Differential Geometry.

I. Curves. (Ch. 2 of [Kühnel].)

S.C. Lau

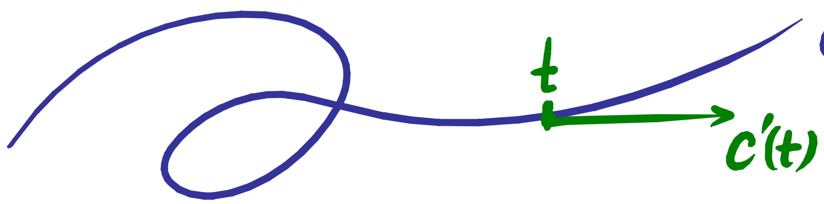
- Regular curve.
- Arc-length parametrization.
- Examples.
- Frenet frame and Frenet equations.
- dim $= 2, 3$.
- Approximations.
- Fundamental theorem of curves.

$\mathbb{R}^n$: vector space $/ \mathbb{R}$.

Def. 2.1 : A parametrized curve in $\mathbb{R}^n$ is

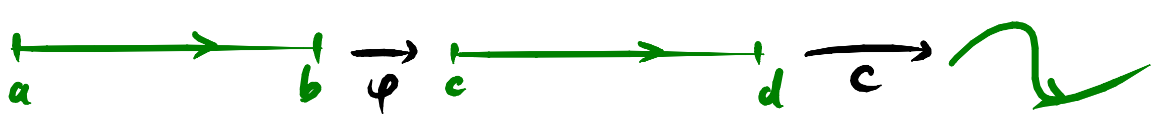
$$c : [a,b] \xrightarrow[C^\infty]{} \mathbb{R}^n .$$

$$t \longmapsto (c_1(t), \dots, c_n(t)).$$



- It is regular (reg.) if $\dot{c} \neq 0$ everywhere.
- For $\varphi : [a,b] \xrightarrow[\text{smooth}]{\sim} [c,d]$, $\dot{\varphi} > 0$ (oriented diffeomorphism),

get $c \xrightarrow[\text{reparametrization}]{} c \circ \varphi$.



Def.: $c_1 \sim c_2$ if

$$\exists\, \varphi : \mathrm{Dom}(c_2) \xrightarrow[\text{smooth}]{\simeq} \mathrm{Dom}(c_1),\ \dot{\varphi} > 0 \text{ s.t.}$$

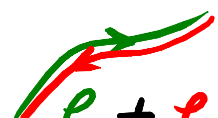
$$c_2 = c_1 \circ \varphi.$$

Note: $c_1 \sim c_2$, c_1 regular $\Rightarrow c_2$ regular.

ex. This is an equivalence relation.

Def.: A reg. curve is an equivalence class of reg. parametrized curves.

Note: A reg. curve comes with orientation.



$\mathcal{C}_1 \neq \mathcal{C}_2$

• Equip $\mathbb{R}^n$ with the standard inner product.

Def: Length of a reg. curve $\mathcal{C} = [c]$

$$\triangleq \int_{\mathrm{Dom}(c)} \|\dot{c}(t)\|\, dt.$$

ex. Length is well-defined, i.e. independent of parametrization.

Lemma 2.2: $\forall$ reg. curve $\mathcal{C}$,

curve has no local invariant.
$\Uparrow$

(canonical representative by using metric)

$$\exists\, c \in \mathcal{C} \text{ s.t. } \|\dot c\| \equiv 1.$$

(Unique up to $c(t) \longmapsto c(t+a)$.)

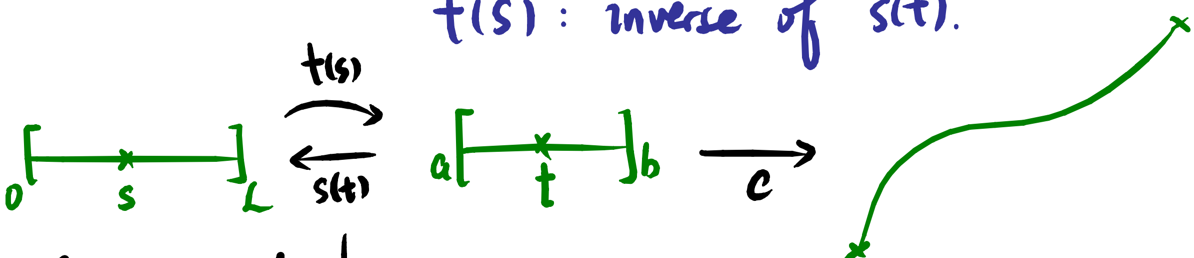
c is the arc-length parametrization of $\mathcal{C}$.

Pf: Take any parametrization $c : [a,b] \to \mathbb{R}^n$.

$$s : [a,b] \xrightarrow{\sim} [0,L]$$

$$s(t) \triangleq \int_a^t \|\dot c(\tau)\|\, d\tau. \quad (s' > 0.)$$

$t(s)$: inverse of $s(t)$.



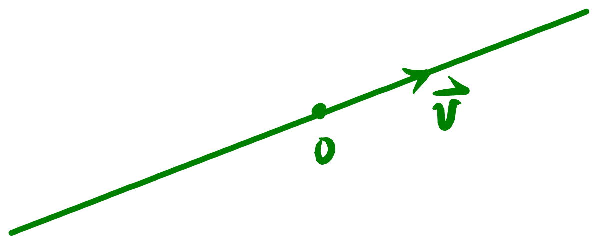
$$\dot{\tilde c}(s) \underset{\text{chain rule}}{=} \left.\frac{dc}{d\tau}\right|_{\varphi(s)} \cdot \underbrace{\frac{d\varphi}{ds}} = \left.\frac{ds}{dt}\right|^{-1}_{t=\varphi(s)} \underset{\text{funda. thm. of calculus}}{=} \left.\|\dot c\|^{-1}\right|_{\varphi(s)}$$

$$\therefore\ \|\dot{\tilde c}(s)\| = \|\dot c\|_{\varphi(s)} \|\dot c\|^{-1}_{\varphi(s)} = 1. \ \#$$

Examples.

