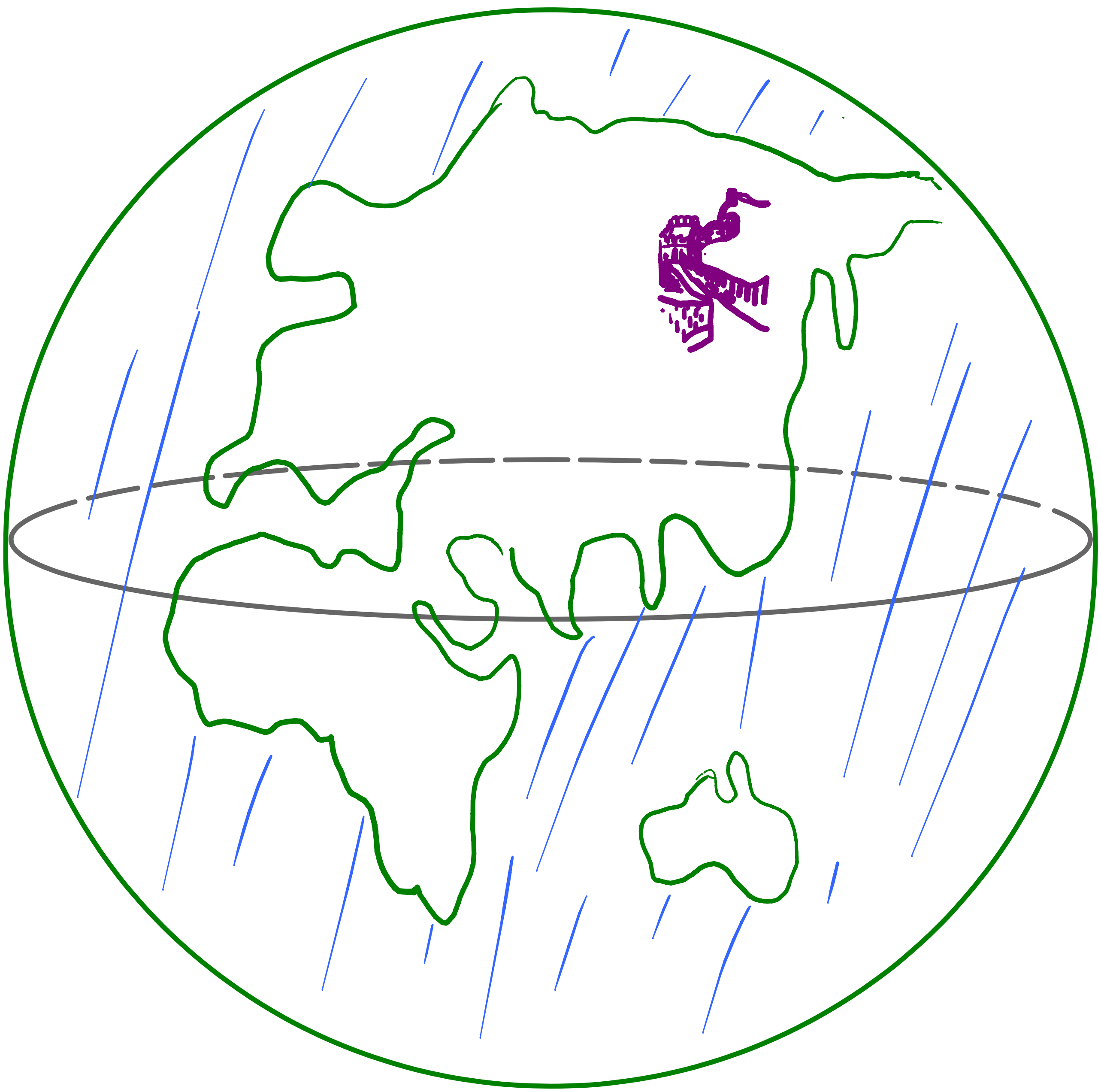
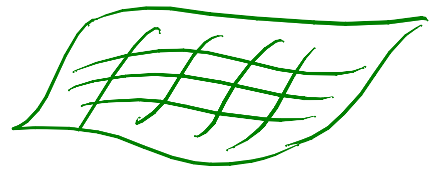
Differential Geometry

II. Surfaces (Ch. 3 of [Kühnel].)

S.C. Lau



Overview.

Surface : locally like $\mathbb{R}^2$ 

topologically !

NOT isometrically : $\nexists$ planar map isometric to Earth.

The 'shape' of surface is important :

captured by Gauss map

(i.e. normal vectors).

Can cook up intrinsic invariant called Gauss curvature.

- Regular surface.
- Tangent and normal bundles, vector fields.
- First fundamental form.
- Second fundamental form.
- Geodesics in surface
- Curvatures (principal, Gauss, mean)

Def.: A (local) parametrized surface is 

$$f : \underset{\substack{\cap \text{ open} \\ \mathbb{R}^2}}{U} \longrightarrow \mathbb{R}^3 .$$

Regular $\triangleq \operatorname{rk}(\underbrace{df}_{3\times 2 \text{ matrix}}) \equiv 2$. (i.e. f is immersed.)

(note: we allow self-intersection here, since we only care about local behaviour.)

A regular surface is an equiv. class of (oriented) parametrized regular surfaces up to (oriented) reparametrizations.

$$\tilde{U} \underset{\varphi}{\xrightarrow{\sim}} U \underset{f}{\longrightarrow} \mathbb{R}^3 .$$

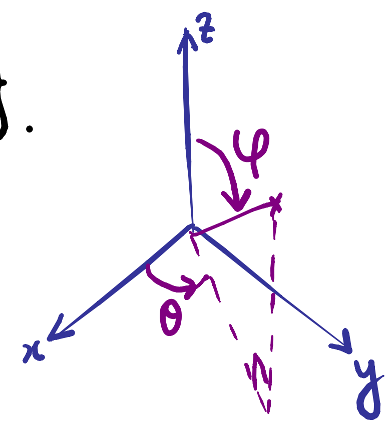
reparametrization $\to \varphi$

(oriented if $\det(d\varphi) > 0$.)

e.g. $S^2 \subset \mathbb{R}^3$.

