$
7. $\mathbf{u} = -8\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
8. $\mathbf{u} = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$

In Exercises 9–14, sketch the coordinate axes and then include the vectors **u**, **v**, and $\mathbf{u}\times\mathbf{v}$ as vectors starting at the origin.

9. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$
10. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j}$
11. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
12. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$
13. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$
14. $\mathbf{u} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i}$

Triangles in Space

In Exercises 15–18,

a. Find the area of the triangle determined by the points P, Q, and R.

b. Find a unit vector perpendicular to plane PQR.

15. $P(1, -1, 2)$, $Q(2, 0, -1)$, $R(0, 2, 1)$
16. $P(1, 1, 1)$, $Q(2, 1, 3)$, $R(3, -1, 1)$
17. $P(2, -2, 1)$, $Q(3, -1, 2)$, $R(3, -1, 1)$
18. $P(-2, 2, 0)$, $Q(0, 1, -1)$, $R(-1, 2, -2)$

Triple Scalar Products

In Exercises 19–22, verify that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$ and find the volume of the parallelepiped (box) determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

	$\mathbf{u}$	$\mathbf{v}$	$\mathbf{w}$
19.	$2\mathbf{i}$	$2\mathbf{j}$	$2\mathbf{k}$
20.	$\mathbf{i} - \mathbf{j} + \mathbf{k}$	$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
21.	$2\mathbf{i} + \mathbf{j}$	$2\mathbf{i} - \mathbf{j} + \mathbf{k}$	$\mathbf{i} + 2\mathbf{k}$
22.	$\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} - \mathbf{k}$	$2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

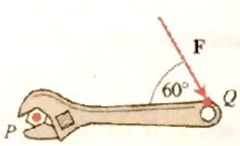
Theory and Examples

23. **Parallel and perpendicular vectors** Let $\mathbf{u} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} - 5\mathbf{k}$, $\mathbf{w} = -15\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$. Which vectors, if any, are **(a)** perpendicular? **(b)** Parallel? Give reasons for your answers.

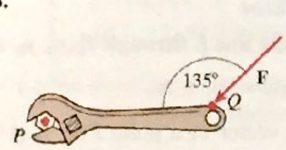
24. **Parallel and perpendicular vectors** Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{k}$, $\mathbf{r} = -(\pi/2)\mathbf{i} - \pi\mathbf{j} + (\pi/2)\mathbf{k}$. Which vectors, if any, are **(a)** perpendicular? **(b)** Parallel? Give reasons for your answers.

In Exercises 25 and 26, find the magnitude of the torque exerted by $\mathbf{F}$ on the bolt at P if $|\overrightarrow{PQ}| = 8$ in. and $|\mathbf{F}| = 30$ lb. Answer in foot-pounds.

25.



26.



27. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.
    a. $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
    b. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|$
    c. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
    d. $\mathbf{u} \times (-\mathbf{u}) = \mathbf{0}$
    e. $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$
    f. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
    g. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$
    h. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

28. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.
    a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
    b. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
    c. $(-\mathbf{u}) \times \mathbf{v} = -(\mathbf{u} \times \mathbf{v})$
    d. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ (any number c)
    e. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$ (any number c)
    f. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
    g. $(\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = 0$
    h. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$

29. Given nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, use dot product and cross product notation, as appropriate, to describe the following.
    a. The vector projection of $\mathbf{u}$ onto $\mathbf{v}$
    b. A vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$
    c. A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and $\mathbf{w}$
    d. The volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$
    e. A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u} \times \mathbf{w}$
    f. A vector of length $|\mathbf{u}|$ in the direction of $\mathbf{v}$

30. Compute $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$ and $\mathbf{i} \times (\mathbf{j} \times \mathbf{j})$. What can you conclude about the associativity of the cross product?

31. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors. Which of the following make sense, and which do not? Give reasons for your answers.
    a. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
    b. $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$
    c. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$
    d. $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

32. **Cross products of three vectors** Show that except in degenerate cases, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ lies in the plane of $\mathbf{u}$ and $\mathbf{v}$, whereas $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of $\mathbf{v}$ and $\mathbf{w}$. What *are* the degenerate cases?

33. **Cancelation in cross products** If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \neq 0$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

34. **Double cancelation** If $\mathbf{u} \neq \mathbf{0}$ and if $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

Area of a Parallelogram

Find the areas of the parallelograms whose vertices are given in Exercises 35–40.

35. $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, $D(0, -1)$
36. $A(0, 0)$, $B(7, 3)$, $C(9, 8)$, $D(2, 5)$
37. $A(-1, 2)$, $B(2, 0)$, $C(7, 1)$, $D(4, 3)$
38. $A(-6, 0)$, $B(1, -4)$, $C(3, 1)$, $D(-4, 5)$
39. $A(0, 0, 0)$, $B(3, 2, 4)$, $C(5, 1, 4)$, $D(2, -1, 0)$
40. $A(1, 0, -1)$, $B(1, 7, 2)$, $C(2, 4, -1)$, $D(0, 3, 2)$

Area of a Triangle

Find the areas of the triangles whose vertices are given in Exercises 41–47.