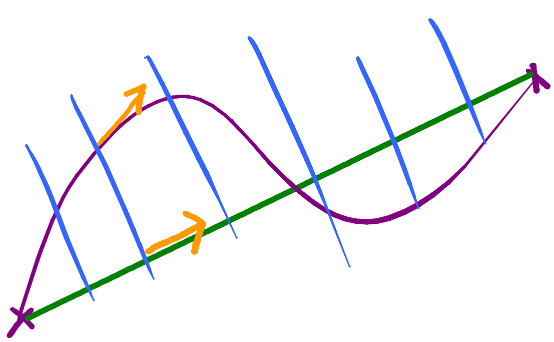
1. Straight line in $\mathbb{R}^n$.

$c(t) = t\vec{v}$ for $t \in (-\infty, +\infty)$.



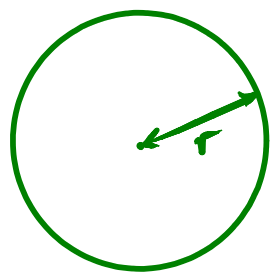
Ex. P.49 Q.4.

4. A regular curve between two points p, q in $\mathbb{R}^n$ with minimal length is necessarily the line segment from p to q. Hint: Consider the Schwarz inequality $\langle X, Y\rangle \leq ||X|| \cdot ||Y||$ for the tangent vector and the difference vector $p - q$.



2. Circle in $\mathbb{R}^2$.

$$c(\theta) = r(\cos\theta, \sin\theta) \quad \text{for } \theta \in [0, 2\pi].$$

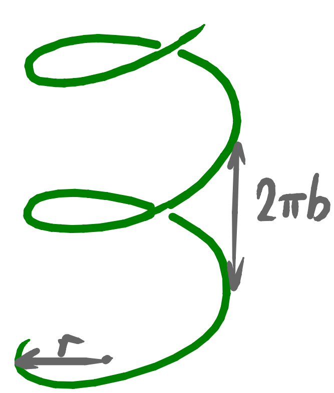


By any $\mathbb{R}^2 \xrightarrow[\text{affine linear}]{L} \mathbb{R}^n$

$L \circ c$ is a circle in $\mathbb{R}^n$.

3. Helix in $\mathbb{R}^3$.

$$c(t) = (r\cos t, r\sin t, bt).$$

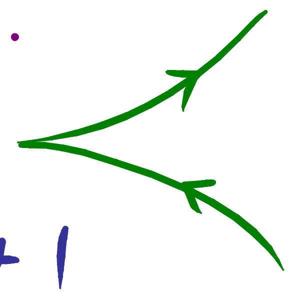


4. Neil parabola.

$$c(t) = (t^2, t^3).$$

coordinates are smooth,

but the image is not!



$$c'(0) = 0.$$

Ex. P.50 5

5. If all tangent vectors to the curve $c(t) = (3t, 3t^2, 2t^3)$ are drawn from the origin, then their endpoints are on the surface of a circular cone with axis the line $x - z = y = 0$.

Moving frame adapted to the curve

Def 2.4: c: reg. arc-length-parametrized.

- c is a Frenet curve if $\{c'(t), c''(t), \dots, c^{(n-1)}(t)\}$ is linearly independent (l.i.) $\forall t$.
- Frenet frame is $\{e_1(t), \dots, e_n(t)\}$ s.t. $\forall t$,

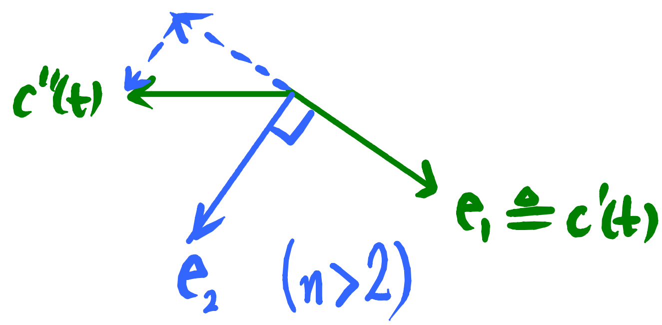
  i. $\{e_1(t), \dots, e_n(t)\}$ orthonormal basis (o.n.b.) and has $\underbrace{\text{(+) orientation}}_{\det(e_1 \dots e_n) > 0}$. (ori.)

  ii. $\mathrm{Span}(e_1 \dots e_k) = \mathrm{Span}(c', \dots, c^{(k)})$ $\forall k$.

  iii. $\langle c^{(k)}, e_k \rangle > 0$ $\forall k$.

• Frenet frame exists by
Gram-Schmidt orthogonalization.

e.g.



Ex.: P.49 Q.1.

1. The curvature and the torsion of a Frenet curve $c(t)$ in $\mathbb{R}^3$ are given by the formulas

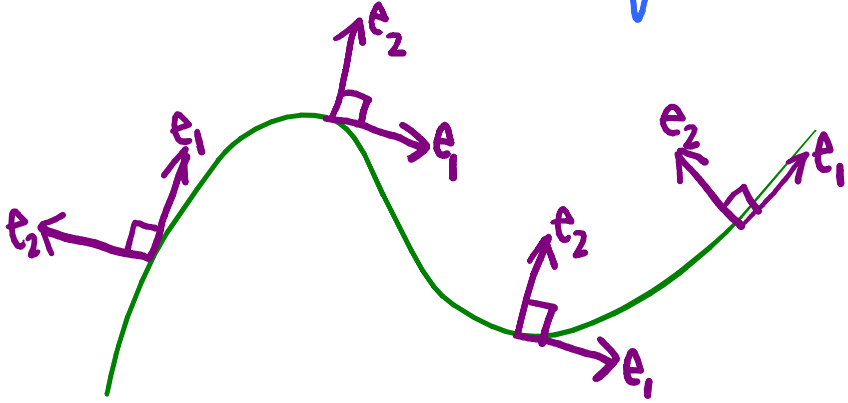
$$\kappa(t) = \frac{||\dot{c} \times \ddot{c}||}{||\dot{c}||^3} \quad \text{and} \quad \tau(t) = \frac{\mathrm{Det}(\dot{c}, \ddot{c}, \dddot{c})}{||\dot{c} \times \ddot{c}||^2}$$

for an arbitrary parametrization. For a plane curve we have $\kappa(t) = \mathrm{Det}(\dot{c}, \ddot{c})/||\dot{c}||^3$.