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (\varphi, \theta) &\longmapsto (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi). \end{aligned}$$

$\text{Image} = S^2 \triangleq \{p \in \mathbb{R}^3 : \|p\| = 1\}$.



$$\begin{aligned} \{u^2+v^2<1\} = D &\longrightarrow \mathbb{R}^3 \\ (u,v) &\longmapsto (u, v, \sqrt{1-u^2-v^2}) \end{aligned}$$

$\text{Image} = \text{upper hemisphere} = \{(p_1, p_2, p_3) \in \mathbb{R}^3 : p_3 > 0\}$.

ex. Find all of their immersed points.

e.g. Graph of $h(u,v)$.

$$f(u,v) \triangleq (u, v, h(u,v)).$$

$\frac{\partial f}{\partial u} = (1, 0, \frac{\partial h}{\partial u})$, $\frac{\partial f}{\partial v} = (0, 1, \frac{\partial h}{\partial v})$ are linearly independent

$\Rightarrow f$ is immersed. #

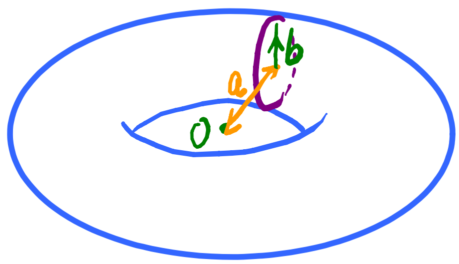
10. Investigate for which parameters the 3-sphere

ex. $$f(\phi, \psi, \theta) = (\cos\phi\cos\psi\cos\theta, \sin\phi\cos\psi\cos\theta, \sin\psi\cos\theta, \sin\theta)$$

is an immersion. Compare your results with the case of the two-dimensional sphere.

e.g. Torus. $0 < b < a$ constant.

$$f(u,v) = \big((a+b\cos u)\cos v, (a+b\cos u)\sin v, b\sin u\big)$$

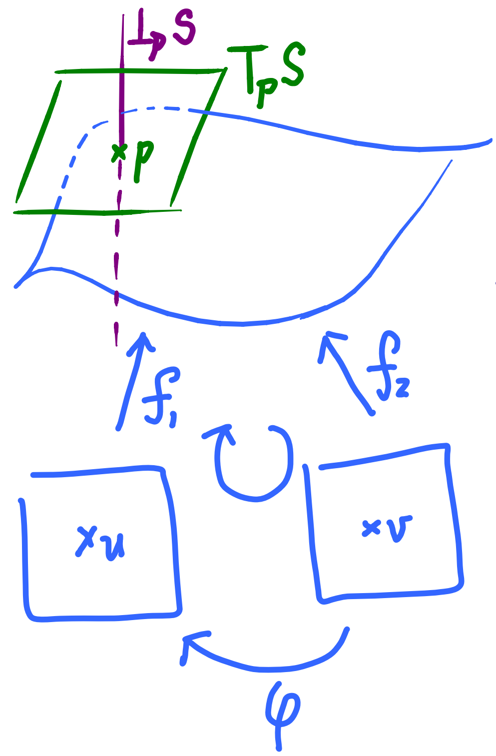


S: regular surface.

$$T_pS \triangleq \mathrm{Im}\left(df|_u\right). \quad \text{tangent space}$$

$$\perp_pS \triangleq \mathrm{Im}\left(df|_u\right)^{\perp}. \quad \text{normal space.}$$

$(p = f(u).)$



Independent of parametrization:

$$f_2(v) = f_1(\varphi(v)).$$

$$df_2|_v = df_1|_{\varphi(v)} \circ \underbrace{d\varphi|_v}_{\text{isom.}}$$

has the same image as $df_1|_{\varphi(v)}$.

$\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right\}$: standard basis of $\mathbb{R}^2$.

$\Rightarrow \left\{\frac{\partial f(u)}{\partial u_1}, \frac{\partial f(u)}{\partial u_2}\right\}$ basis of T_uS. also denoted as $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}$.

$$TS \triangleq \coprod_p T_pS \quad \text{tangent bundle.}$$

$$\perp S \triangleq \coprod_p \perp_pS \quad \text{normal bundle.}$$

Cotangent space

Recall dual vector space

$$V^* \triangleq \mathrm{Hom}(V, \mathbb{R}).$$

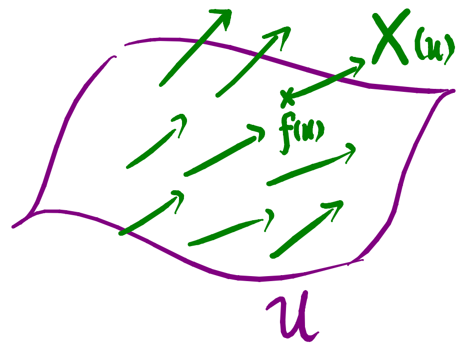
$$T_u^* f \triangleq (T_u f)^* \quad \text{cotangent space.}$$

$\{du_1, du_2\}$: dual basis of $\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right\}$.

$$\left(du_i \left(\frac{\partial}{\partial u_j} \right) \triangleq \delta_{ij}. \right)$$

$$T^* f \triangleq \bigcup_u T_u^* f \quad \text{cotangent bundle}$$

Vector fields. (v.f.)