41. $A(0, 0)$, $B(-2, 3)$, $C(3, 1)$
42. $A(-1, -1)$, $B(3, 3)$, $C(2, 1)$
43. $A(-5, 3)$, $B(1, -2)$, $C(6, -2)$
44. $A(-6, 0)$, $B(10, -5)$, $C(-2, 4)$
45. $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, -1)$
46. $A(0, 0, 0)$, $B(-1, 1, -1)$, $C(3, 0, 3)$
47. $A(1, -1, 1)$, $B(0, 1, 1)$, $C(1, 0, -1)$

The points on the plane easiest to find from the plane's equation are the intercepts. If we take P to be the y-intercept $(0, 3, 0)$, then

$$\begin{aligned}\overrightarrow{PS} &= (1-0)\mathbf{i} + (1-3)\mathbf{j} + (3-0)\mathbf{k}\\ &= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},\\ |\mathbf{n}| &= \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.\end{aligned}$$

The distance from S to the plane is

$$\begin{aligned}d &= \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| && \text{Length of } \operatorname{proj}_{\mathbf{n}} \overrightarrow{PS}\\ &= \left|(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)\right|\\ &= \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}.\end{aligned}$$

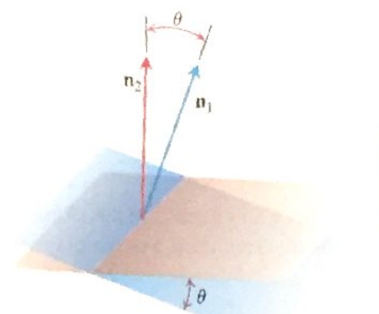


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.

Angles Between Planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors (Figure 12.42).

EXAMPLE 12 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The vectors

$$\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \qquad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

are normals to the planes. The angle between them is

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)\\ &= \cos^{-1}\left(\frac{4}{21}\right)\\ &\approx 1.38 \text{ radians.} && \text{About 79 degrees}\end{aligned}$$

Exercises 12.5

Lines and Line Segments

Find parametric equations for the lines in Exercises 1–12.

1. The line through the point $P(3, -4, -1)$ parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$
2. The line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$
3. The line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$
4. The line through $P(1, 2, 0)$ and $Q(1, 1, -1)$
5. The line through the origin parallel to the vector $2\mathbf{j} + \mathbf{k}$
6. The line through the point $(3, -2, 1)$ parallel to the line $x = 1 + 2t, y = 2 - t, z = 3t$
7. The line through $(1, 1, 1)$ parallel to the z-axis
8. The line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$
9. The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$
10. The line through $(2, 3, 0)$ perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$
11. The x-axis
12. The z-axis

Find parametrizations for the line segments joining the points in Exercises 13–20. Draw coordinate axes and sketch each segment, indicating the direction of increasing t for your parametrization.

13. $(0, 0, 0)$, $(1, 1, 3/2)$
14. $(0, 0, 0)$, $(1, 0, 0)$
15. $(1, 0, 0)$, $(1, 1, 0)$
16. $(1, 1, 0)$, $(1, 1, 1)$
17. $(0, 1, 1)$, $(0, -1, 1)$
18. $(0, 2, 0)$, $(3, 0, 0)$
19. $(2, 0, 2)$, $(0, 2, 0)$
20. $(1, 0, -1)$, $(0, 3, 0)$

Planes

Find equations for the planes in Exercises 21–26.

21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

22. The plane through $(1, -1, 3)$ parallel to the plane

$$3x + y + z = 7$$

23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$

24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$

25. The plane through $P_0(2, 4, 5)$ perpendicular to the line

$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$

26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A

27. Find the point of intersection of the lines $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, and $x = s + 2$, $y = 2s + 4$, $z = -4s - 1$, and then find the plane determined by these lines.

28. Find the point of intersection of the lines $x = t$, $y = -t + 2$, $z = t + 1$, and $x = 2s + 2$, $y = s + 3$, $z = 5s + 6$, and then find the plane determined by these lines.

In Exercises 29 and 30, find the plane containing the intersecting lines.

29. $L1: x = -1 + t, \quad y = 2 + t, \quad z = 1 - t; \quad -\infty < t < \infty$

$L2: x = 1 - 4s, \quad y = 1 + 2s, \quad z = 2 - 2s; \quad -\infty < s < \infty$

30. $L1: x = t, \quad y = 3 - 3t, \quad z = -2 - t; \quad -\infty < t < \infty$

$L2: x = 1 + s, \quad y = 4 + s, \quad z = -1 + s; \quad -\infty < s < \infty$

31. Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3$, $x + 2y + z = 2$.

32. Find a plane through the points $P_1(1, 2, 3)$, $P_2(3, 2, 1)$ and perpendicular to the plane $4x - y + 2z = 7$.

Distances

In Exercises 33–38, find the distance from the point to the line.