'a section of frame bundle'

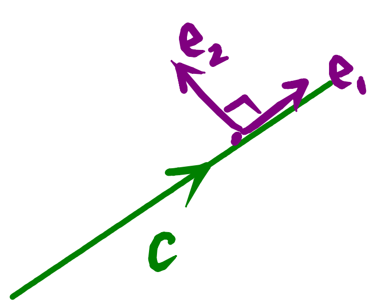
- **Moving frame** is just basis of $\mathbb{R}^n$ which varies with domain point $t \in \mathrm{Dom}(c)$.

It simplifies the equation of motions (Frenet eqn below).



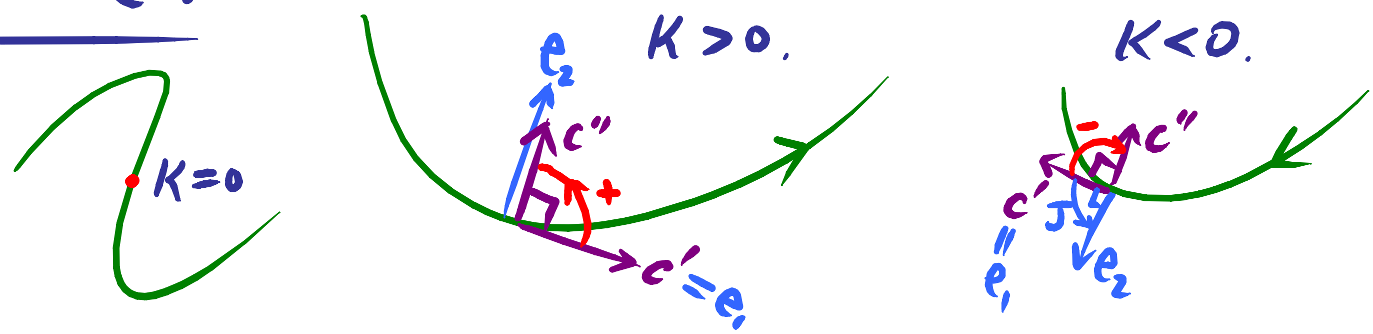
- **Care**: The last vector e_n is NOT defined by $c^{(n)}(t)$! It is determined by orientation and metric.

e.g.



$c'' = 0$, yet we can still define $\{e_1, e_2\}$.

$n=2$.



Prop.: (1). Every reg. curve is Frenet.

(2). c: Arc-length parametrized.

$e_1 = c'$, $e_2 = J\cdot e_1$. $J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ rotate by $\frac{\pi}{2}$.

$\forall t$, $c''(t) = K(t)\, e_2(t)$ for some $K(t) \in \mathbb{R}$.

K is curvature of c.

Pf: (1). $\|c'\| = 1$, so $\{c'\}$ is l.i.

(2). $\langle c', c' \rangle \equiv 1$.

$$\therefore\ 0 = (\langle c', c' \rangle)' = 2\langle \underbrace{c'}_{e_1}, c'' \rangle.$$

$$c'' = a e_1 + K e_2$$

and $a = \langle c'', e_1 \rangle = 0$. #

Care: This notion of curvature K is NOT intrinsic to $\mathcal{C}$ itself.

An ant walking on $\mathcal{C}$ (without external field from $\mathbb{R}^2$) CANNOT detect K.

Prop.: (Frenet equations, ODE)

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}' = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

It has a simple form because Frenet frame is taken.

Pf:

$$e_1' = c'' = K e_2.$$

$$\langle e_2', e_1 \rangle = \cancelto{0}{\langle e_2, e_1 \rangle'} - \langle e_2, e_1' \rangle = -K.$$

$$\langle e_2', e_2 \rangle = \frac{1}{2} \langle e_2, e_2 \rangle' = 0. \quad \#$$

Note: don't need to differentiate $\langle , \rangle$ since it is constant. Need to take care of this point for 'manifolds'. (general relativity)

Ex. P.50 8, 9.

8. The Frenet two-frame of a plane curve with given curvature function $\kappa(s)$ can be described by the exponential series for the matrix

$$\begin{pmatrix} 0 & \int_0^s \kappa(t)dt \\ -\int_0^s \kappa(t)dt & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix} \downarrow i \int_0^s K$$

It follows that

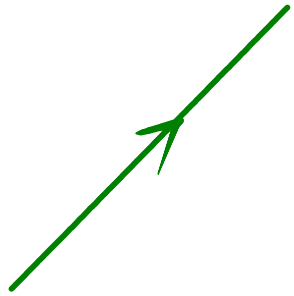
$$\begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix} = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{pmatrix} 0 & \int_0^s \kappa \\ -\int_0^s \kappa & 0 \end{pmatrix}^i. = e^{i\int_0^s K} = \begin{pmatrix} \cos\int^s K & -\sin\int^s K \\ \sin\int^s K & \cos\int^s K \end{pmatrix}$$

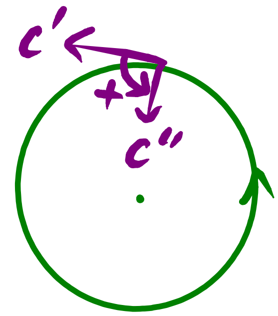
9. Let a plane curve be given in polar coordinates (r, φ) by $r = r(\varphi)$. Using the notation $r' = \frac{dr}{d\varphi}$, the arc length in the interval $[\varphi_1, \varphi_2]$ can be calculated as $s = \int_{\varphi_1}^{\varphi_2} \sqrt{r'^2 + r^2}\, d\varphi$, and the curvature is given by

$$\kappa(\varphi) = \frac{2r'^2 - rr'' + r^2}{(r'^2 + r^2)^{3/2}}.$$

e.g. Line.

$$c'' = 0$$
$$\Rightarrow K = 0.$$





e.g. Circle. $c = p + r(\cos\theta, \sin\theta)$.

NOT arc-length parametrized.

Suppose $\tilde{c}(s) = c(\theta(s))$ arc length parametrized.

$$c' = r(-\sin\theta, \cos\theta).$$

$$\tilde{c}'(s) = \frac{c'(\theta(s))}{\|c'(\theta(s))\|} = (-\sin\theta(s), \cos\theta(s)).$$

$$\tilde{c}''(s) = \underbrace{(-\cos\theta(s), -\sin\theta(s))}_{e_2} \cdot \underbrace{\theta'(s)}_{= 1/s'(\theta) = 1/\|c'(\theta)\| = \frac{1}{r}}$$

$$= \frac{1}{r} e_2.$$

$$\therefore K = \frac{1}{r}. \quad \#$$

(Alternatively: use explicit arc-length parametrization $p + r(\cos\frac{\theta}{r}, \sin\frac{\theta}{r})$.)