Def. 3.5: A vector field along f is

$$X : \mathcal{U} \longrightarrow \mathbb{R}^3 .$$

X is a *tangent vector field* if

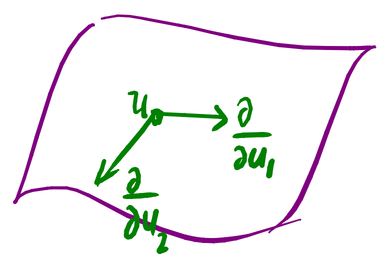
$$X(u) \in T_u f \quad \forall u .$$

X is a *normal vector field* if

$$X(u) \in \perp_u f \quad \forall u .$$

Tangent vector field can be written as

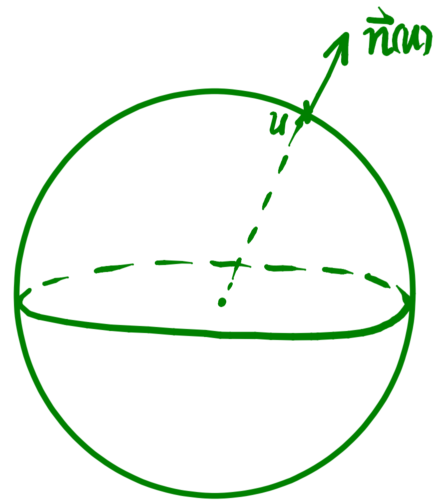
$$X(u) = \alpha_1(u) \frac{\partial}{\partial u_1} + \alpha_2(u) \frac{\partial}{\partial u_2} .$$



Normal vector field can be written as

$$X(u) = \alpha(u) \frac{\partial}{\partial u_1} \times \frac{\partial}{\partial u_2} .$$

e.g. Unit normal vector field on S^2.



$$X(\varphi,\theta) = f(\varphi,\theta)$$
$$= (\cos\varphi\cos\theta,\ \sin\varphi\cos\theta,\ \sin\theta).$$

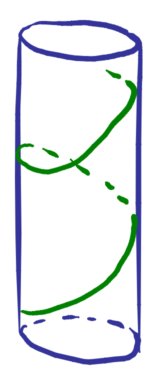
e.g. Helix contained in cylinder.

cylinder $f(\varphi,x) = (\cos\varphi,\ \sin\varphi,\ x)$.

helix $(\cos t,\ \sin t,\ tx_0 + c)$.

Integral curves of the tangent vector field

$$(-\sin\varphi,\ \cos\varphi,\ x_0).$$



First fundamental form. (metric : measure length and angles)

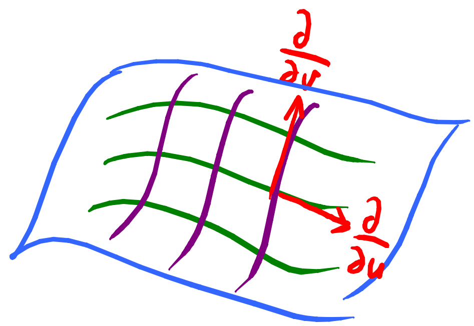
Equip $\mathbb{R}^3$ with standard inner product $\langle , \rangle$.

For $X, Y \in T_u f \subset \mathbb{R}^3$,

have $g(X,Y) \triangleq \langle X, Y \rangle$.

symmetric, bilinear, positive def.
(sym.) (bilin.) (pos. def.)

$$g = g_{ij} \triangleq \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$$ 2×2 matrix.
sym. pos. def.



$$ds^2 \triangleq g_{11}\, du_1^2 + 2 g_{12}\, du_1 du_2 + g_{22}\, du_2^2 .$$

(symmetric 2-tensor.) $(du_1 du_2 = du_2 du_1)$

$$g(X,Y) = (X_1 \;\; X_2) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

e.g. S^2.

$$f = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta)$$

$$\frac{\partial f}{\partial \varphi} = (-\sin\varphi\cos\theta, \cos\varphi\cos\theta, 0).$$

$$\frac{\partial f}{\partial \theta} = (-\cos\varphi\sin\theta, -\sin\varphi\sin\theta, \cos\theta).$$

$$(g_{ij}) = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & 1 \end{pmatrix}.$$

e.x. 1. Verify that the matrix g_{ij} of the first fundamental form of $f : U \to \mathbb{R}^{n+1}$ can be written as a matrix product $(Df)^T \circ (Df)$.

9. The *Mercator projection* (see Figure 3.30)

$$f(u, \varphi) = \frac{1}{\cosh u}(\cos\varphi, \sin\varphi, \sinh u)$$

is a parametrization of the surface of the sphere without the north and the south pole. Show that this parametrization is angle preserving, i.e., that u, φ are isothermal parameters. In the science of cartography, a map with this property is referred to as *angle preserving* or *conformal*. For more information concerning mathematical cartography, compare [**5**], §§66,67, or [**8**].

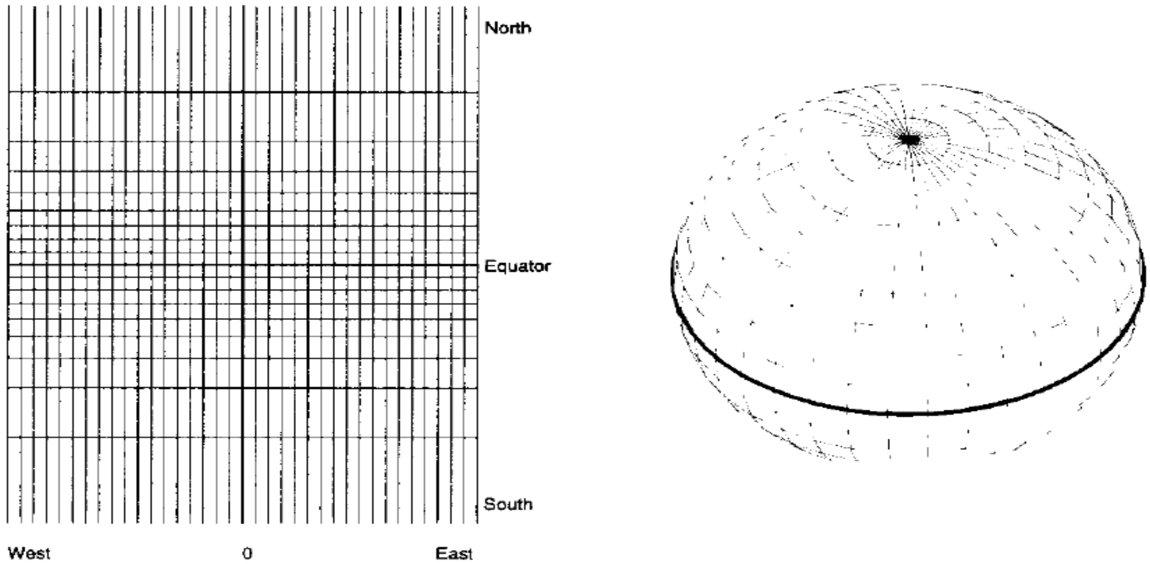


Figure 3.30. Coordinate grid of the Mercator projection

Lemma 3.3. Under reparametrization $\tilde{f} = f \circ \varphi$,

$$\left(\tilde{g}\right) = \left(D\varphi\right)^t \left(g\right)\left(D\varphi\right).$$