33. $(0, 0, 12)$; $x = 4t, \quad y = -2t, \quad z = 2t$

34. $(0, 0, 0)$; $x = 5 + 3t, \quad y = 5 + 4t, \quad z = -3 - 5t$

35. $(2, 1, 3)$; $x = 2 + 2t, \quad y = 1 + 6t, \quad z = 3$

36. $(2, 1, -1)$; $x = 2t, \quad y = 1 + 2t, \quad z = 2t$

37. $(3, -1, 4)$; $x = 4 - t, \quad y = 3 + 2t, \quad z = -5 + 3t$

38. $(-1, 4, 3)$; $x = 10 + 4t, \quad y = -3, \quad z = 4t$

In Exercises 39–44, find the distance from the point to the plane.

39. $(2, -3, 4)$, $\quad x + 2y + 2z = 13$

40. $(0, 0, 0)$, $\quad 3x + 2y + 6z = 6$

41. $(0, 1, 1)$, $\quad 4y + 3z = -12$

42. $(2, 2, 3)$, $\quad 2x + y + 2z = 4$

43. $(0, -1, 0)$, $\quad 2x + y + 2z = 4$

44. $(1, 0, -1)$, $\quad -4x + y + z = 4$

45. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.

46. Find the distance from the line $x = 2 + t$, $y = 1 + t$, $z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

Angles

Find the angles between the planes in Exercises 47 and 48.

47. $x + y = 1, \quad 2x + y - 2z = 2$

48. $5x + y - z = 10, \quad x - 2y + 3z = -1$

T Use a calculator to find the acute angles between the planes in Exercises 49–52 to the nearest hundredth of a radian.

49. $2x + 2y + 2z = 3, \quad 2x - 2y - z = 5$

50. $x + y + z = 1, \quad z = 0$ (the xy-plane)

51. $2x + 2y - z = 3, \quad x + 2y + z = 2$

52. $4y + 3z = -12, \quad 3x + 2y + 6z = 6$

Intersecting Lines and Planes

In Exercises 53–56, find the point in which the line meets the plane.

53. $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$

54. $x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$

55. $x = 1 + 2t, \quad y = 1 + 5t, \quad z = 3t; \quad x + y + z = 2$

56. $x = -1 + 3t, \quad y = -2, \quad z = 5t; \quad 2x - 3z = 7$

Find parametrizations for the lines in which the planes in Exercises 57–60 intersect.

57. $x + y + z = 1, \quad x + y = 2$

58. $3x - 6y - 2z = 3, \quad 2x + y - 2z = 2$

59. $x - 2y + 4z = 2, \quad x + y - 2z = 5$

60. $5x - 2y = 11, \quad 4y - 5z = -17$

Given two lines in space, either they are parallel, they intersect, or they are skew (lie in parallel planes). In Exercises 61 and 62, determine whether the lines, taken two at a time, are parallel, intersect, or are skew. If they intersect, find the point of intersection. Otherwise, find the distance between the two lines.

61. $L1: x = 3 + 2t, \ y = -1 + 4t, \ z = 2 - t; \quad -\infty < t < \infty$

$L2: x = 1 + 4s, y = 1 + 2s, z = -3 + 4s; \quad -\infty < s < \infty$

$L3: x = 3 + 2r, \ y = 2 + r, \ z = -2 + 2r; \quad -\infty < r < \infty$

62. $L1: x = 1 + 2t, \quad y = -1 - t, \quad z = 3t; \quad -\infty < t < \infty$

$L2: x = 2 - s, \quad y = 3s, \quad z = 1 + s; \quad -\infty < s < \infty$

$L3: x = 5 + 2r, \ y = 1 - r, \ z = 8 + 3r; \quad -\infty < r < \infty$

Theory and Examples

63. Use Equations (3) to generate a parametrization of the line through $P(2, -4, 7)$ parallel to $\mathbf{v}_1 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Then generate another parametrization of the line using the point $P_2(-2, -2, 1)$ and the vector $\mathbf{v}_2 = -\mathbf{i} + (1/2)\mathbf{j} - (3/2)\mathbf{k}$.

64. Use the component form to generate an equation for the plane through $P_1(4, 1, 5)$ normal to $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Then generate another equation for the same plane using the point $P_2(3, -2, 0)$ and the normal vector $\mathbf{n}_2 = -\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}$.

65. Find the points in which the line $x = 1 + 2t$, $y = -1 - t$, $z = 3t$ meets the coordinate planes. Describe the reasoning behind your answer.

66. Find equations for the line in the plane $z = 3$ that makes an angle of $\pi/6$ rad with $\mathbf{i}$ and an angle of $\pi/3$ rad with $\mathbf{j}$. Describe the reasoning behind your answer.

67. Is the line $x = 1 - 2t$, $y = 2 + 5t$, $z = -3t$ parallel to the plane $2x + y - z = 8$? Give reasons for your answer.