Ex. P.50 10, 11.

polar form

$\gamma(\varphi) = r(\varphi)(\cos\varphi, \sin\varphi)$.

10. Calculate the curvature of the curve given by $r(\varphi) = a\varphi$ (a constant), the so-called *Archimedean spiral*, see Figure 2.12.

11. Show the following: (i) The length of the curve given in polar coordinates by $r(t) = \exp(t), \varphi(t) = at$ with a constant a (the *logarithmic spiral*) in the interval $(-\infty, t]$ is proportional to the radius $r(t)$, see Figure 2.12. (ii) The position vector of the logarithmic spiral has a constant angle with the tangent vector.

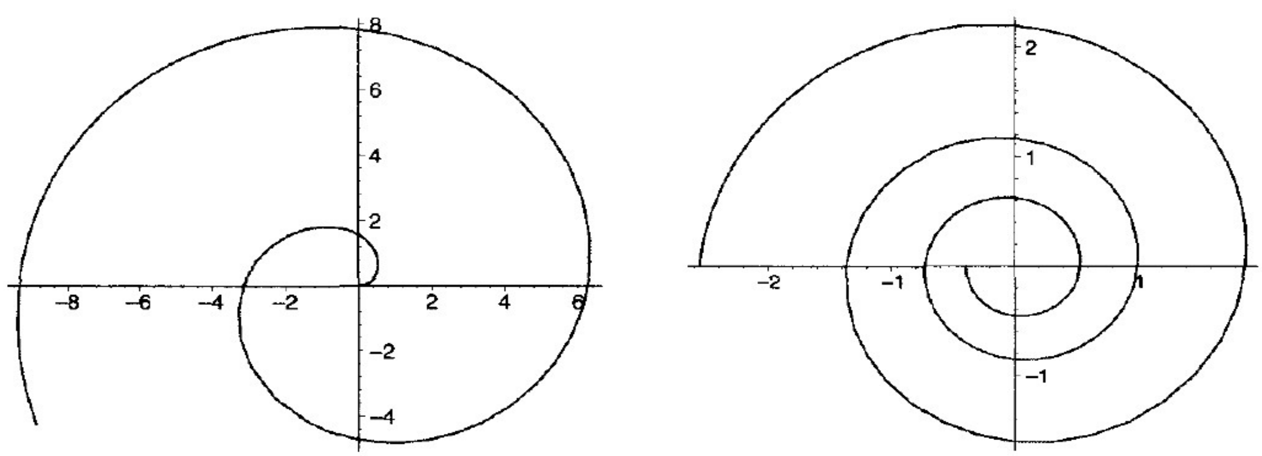


Figure 2.12. Archimedean spiral and logarithmic spiral

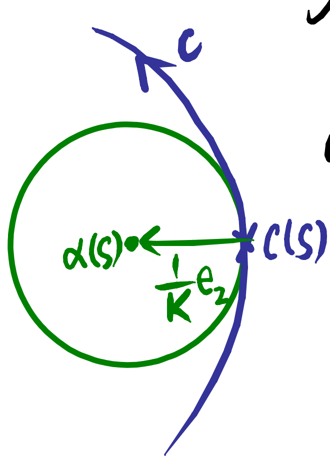
Thm. 2.6: κ constant $\Leftrightarrow$ line or circle.

$(\kappa \equiv 0)$ $\quad$ $\left(|\kappa| = \frac{1}{r}\right)$.

Pf: $\Leftarrow$) From previous direct calculation.

$\Rightarrow$) Follows from uniqueness of solutions to $\underbrace{\text{Frenet equation}}_{1^{st}\text{ order system}}$.

Alternatively, consider focal curve (may not be regular)



$$\alpha(s) = c(s) + \frac{1}{\kappa} e_2(s).$$

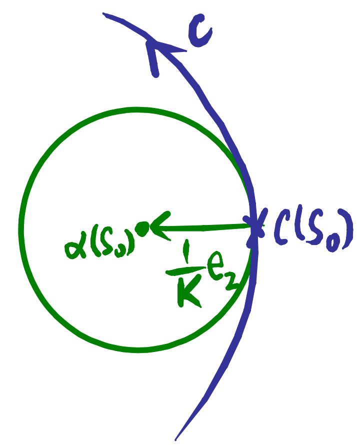
$$\alpha' = e_1 + \frac{1}{\kappa} \underbrace{e_2'}_{-\kappa e_1} = 0.$$

$$\therefore \alpha \equiv \text{const } p \Rightarrow c(s) = p - \underbrace{\frac{1}{\kappa}}_{\text{constant length}} e_2(s). \#$$

Osculating circle of c at s_0

$$\gamma(\theta) = \alpha(s_0) + \frac{1}{|K(s_0)|}(\cos\theta, \sin\theta).$$ (not arc-length parametrized)

$$\left.\frac{d^{(i)}}{ds^{(i)}}\right|_{s=s_0} \|c(s) - \alpha(s_0)\| = 0 \quad \text{for } i = 1, 2:$$



$\|c(s) - \alpha(s_0)\| > 0$ for $s \sim s_0$.

$\therefore$ It is smooth for $s \sim s_0$.

$$\frac{d}{ds}\left(\|c(s) - \alpha(s_0)\|^2\right) = \langle \underbrace{c'(s)}_{e_1(s)}, \underbrace{c(s) - \alpha(s_0)}_{-\frac{1}{K(s_0)^2} e_2(s_0) \text{ for } s=s_0} \rangle \overset{s=s_0}{=} 0$$

$$\frac{d^2}{ds^2}\left(\|c(s) - \alpha(s_0)\|^2\right) = \langle c''(s), c(s) - \alpha(s_0)\rangle + \underbrace{\langle c'(s), c'(s)\rangle}_{1}$$

$$\underset{s=s_0}{=} \underbrace{\langle c''(s_0), -\frac{1}{K(s_0)} e_2(s_0)\rangle}_{-\frac{1}{K(s_0)} \cdot K(s_0)} + 1$$

$$= 0. \ \#$$

Contact order k:
Regard c as a function f over γ for $s \sim s_0$ by orthogonal projection.
Then f has zero of order k at s_0.