Pf :

$$\begin{array}{ccccc} \tilde{U} & \xrightarrow[\varphi]{} & U & \xrightarrow[f]{} & \mathbb{R}^3. \\ \Psi & & \Psi & & \\ \tilde{y} & & y & & \end{array}$$

$$\tilde{g}_{ij} = \left\langle \frac{\partial}{\partial \tilde{y}_i}, \frac{\partial}{\partial \tilde{y}_j} \right\rangle$$

$$\underset{\text{chain rule}}{=} \sum_{k,\ell} \left\langle \frac{\partial y_k}{\partial \tilde{y}_i} \frac{\partial}{\partial y_k}, \frac{\partial y_\ell}{\partial \tilde{y}_j} \frac{\partial}{\partial y_\ell} \right\rangle$$

$$= \sum_{k,\ell} \frac{\partial y_k}{\partial \tilde{y}_i} \frac{\partial y_\ell}{\partial \tilde{y}_j} g_{k\ell}$$

$$D\varphi = \left(\frac{\partial y_k}{\partial \tilde{y}_i}\right) \downarrow i, \quad \xrightarrow{k}$$

$$= \left[\left(D\varphi\right)^t \left(g\right)\left(D\varphi\right)\right]_{ij}.$$

ex 6. Let $f\colon [0, A] \times [0, B] \longrightarrow I\!R^3$ be a parametrized surface element. Show that the following conditions (i) and (ii) are equivalent:

(i) For each rectangle $R = [u_1, u_1 + a] \times [u_2, u_2 + b] \subset U$, the opposite sides of $f(R)$ are of equal length.

(ii) One has $\frac{\partial g_{11}}{\partial u_2} = \frac{\partial g_{22}}{\partial u_1} = 0$ in all of U.

The coordinate grid (or two-parameter family of curves) formed by the u_1 and the u_2 lines is called a *Tchebychev grid*. Show that under these conditions there is a parameter transformation $\varphi\colon U \to \widetilde{U}$ such that for $\widetilde{f} = f \circ \varphi^{-1}$ the first fundamental form can be written as

$$(\widetilde{g}_{ij}) = \begin{pmatrix} 1 & \cos\vartheta \\ \cos\vartheta & 1 \end{pmatrix},$$

where ϑ is the angle between the coordinate lines.

Hint: Set $\varphi(u_1, u_2) = \left(\int \sqrt{g_{11}}\,du_1, \int \sqrt{g_{22}}\,du_2\right)$.

Einstein summation. (Important to play with indices in Riemannian geometry!)

Upper/Lower index:

coordinates use upper: $(u^i) = (u^1, u^2)$.

basis of tangent space: $\left\{ \frac{\partial}{\partial u^i} \right\}$ (lower index)

basis of cotangent vector: $\{du^i\}$ (upper index)

tangent vector $= \sum_{i=1}^{2} a^i \frac{\partial}{\partial u^i} \triangleq a^i \frac{\partial}{\partial u^i}$.

written as $\begin{pmatrix} a^1 \\ a^2 \end{pmatrix}$

upper

Einstein's lazy convention

cotangent vector $= \alpha_i \, du^i$

written as (α_1, α_2)

lower

metric $g(u,v) = g_{ij} u^i v^j$.

Metric g identifies Tf with T^*f.

$$v \longmapsto g(v,\cdot) \triangleq \nu.$$

'lower the index' $v^i \longmapsto g_{ji} v^i = \nu_j$

It is also convenient to write

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \triangleq \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}. \quad g^{ij} g_{jl} = \delta_{il}.$$

$$\underbrace{g}_{\text{metric on } T^*f}(\overbrace{\mu,\nu}^{\text{identified as vectors} \in Tf \text{ as above}}) \triangleq g(u,v) = g_{ij} u^i v^j$$

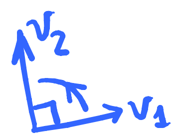
$$= g_{ij} (\mu_k g^{ki})(\nu_l g^{lj})$$

$$= \mu_j \nu_l g^{lj}.$$

$\therefore$ (g^{jl}) is the metric on T^*f.

Area form.

f : oriented regular surface.



Take oriented orthonormal basis $\{v_1, v_2\}$ of $T_u f$.

Let $\{\vartheta_1, \vartheta_2\}$ be the dual basis of $T_u^* f$.

Define $\vartheta_1 \wedge \vartheta_2$ (2-form) to be the area form at u.

wedge product : bilinear, skew-symmetric.

Independent of choice of oriented orthonormal basis :
(o.o.n.b.)

Let $\{(a\vartheta_1 + b\vartheta_2), (c\vartheta_1 + d\vartheta_2)\}$ be another o.o.n.b.

Then $(a\vartheta_1 + b\vartheta_2) \wedge (c\vartheta_1 + d\vartheta_2) = \underbrace{(ad - bc)}_{\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = 1} \vartheta_1 \wedge \vartheta_2 = \vartheta_1 \wedge \vartheta_2$.

#

In terms of coordinates:

Prop.: Area form $= \sqrt{\det g}\, du_1 \wedge du_2$.

$\{du_1, du_2\}$ defined as dual basis of $\{\frac{\partial}{\partial u_i}\}$.

Pf: Write o.o.n.b. as $(e_1 \ \ e_2) = (\partial_u \ \partial_v) \cdot A$.

Area form $\underbrace{k}_{\text{want to find}} \cdot du_1 \wedge du_2$.

$$1 = k\, du_1 \wedge du_2 (e_1, e_2) = k \cdot \det A.$$

$$\mathrm{Id} = \begin{pmatrix} e_1^t \\ e_2^t \end{pmatrix} \cdot (e_1 \ \ e_2) = A^t \underbrace{\begin{pmatrix} \partial_u f^t \\ \partial_v f^t \end{pmatrix} \cdot (\partial_u f \ \partial_u f)}_{g} \cdot A$$

$$= A^t \cdot g \cdot A.$$

$$\therefore (\det A)^2 (\det g) = 1.$$

$\therefore$ Area form $= \sqrt{\det g}\, du_1 \wedge du_2$. Denote by dA.

Note : $\det g = \left\| \frac{\partial}{\partial u_1} \times \frac{\partial}{\partial u_2} \right\|^2.$

$$\frac{\partial}{\partial u_1} = a v_1 + \alpha v_2.$$

$$\frac{\partial}{\partial u_2} = b v_1 + \beta v_2.$$

$$\frac{\partial}{\partial u_1} \times \frac{\partial}{\partial u_2} = (a\beta - b\alpha)\, v_1 \times v_2.$$

$$\therefore \| \;''\; \| = (a\beta - b\alpha)^2 = \det g. \quad \#$$

Def 3.4: For compact $Q \subset \mathcal{U}$ and $\alpha : f(\mathcal{U}) \to \mathbb{R}$.

$$\text{surface integral} \triangleq \int_{f(Q)} \alpha \, dA$$

$$= \int_Q \alpha \circ f \sqrt{\det g} \; du_1 \wedge du_2.$$

Independent of parametrization:

$$\begin{tikzcd}
\tilde{\mathcal{U}} \arrow[r, "\sim"', "\varphi"] & \mathcal{U} \arrow[r, "f"'] & \mathbb{R}^3 \\
\tilde{Q} \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\sim"] & Q \arrow[u, phantom, sloped, "\subset"] &
\end{tikzcd}$$

$$\int_{f(Q)} \alpha \, dA \underset{\substack{\text{change} \\ \text{of variable}}}{=} \int_{\tilde{f}(\tilde{Q})} \alpha \, dA.$$

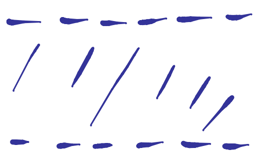
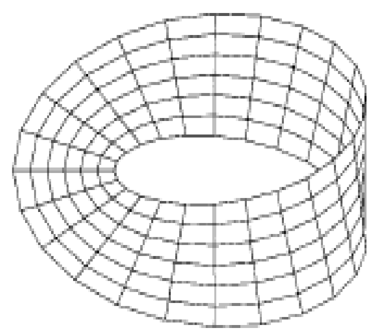
(dA is independent of choice of coordinates by defn.)

Gauss map.

In general $\vec{\nu} = \left(\frac{\partial}{\partial u_1} \times \frac{\partial}{\partial u_2}\right) \Big/ \left\| \frac{\partial}{\partial u_1} \times \frac{\partial}{\partial u_2} \right\|$ is a unit normal v.f. to a parametrized surface.

Def. 3.8: $\vec{\nu} : \mathcal{U} \to S^2$ is called the Gauss map.

Note: 1. It is defined over $\mathcal{U}$ instead of $f(\mathcal{U})$.

e.g.  $\xrightarrow{f}$  Fig. 3.5

$$f(u, v) = \left(\sin u + v \sin \frac{u}{2} \sin u, \cos u + v \sin \frac{u}{2} \cos u, v \cos \frac{u}{2}\right)$$

Möbius strip.

$\nexists$ unit normal v.f. over $f(\mathcal{U})$.

2. Two choices exist, depending on parametrization.

$(u_1, u_2) \longmapsto (u_2, u_1) \Rightarrow \vec{\nu} \longmapsto -\vec{\nu}$.

Def.: An oriented regular surface is an equiv. class of parametrized regular surfaces up to oriented reparametrizations. $\tilde{U} \xrightarrow[\varphi]{} U \xrightarrow[f]{} \mathbb{R}^3$. $\det d\varphi > 0$.

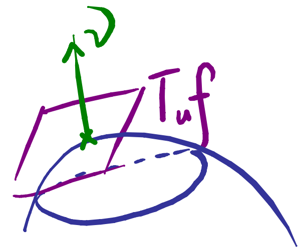
e.x. $\vec{\nu} = \left(\frac{\partial}{\partial u_1} \times \frac{\partial}{\partial u_2}\right) \Big/ \left\| \frac{\partial}{\partial u_1} \times \frac{\partial}{\partial u_2} \right\|$ is independent of oriented reparametrizations.

called outward unit normal

$-\vec{\nu}$ is inward " .

Weingarten map.

(analog of c'' in curve case.)



$\nu : U \longrightarrow S^2$ Gauss map.

$$\begin{tikzcd} T_uU \arrow[r, "d\nu_u"] \arrow[d, "df", "\wr"'] & T_{\nu(u)}S^2 = \perp(\nu(u)) = T_uf \\ T_uf \arrow[ur, "\text{define as } -L_u"'] & \end{tikzcd}$$

Weingarten map

Note the negative sign (a convention)

$$\langle \nu, \nu \rangle = 1$$

$$\Rightarrow X \cdot \langle \nu, \nu \rangle = 0 \quad \forall X \in T_uU$$

$$\| $$

$$2 \langle d\nu_u(X), \nu \rangle$$

$\therefore \operatorname{Im}(d\nu) \perp \nu(u).$

measure the rotation of tangent space.

Lemma 3.9. $\mathcal{L}$ is self-adjoint with respect to g.