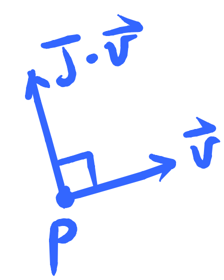
$\therefore \gamma$ has contact order 2 with c at $s = s_0$.

Rmk 2.7 : $K(s)$ determines c up to translation and rotation.

Pf: By existence and uniqueness of solution to Frenet equations:

(soln) (eqn)

$$\begin{cases} \begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ \begin{pmatrix} e_1(0) \\ e_2(0) \end{pmatrix} = \begin{pmatrix} \vec{v} \\ J\cdot\vec{v} \end{pmatrix}. \end{cases}$$



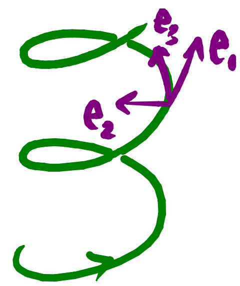
$\exists!$ soln. e_1, e_2.

$$\begin{cases} c'(s) = e_1(s) \\ c(0) = p. \end{cases}$$

Need the initial data $(p, \vec{v})$.

$\exists!$ soln c.

$n=3.$



Prop. : c: Arc-length parametrized.

(1). c is Frenet

$\Leftrightarrow c''(t) \neq 0 \quad \forall t.$

$\langle c', c'' \rangle = \frac{1}{2}\langle c', c' \rangle' \equiv 0.$

(2). $e_1 = c'$, $e_2 = \dfrac{c''}{\|c''\|}$, $e_3 = e_1 \times e_2$.

measure how tangent vector turns

$\kappa \triangleq \|c''\| > 0$: curvature of c.

(Then $c'' = \kappa e_2$ as in $n=2$ case.)

$\forall t$. $e_3'(t) = -\tau(t)\, e_2(t)$ for some $\tau(t) \in \mathbb{R}$.

measure how the plane containing c', c'' turns

τ : torsion of c.

(3) Frenet equations:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

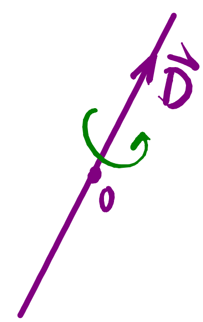
Darboux rotation vector.

Frenet matrix $\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$ is skew-symmetric

$\Rightarrow$ it is an infinitesimal rotation $\mathbb{R}^3 \circlearrowleft$

$\underset{\dim=3}{\Rightarrow}$ It is given by certain $\vec{D} \times (\cdot) : \mathbb{R}^3 \circlearrowleft$.

Finding $\vec{D} = a e_1 + b e_2 + c e_3$:

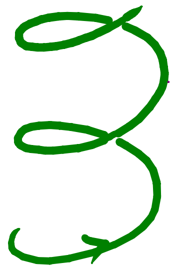


$$\begin{pmatrix} \vec{D} \times e_1 \\ \vec{D} \times e_2 \\ \vec{D} \times e_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$\Rightarrow \vec{D} = \tau e_1 + \kappa e_3$ Darboux rotation vector

$\|\vec{D}\| = \sqrt{\tau^2 + \kappa^2}$ speed of rotation.

Example : Helix.



$$c(t) = (a\cos\alpha t,\ a\sin\alpha t,\ bt).$$

$$a>0,\ \alpha\neq 0.$$

$$e_1 = c' = (-a\alpha\sin\alpha t,\ a\alpha\cos\alpha t,\ b).$$

Suppose $\|c'\|^2 = a^2\alpha^2+b^2 \equiv 1.$

$$c'' = (-a\alpha^2\cos\alpha t,\ -a\alpha^2\sin\alpha t,\ 0).$$

$$\kappa = \|c''\| = a\alpha^2.$$

$$e_2 = (-\cos\alpha t,\ -\sin\alpha t,\ 0).$$

$$e_3 = (b\sin\alpha t,\ -b\cos\alpha t,\ a\alpha).$$

$$e_3' = (b\alpha\cos\alpha t,\ b\sin\alpha t,\ 0) = -b\alpha\, e_2.$$

$$\tau = b\alpha.$$ #

Every c with const κ, τ is (part of) helix.

Darboux vector $= \tau e_1 + \kappa e_3 = (0, 0, \alpha).$

Ex. P.52 Q 17, 18, 19, 20, 21.

17. In the orthogonal (but not normal) three-frame $c', c'', c' \times c''$ the Frenet equations of a space curve take the equivalent form

$$\begin{pmatrix} c' \\ c'' \\ c' \times c'' \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{pmatrix} \begin{pmatrix} c' \\ c'' \\ c' \times c'' \end{pmatrix}.$$

Here the entries of the matrix depend in some sense rationally (i.e., without roots) on $\kappa^2 = \langle c'', c'' \rangle$ and τ (because of the relation $\kappa'/\kappa = \frac{1}{2}(\log(\kappa^2))')$.

18. Show the that the Frenet equations for a space curve are equivalent to the *Darboux equations* $e_i' = D \times e_i$ for $i = 1, 2, 3$, where $D = \tau e_1 + \kappa e_3$ is the *Darboux rotation vector.*

19. Show that the Darboux rotation vector D is perpendicular to e_1', e_2', e_3', and because of this lies in the kernel of the Frenet matrix. The normal form of the Frenet matrix is

$$\begin{pmatrix} 0 & \sqrt{\kappa^2 + \tau^2} & 0 \\ -\sqrt{\kappa^2 + \tau^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this normal form, the Darboux vector points in the direction of the third coordinate axis. Since the Frenet matrix is the derivative of the rotation of the Frenet three-frame, it follows that the Darboux vector points in this direction, and its length is the angular velocity. Similarly, the Darboux vector describes the accompanying screw-motion around that axis.

20. Show the following: c is a helix if and only if D is constant. c is a slope line if and only if $D/||D||$ is constant.

21. The axis of the accompanying screw-motion at a point $c(0)$ is the line in the direction of the Darboux vector $D(0) = \tau(0)e_1(0) + \kappa(0)e_3(0)$ through the point

$$P(0) = c(0) + \frac{\kappa(0)}{\kappa^2(0) + \tau^2(0)} e_2(0).$$

Show that under these circumstances the tangent to the curve which passes through all of these points, namely

$$P(s) = c(s) + \frac{\kappa}{\kappa^2 + \tau^2} e_2(s),$$

is proportional to $D(s)$ if and only if $\kappa/(\kappa^2 + \tau^2)$ is constant.

Pf : (1.) $\langle c', c'' \rangle = \frac{1}{2}(\underbrace{\langle c', c' \rangle}_{1})' = 0.$

$\therefore\ c'' \neq 0 \Rightarrow \{c', c''\}$ l.i. #

(2.) $e_3' = \underbrace{e_1' \times \underbrace{e_2}_{\propto e_1'}}_{0} + e_1 \times e_2' = e_1 \times e_2'.$

$\therefore\ \langle e_3', e_1 \rangle = \langle e_1 \times e_2', e_1 \rangle = 0.$

$\langle e_3', e_3 \rangle = \frac{1}{2}(\underbrace{\langle e_3, e_3 \rangle}_{1})' = 0.$

$\therefore\ e_3' = -\tau e_2$ for some τ. #

(3.) $e_1' = \|e_1'\|\, e_2 = \kappa e_2.$

$\langle e_2', e_1 \rangle = -\langle e_2, \underbrace{e_1'}_{\kappa e_2} \rangle = -\kappa.$

$\langle e_2', e_2 \rangle = \frac{1}{2}\langle e_2, e_2 \rangle' = 0.$

$\langle e_2', e_3 \rangle = -\langle e_2, \underbrace{e_3'}_{-\tau e_2} \rangle = \tau.$

$\therefore\ e_2' = -\kappa e_1 + \tau e_3.$ #

Approximation

$$\underset{\text{assume analytic}}{c(s)} = c(0) + \underbrace{c'(0)}_{e_1(0)} s + \frac{\overbrace{c''(0)}^{K(0)\,e_2(0)}}{2!} s^2 + \frac{\overbrace{c'''(0)}^{?}}{3!} s^3 + \ldots$$

$$c'' = K e_2.$$

$$c''' = K' e_2 + K \underbrace{e_2'}_{-Ke_1 + \tau e_3} = -K^2 e_1 + K' e_2 + K\tau e_3.$$

$$\therefore\ c(s) = c(0) + \left(s - K^2 \cdot \frac{s^3}{6}\right) e_1(0) + \left(K(0) \cdot \frac{s^2}{2} + K' \cdot \frac{s^3}{6}\right) e_2(0)$$

(Cor. 2.9)

$$+ K\tau \cdot \frac{s^3}{6} e_3(0) + o(s^3).$$

#

Rmk: $\tau = 0 \Rightarrow \overset{e_1 \times e_2}{\overset{\shortparallel}{e_3}} \equiv \text{const}$

$\Rightarrow \langle e_1, \text{const} \rangle = 0$

$\Rightarrow c \subset \text{plane}.$ #

Ex. P.49 Q2. P.51 Q.14

2. At every point p of a regular plane curve c with $c''(p) \neq 0$ (or, equivalently, $\kappa(p) \neq 0$) there is a parabola which has a point of third order contact with the curve at p. The point of contact is the vertex of the parabola if and only if $\kappa'(p) = 0$.
Hint: There is a two-parameter family of parabolas which have a given point as a point of contact on a given line. If we choose this line to be the tangent of a given curve at p, then by prescribing $\kappa(p)$ and $\kappa'(p)$, a unique parabola of the two-dimensional family is determined. The curvature of the parabola given by $x \mapsto (x, \frac{a}{2}x^2)$ calculates by Exercise 1 to $\kappa(x) = a(1 + a^2x^2)^{-3/2}$. This implies $\kappa'(x) = \frac{d\kappa}{dx} \cdot \frac{dx}{ds} = -3ax\kappa^2$. Consequently one can express a and x by κ und κ': $a = \kappa\left(1 + \frac{\kappa'^2}{9\kappa^4}\right)^{3/2}$ and $x = -\frac{\kappa'}{3a\kappa^2}$.

14. Show that the *osculating cubic parabola* of a Frenet curve c in $I\!R^3$, defined by

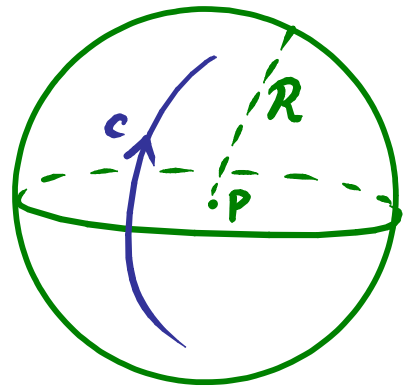
$$s \mapsto c(o) + se_1(0) + \frac{s^2}{2}\kappa(0)e_2(0) + \frac{s^3}{6}\kappa(0)\tau(0)e_3(0),$$

has at the point $s = 0$ the same curvature $\kappa(0)$ and torsion $\tau(0)$ as c itself.

Spherical curve

Suppose $\mathrm{Im}(c) \subset S^2_R = \{\|x\| = R\}$.

$$\langle c, c \rangle = R^2.$$



$$\langle \overbrace{c'}^{e_1}, c \rangle = 0.$$

$$\langle \underbrace{c''}_{\kappa e_2}, c \rangle = -\langle c', c' \rangle = -1.$$

$$\langle c, e_2 \rangle = -\frac{1}{\kappa}.$$

$$\langle \kappa' e_2 + \kappa \underbrace{e_2'}_{-\kappa e_1 + \tau e_3}, c \rangle = -\langle \kappa e_2, \underbrace{c'}_{e_1} \rangle = 0.$$

$$-\frac{\kappa'}{\kappa} + \kappa \tau \langle e_3, c \rangle = 0.$$

$$\langle c, e_3 \rangle = \frac{\kappa'}{\kappa^2 \tau}.$$

$$\therefore\ c = -\frac{1}{\kappa} e_2 + \frac{\kappa'}{\kappa^2 \tau} e_3.$$

$$\therefore\ \mathrm{Im}(c) \subset S^2_R(\overset{\text{center}}{p}) \Rightarrow c + \frac{1}{\kappa} e_2 - \frac{\kappa'}{\kappa^2 \tau} e_3 = p.$$