Pf:

$$g(\mathcal{L}(df(X)), df(Y)) = \langle d\nu(X), df(Y) \rangle_{\mathbb{R}^3} = \langle X \cdot \nu, Y \cdot f \rangle$$
$$= -\langle \nu, X \cdot Y \cdot f \rangle$$
$$\parallel$$
$$g(df(X), \mathcal{L}(df(Y))) = \langle df(X), d\nu_u(Y) \rangle_{\mathbb{R}^3} = -\langle \nu, Y \cdot X \cdot f \rangle$$

#

Def: Second fundamental form (extrinsic curvature) is

$$h(X, Y) = g(\mathcal{L} \cdot X, Y) = h(Y, X) = -\langle X, d\nu(Y) \rangle.$$

h is represented by symmetric matrix

$$h_{ij} \triangleq h\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = \sum_{k=1}^{2} g_{kj} \underbrace{\mathcal{L}_i^k}.$$

$$\mathcal{L} \cdot \frac{\partial}{\partial u_i} \triangleq \mathcal{L}_i^k \frac{\partial}{\partial u_k}.$$

Third fundamental form is

$$\mathrm{III}(X, Y) = g(\mathcal{L} \cdot X, \mathcal{L} \cdot Y) = \mathrm{III}(Y, X).$$

Prop. $\mathrm{III} - \mathrm{Tr}(L)h + \det(L)\cdot g = 0.$

Thus (g, h) is enough to determine III.

Pf: L is self-adjoint

$\Rightarrow \exists$ o.n.b. $\{v_1, v_2\}$ at $u \in U$ s.t.

$$L(v_i) = \lambda_i v_i \quad \text{for } i = 1, 2.$$

Then $\mathrm{Tr}(L) = \lambda_1 + \lambda_2$, $\det L = \lambda_1 \lambda_2$.

Suffices to check the equality for basis.

$$\mathrm{III}(v_i, v_j) = g(L \cdot v_i, L \cdot v_j)$$

$$= \lambda_i \lambda_j \delta_{ij}.$$

$$h(v_i, v_j) = g(L \cdot v_i, v_j) = \lambda_i \delta_{ij}.$$

$$(\mathrm{III} - \mathrm{Tr}(L)h + \det(L)\cdot g)(v_i, v_j)$$

$$= \lambda_i \lambda_j \delta_{ij} - (\lambda_1 + \lambda_2)\lambda_i \delta_{ij} + \lambda_1 \lambda_2 \delta_{ij}$$

$$= \begin{cases} 0 & i \neq j \\ \lambda_i^2 - (\lambda_1 + \lambda_2)\lambda_i + \lambda_1 \lambda_2 = 0 & i = j. \end{cases} \#$$

e.g. $\mathbb{S}^2$.

Gauss map $\nu = f$.

$$\begin{tikzcd}
TU \arrow[r, "-d\nu = -df"] & Tf \\
Tf \arrow[u, "\wr"', "(df)^{-1}"] \arrow[ur, "L = -\mathrm{Id}"'] &
\end{tikzcd}$$

$\therefore \; g = -h = \mathrm{III}$.

Curves in surfaces.

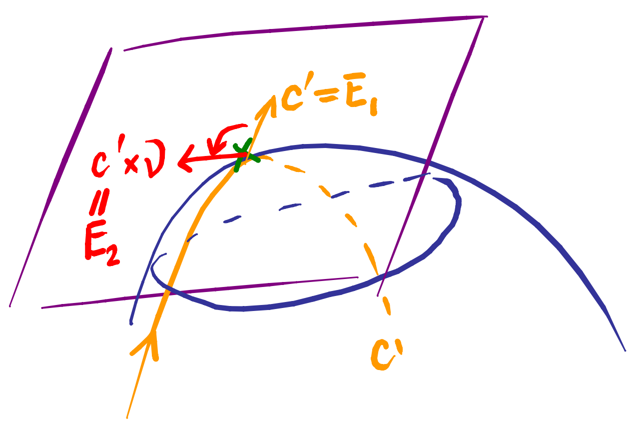
ex. 3. Let c be a curve parametrized by arc length, and suppose that its image is contained in a surface element $f\colon U \to I\!R^3$. The *Darboux three-frame* E_1, E_2, E_3 is then defined by the relations $E_1(s) = c'(s)$, $E_3(s) = \nu(c(s))$, $E_2(s) = E_3(s) \times E_1(s)$. Here, as usual, ν denotes the unit normal on the surface f.

Derive the following derivative equations for this three-frame, which correspond to the Frenet equations:

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_g & \kappa_\nu \\ -\kappa_g & 0 & \tau_g \\ -\kappa_\nu & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}.$$

The notations are as follows. The *geodesic curvature* is $\kappa_g = \langle c'', E_2 \rangle$, the *normal curvature* is $\kappa_\nu = II(c', c')$, and τ_g denotes a certain *geodesic torsion*.

$K_g \triangleq \langle c'', E_2 \rangle$ geodesic curvature.



Levi-Civita connection

Denote $\nabla_v \triangleq \underbrace{pr_T}_{\text{projection on tangent space } T} \circ \dfrac{\partial}{\partial v}$.

Then $\nabla_{\frac{d}{dt}} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} 0 & K_g \\ -K_g & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$.

(projection forget E_3 direction)

Def.: A regular curve in surface is called geodesic if $\nabla_{\frac{d}{dt}} c' \equiv 0$, i.e. $K_g \equiv 0$.

Write $c'' = \sum_{i=1}^{3} a_i E_i$. $\begin{cases} a_1 = \langle c'', c' \rangle = 0. \\ a_2 = K_g. \\ a_3 = K_\nu. \end{cases}$

$$= K_g E_2 + \underbrace{K_\nu}_{} E_3.$$

normal curvature $\triangleq \langle c'', \nu \rangle$.

$$K_\nu \triangleq \langle c'', \nu \rangle = -\langle c', \tfrac{d}{dt}\nu \rangle = -\langle c', d\nu(c') \rangle = h(c', c')$$

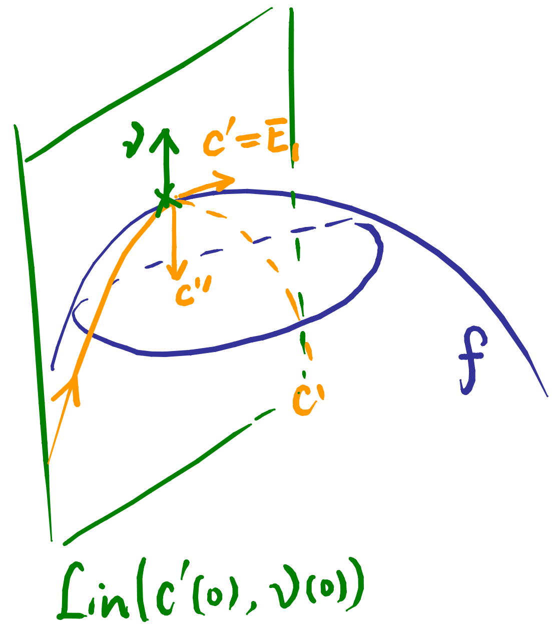
only depends on c'. $\therefore h$ can be identified as K_ν of a curve.