Conversely if $\quad p(s) \triangleq c + \frac{1}{\kappa} e_2 - \frac{\kappa'}{\kappa^2 \tau} e_3 \equiv \text{const } p,$

$$\text{i.e. } \; 0 \equiv p' = \cancel{c'} + \cancel{\frac{-\kappa'}{\kappa^2} e_2} + \frac{1}{\kappa}\left(\cancel{-\kappa e_1} + \tau e_3\right) + \cancel{\frac{\kappa'}{\kappa^2 \tau}\cdot \tau e_2} - \left(\frac{\kappa'}{\kappa^2 \tau}\right)' e_3$$

$$= \left(\frac{\tau}{\kappa} - \left(\frac{\kappa'}{\kappa^2 \tau}\right)'\right) e_3,$$

then $\quad \langle c-p, c-p \rangle' = 2 \underbrace{\langle c-p, c' \rangle}_{\langle -\frac{1}{\kappa} e_2 + \frac{\kappa^2}{\kappa^2 \tau} e_3,\; e_1 \rangle}$

$$= 0.$$

$$\therefore \quad \|c-p\| = \text{const}$$

$$\Rightarrow c \in S^2_R(p).$$

$\therefore \quad \mathrm{Im}(C) \subset S^2_R(p)$ for some p, R

$\Leftrightarrow \frac{\tau}{\kappa} - \left(\frac{\kappa'}{\kappa^2 \tau}\right)' = 0.$ (Thm 2.10 (iii)).

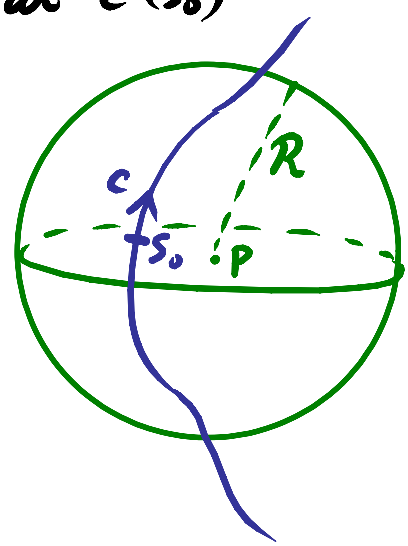
In general for any Frenet curve and $s = s_0$,

consider $\underbrace{S_R^2(p)}_{\text{osculating sphere}}$ for $p = \left.\left(c + \frac{1}{K} e_2 - \frac{K'}{K^2 \tau} e_3\right)\right|_{s=s_0}$.

$$R = \|c(s_0) - p\| = \left(\frac{1}{K^2} + \left(\frac{K'}{K^2 \tau}\right)^2\right)^{1/2}.$$

Thm. 2.10 (i). c and its osculating sphere S at $c(s_0)$ intersect w/ contact order 3,

i.e. $\left.\frac{d^{(i)}}{ds^{(i)}}\right|_{s=s_0} \|c(s) - p\| = 0 \quad \forall\, i = 1, 2, 3.$



Pf.: $\|c(s) - p\| \neq 0 \quad \forall s \sim s_0$.

$\therefore$ It is a smooth fcn.

Let $r(s) = \|c(s) - p\|^2$.

$$r' = 2 \langle c(s) - p, \underbrace{c'(s)}_{e_1} \rangle$$

$$r'' = 2\langle c-p, \underbrace{c''}_{\kappa e_2}\rangle + 2.$$

$$r''' = 2\langle c-p, c'''\rangle + \cancel{2\langle c', c''\rangle}^{\,0}.$$

$$r'(s_0) = 2\left\langle -\frac{1}{\kappa}e_2 + \frac{\kappa'}{\kappa^2\tau}e_3 \;,\; e_1\right\rangle = 0.$$

$$r''(s_0) = 2\left(\left\langle -\frac{1}{\kappa}e_2 + \frac{\kappa'}{\kappa^2\tau}e_3, \kappa e_2\right\rangle + 1\right) = 0.$$

$$r'''(s_0) = 2\left\langle -\frac{1}{\kappa}e_2 + \frac{\kappa'}{\kappa^2\tau}e_3 \,,\, \underbrace{c'''}_{\kappa' e_2 + \kappa e_2'}\right\rangle$$

$$= 2\left(-\frac{\kappa'}{\kappa} + \left(\frac{\kappa'}{\kappa^2\tau}\right)\kappa\tau\right) = 0.$$

$$\therefore \; \frac{d^{(i)}}{ds^{(i)}}\left(\|c(s)-p\|\right)\Big|_{s=s_0} = 0 \quad \forall i = 1, 2, 3. \quad \#$$

ex. For $\mathrm{Im}(c) \subset S^2_R$, derive a formula for κ and τ in terms of $j = \langle c \times c', c''\rangle$ similar to Thm 2.10 (iii).

Ex. P.51 Q.15.

15. In spherical coordinates φ, ϑ, let a regular curve be given by the functions $(\varphi(s), \vartheta(s))$ inside the sphere with parametrization $(\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, \sin\vartheta)$. For $s = 0$ the tangent to this curve is tangent to the equator $\vartheta = 0$, i.e., $\vartheta'(0) = 0$. Then the geodesic curvature is given by $\vartheta''(0) = \frac{d^2\vartheta}{ds^2}|_{s=0}$, and the curvature is consequently

$$\kappa(0) = \sqrt{1 + (\vartheta''(0))^2}.$$

Hint: 2.10 (iii), where the geodesic curvature is denoted J.

Can also approximate a Frenet curve c locally by helix.

$(\kappa(0), \tau(0))$ of c determines Helix $h = (a \cos\alpha s,\ a \sin\alpha s,\ bs)$

with $\alpha = (\kappa^2 + \tau^2)^{1/2}$

$a = \kappa / (\kappa^2 + \tau^2)$

$b = \pm \tau / (\kappa^2 + \tau^2)$.

Then match

$$\left(c(0), \begin{pmatrix} e_1^c \\ e_2^c \\ e_3^c \end{pmatrix}_{(0)} \right) \quad \text{with} \quad \left(h(0), \begin{pmatrix} e_1^h \\ e_2^h \\ e_3^h \end{pmatrix}_{(0)} \right)$$

by $SO(3) \ltimes \mathbb{R}^3$.

(rotation and translation)

General n.

Frenet eqn.

Thm 2.13

$$\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}' = \begin{pmatrix} 0 & K_1 & & & \\ -K_1 & \ddots & K_2 & & \\ & -K_2 & \ddots & \ddots & \\ & & \ddots & \ddots & K_{n-1} \\ & & & -K_{n-1} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

with $K_1, \dots, K_{n-2} > 0$.

Pf: $e_1 = c'$.

$e_1' = c''$. $\langle e_1', e_1 \rangle = \langle c'', c' \rangle = 0$

and $Lin(e_1, e_2) = Lin(c', c'')$

$\Rightarrow e_1' = K_1 e_2$.

$\langle c'', e_2 \rangle > 0 \Rightarrow K_1 > 0$.

Define $K_i = \langle e_i', e_{i+1} \rangle$.

$e_i \in Lin(c', \dots, c^{(i)})$

$\Rightarrow e_i' \in Lin(c', \dots, c^{(i+1)}) = Lin(e_1 \dots e_{i+1})$.

$\therefore \langle e_i', e_j \rangle = -\langle e_i, \underbrace{e_j'}_{\in Lin(e_1, \dots, e_{j+1})} \rangle = 0$ for $j < i-1$ and $j > i+1$.

$$\langle e_i', e_{i-1}\rangle = -\langle e_i, e_{i-1}'\rangle = -\kappa_i .$$

$$\kappa_i > 0 : \kappa_i = \langle \underset{\left(\sum_{l=1}^{i} a_l c^{(l)}\right)'}{\underset{\|}{e_i'}}, e_{i+1}\rangle = \underset{0}{\underset{\vee}{a_i}} \langle c^{(i+1)}, e_{i+1}\rangle \underset{0}{\underset{\vee}{}}\ (i+1<n)$$

$$\therefore \kappa_i > 0. \quad \#$$

κ_{n-1} is called the torsion.

$$\kappa_{n-1} = 0 \Rightarrow e_n' = 0$$

$$\Rightarrow e_n \text{ is const.}$$

$$\langle c', e_n\rangle = \langle e_1, e_n\rangle = 0.$$

$$\therefore \langle c', \text{const}\rangle = 0$$

$\kappa_{n-1} = 0 \iff c$ lies in hyperplane.

Ex. P.53 Q 23.

23. Let c be a Frenet curve in $\mathbb{R}^n$. Show that

$$\operatorname{Det}(c', c'', \ldots, c^{(n)}) = \prod_{i=1}^{n-1} (\kappa_i)^{n-i}.$$

Lemma: (Covariance of frame and curvature under rigid motion)
2.14

$$\tilde{c} = Ac + b. \quad A \in SO(n)$$
$$b \in \mathbb{R}^n$$

$$\Rightarrow \quad \tilde{e}_i = Ae_i \quad \forall i,$$
$$\tilde{K}_i = K_i \quad \forall i.$$

Pf: $\{\tilde{e}_i \triangleq Ae_i\}$ ori. o.n.b.,

$$\mathrm{Lin}(\tilde{e}_1, \dots, \tilde{e}_k) = A \cdot \mathrm{Lin}(e_1 \dots e_k)$$
$$= A \cdot \mathrm{Lin}(c^{(1)} \dots c^{(k)})$$
$$= \mathrm{Lin}(A \cdot c^{(1)} \dots A \cdot c^{(k)})$$
$$= \mathrm{Lin}(\tilde{c}^{(1)}, \dots, \tilde{c}^{(k)}).$$

$$\langle \tilde{c}^{(i)}, \tilde{e}_i \rangle = \langle Ac^{(i)}, Ae_i \rangle = \langle c^{(i)}, e_i \rangle$$
$$> 0.$$

$\therefore \{\tilde{e}_i\}$ is Frenet frame for $\tilde{c}$.

$$\tilde{K}_i = \langle \tilde{e}_i', \tilde{e}_{i+1} \rangle = \langle Ae_i', Ae_{i+1} \rangle$$
$$= K_i. \quad \#$$

Fundamental thm of curves

Thm 2.15.

Functions $(\underset{0}{\overset{\vee}{K_1}}, \dots, \underset{0}{\overset{\vee}{K_{n-2}}}, K_{n-1})$, $q \in \mathbb{R}^n$

and an ori. o.n.b. $(e_1^\circ, \dots, e_n^\circ)$ of $\mathbb{R}^n$

determine a Frenet curve c with

$$c(0) = q$$

Frenet frame $(e_1 \dots e_n)\big|_{(0)} = e_1^\circ \dots e_n^\circ$

and Frenet curvature given by K_i's.

Pf: By existence and uniqueness of soln to

$$\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & K_1 & & & \\ -K_1 & \ddots & K_2 & & \\ & -K_2 & \ddots & \ddots & \\ & & \ddots & \ddots & K_{n-1} \\ & & & -K_{n-1} & 0 \end{pmatrix}}_{K} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

get $F = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$. ($e_i$ are row vectors)

Need to check: F is a Frenet frame.

F is o.n.b.: F satisfies

$$\begin{cases} (FF^t)' = F'F^t + F(F')^t. \\ \qquad\quad = K(FF^t) - (FF^t)K. \\ (FF^t)\big|_0 = Id. \end{cases}$$

$FF^t \equiv Id$ is the unique sol.

$\therefore$ $\det F \neq 0$ and $\det F(0) = 1$

$\Rightarrow \det F > 0$. $(e_1 \dots e_n)$ ori.

$\begin{cases} c' = e_1 \\ c(0) = q \end{cases}$ determines c.

$c^{(\ell)} = e_1^{(\ell-1)} \in Lin(e_1 \dots e_\ell) \Rightarrow Lin(c^{(1)} \dots c^{(\ell)})$
$\subset Lin(e_1 \dots e_\ell)$.

$\langle c^{(\ell)}, e_\ell \rangle > 0$: Write $c^{(i)} = \sum_{\ell=1}^{i} a_\ell e_\ell$.

Inductive assumption: $a_i = \langle c^{(i)}, e_i \rangle > 0$.

$$\therefore \langle c^{(i+1)}, e_{i+1} \rangle = \underset{\substack{\vee \\ 0}}{a_i} \underbrace{\langle e_i', e_{i+1} \rangle}_{K_i > 0} > 0.$$

$\therefore$ $Lin(c^{(1)} \dots c^{(\ell)}) = Lin(e_1 \dots e_\ell)$.

$\therefore$ $\{e_1 \dots e_n\}$ is Frenet frame for c. #