($\forall$ curves with the same direction, K_ν is the same.)

$$K = \|c''\| = \sqrt{K_g^2 + K_\nu^2} \geq h(c', c').$$

$\therefore$ c is geodesic $\Leftrightarrow K \equiv K_\nu$.

Suppose the plane $\mathrm{Lin}(c'(0), \nu(0))$ cut a curve c in f.



Then $c''(0) \perp E_2(0)$,

$$\therefore\ K_g(0) = 0.$$

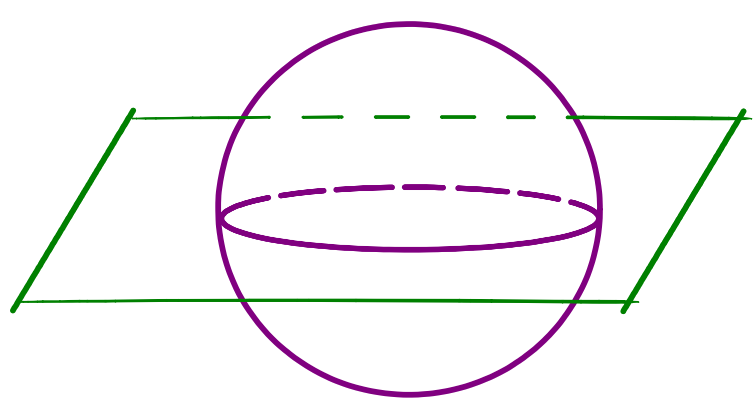
$$\therefore\ \|c''\| = K = |K_\nu| = |h(v,v)|.$$

In case $\nu(c(s))$ lies in the plane $\forall s$, then

$$K_g \equiv 0.$$

$\therefore$ c is a geodesic.

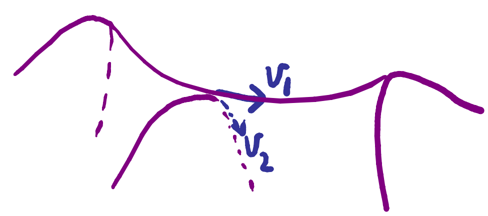
e.g. Great circles in a sphere are geodesics.



Def. 3.12: The eigenvalues $\kappa_1 \le \kappa_2$ of h (or equivalently $d\nu$) are called the principle curvatures.

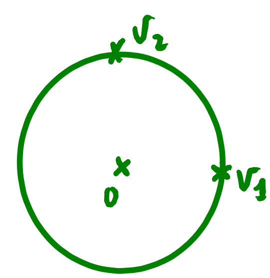
The $\underbrace{\text{eigenvectors}}_{\text{orthogonal to each other}}$ are principle curvature directions.

Let $\{v_1, v_2\} \subset Tf$ be o.n. eigenbasis.



$$h(av_1 + bv_2) = \kappa_1 a^2 + \kappa_2 b^2$$

On $S(Tf) = \{a^2+b^2=1\}$,



h is extremal at (a,b) $\Leftrightarrow$ $dh_{(a,b)} \perp TS(Tf)$

(Lagrangian multiplier)

$$\Leftrightarrow \; \underset{\substack{\| \\ 2(a\kappa_1 da + b\kappa_2 db)}}{dh_{(a,b)}} \sim \underset{\substack{\| \\ 2(a da + b db)}}{d(a^2+b^2)}$$

$$\Leftrightarrow \begin{cases} (a,b) = (1,0) \text{ or } (0,1) & \text{if } \kappa_1 \ne \kappa_2 \\ \text{any } (a,b) & \text{if } \kappa_1 = \kappa_2. \end{cases}$$

$\therefore$ Principle curvatures are max./min. among all directions.

Sign of $h(v,v)$ depends on orientation of f.

$K_1 K_2$ is independent!

Def 3.13: $K \triangleq K_1 K_2$ is called the Gaussian curvature. (intrinsic)

$H \triangleq \frac{1}{2}(K_1+K_2)$ is called the mean curvature. (extrinsic)

A point $p \in U$ is:

elliptic	if	$K>0$
hyperbolic		$K<0$
parabolic		$K=0, H\neq 0$
umbilical		$K_1=K_2$
level point		$K_1=K_2=0$.

$$H^2-K=\frac{1}{4}(K_1-K_2)^2 \geq 0.$$

"=" $\Leftrightarrow$ umbilical

e.x. 4. Show that at a fixed point p on a surface element, the mean curvature is equal to the integral mean of all normal curvatures, i.e.,

$$H(p) = \frac{1}{2\pi}\int_0^{2\pi} \kappa_\nu(\varphi)d\varphi.$$

Here we view κ_ν as a function of the angle φ, which parametrizes the set of unit vectors at this point (for example in some fixed orthonormal basis).

16. The *rotational torus* is given by

$$f(u,v) = \big((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u\big),$$

$0 \leq u, v \leq 2\pi$, cf. Figure 3.3. Here $a > b > 0$ are arbitrary (but fixed) parameters. Calculate the *total mean curvature* of this torus as the surface integral of the function $(H(u,v))^2$, $0 \leq u, v \leq 2\pi$, explicitly as a function of a and b. What is the smallest possible value of the total mean curvature?

Hint: The minimum occurs at $a = \sqrt{2}b$. Note that the integral is invariant under the homotheties $x \mapsto \lambda x$ of space with a fixed number λ.

Remark: The *Willmore conjecture* states that there is no immersed torus in $I\!R^3$ which has a smaller total mean curvature that the above rotational torus, no matter what is looks like geometrically. This conjecture has been verified in many cases, but in general it is still open (see [**17**], 5.1–5.3, 6.5).

17. For a surface element $f: U \to I\!R^3$ we define the *parallel surface* at distance ε by

$$f_\varepsilon(u_1, u_2) := f(u_1, u_2) + \varepsilon \cdot \nu(u_1, u_2),$$

cf. Section 3D. ν is the unit normal of the surface f. Decide for which ε this defines a regular surface, and show the following.

(a) The principal curvatures of f_ε and f have a ratio of $\kappa_i^{(\varepsilon)} = \kappa_i/(1 - \varepsilon\kappa_i)$.

(b) In case f has constant mean curvature $H \neq 0$, f_ε has constant Gaussian curvature for $\varepsilon = \frac{1}{2H}$.

Local expression.

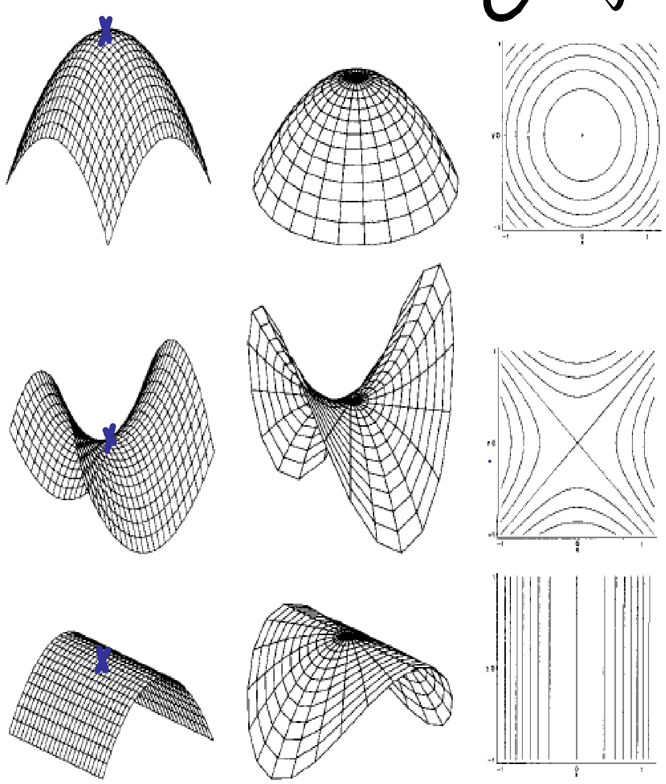
$$K = \det(d\nu) = \frac{\det(h_{ij})}{\det(g_{ij})}.$$

det & tr are numerical invariants of a linear map

$$H = \operatorname{tr}(d\nu)/2 = \frac{1}{2}\operatorname{tr}\Big(\sum_{j=1}^{2} g^{ij}h_{jk}\Big)$$

$$= \frac{1}{2}\sum_{i,j} g^{ij}h_{ji}$$

e.x. Compute K for the following surface at $(x,y)=0$.



elliptic — $a^2x^2+b^2y^2+c^2z^2=1$. ellipsoid

hyperbolic — $x^2+y^2-z^2=1$ hyperboloid

parabolic — $x^2+y^2=1$ cylinder

Figure 3.8. Elliptic, hyperbolic and parabolic points with level curves

Graph (h).

$$f(u_1,u_2) = (u_1, u_2, h(u_1,u_2)) \quad \text{where} \quad h(0)=0$$
$$dh(0) = 0.$$

(By rotation and translation, any surface locally can be made to this form.)

Then $\nu(0) = (0,0,1)$.

2nd fundamental form at $u=0$ $\quad h\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)\Big|_{u=0} = -\left\langle \frac{\partial f}{\partial u_i}, \overbrace{d\nu\left(\frac{\partial}{\partial u_j}\right)}^{\frac{\partial}{\partial u_j}\nu} \right\rangle$

$$= \langle f_{,ij}, \nu \rangle$$
$$= h_{,ij}(0).$$

$\therefore$ $(h_{ij}) = \mathrm{Hess}(h)$.

Note: Any reg. surface locally can be regarded as graph.

$\therefore$ Second fundamental form always has such an interpretation.

Thm 3.14: f: regular surface.

All points are umbilics ($\kappa_1 = \kappa_2$)

$\Leftrightarrow$ $\mathrm{Im}(f) \subset$ plane or sphere.

Pf: $\Leftarrow$) plane: $\kappa_1 = \kappa_2 = 0$.

sphere: $\kappa_1 = \kappa_2 = \frac{1}{r}$ (curvature of great circles)

$\Rightarrow$)

$$d\nu = \lambda \, df.$$

$\lambda = 0 \Rightarrow \nu \equiv \text{const} \Rightarrow \mathrm{Im} f \subset \text{plane}.$

Suppose $\lambda \neq 0$. $\left(\partial_i \nu = \lambda \partial_i f \ , \ 0 = \partial_j(\partial_i \nu) - \partial_i(\partial_j \nu). \right)$

$$0 = d(d\nu) = d\lambda \wedge df = \left(\frac{\partial \lambda}{\partial u_1} \frac{\partial f}{\partial u_2} - \frac{\partial \lambda}{\partial u_2} \frac{\partial f}{\partial u_1} \right) du_1 \wedge du_2$$

$\Rightarrow d\lambda = 0 \Rightarrow \lambda$ is const.

$$d(\nu - \lambda f) = 0 \Rightarrow \underbrace{f - \frac{\nu}{\lambda}}_{\text{center}} \text{ is const.}$$

$$\left\| \frac{\nu}{\lambda} \right\| = \frac{1}{|\lambda|} = \text{const} \Rightarrow \mathrm{Im}(f) \subset \text{sphere}.$$

#

Def. 3.15: A regular curve c is a **line of curvature** of a regular surface f if

$c'(s)$ is a principle curvature direction $\forall s$.

f is parametrized by lines of curvatures

($f(u_1, \cdot)$ and $f(\cdot, u_2)$ are lines of curvatures)

$$\Leftrightarrow \quad g_{ij} = h_{ij} = 0 \quad \forall i \neq j.$$

Surface without umbilics can be parametrized by lines of curvatures.