Differential Geometry

III. Intrinsic geometry of surfaces.

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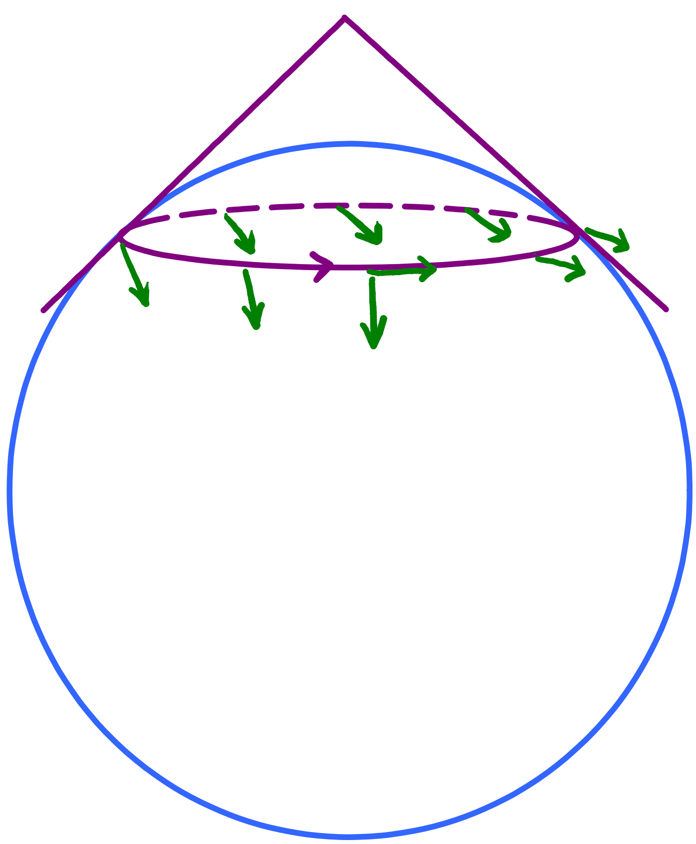
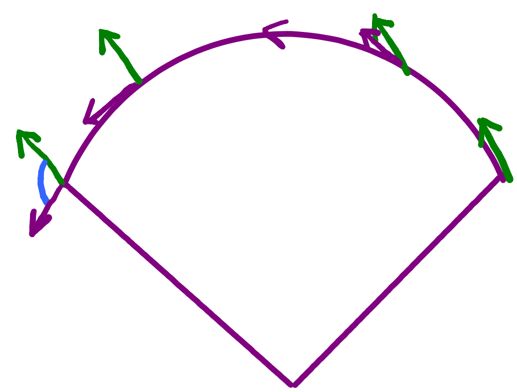
(Ch. 4 of [Kühnel].)





Recall: Y : vector field on $\mathbb{R}^3$ (or an open set).

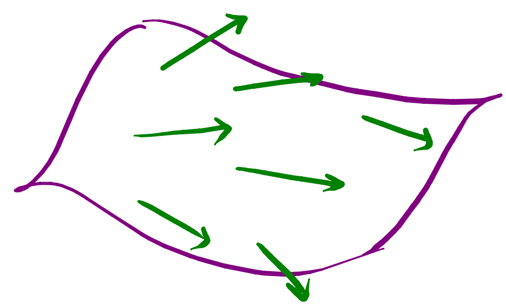
$X \in T_p\mathbb{R}^3$.

Directional derivative : denote $D_X Y$ or $X \cdot Y$.

$f : U \longrightarrow \mathbb{R}^3$ regular surface.

$Y : U \longrightarrow \mathbb{R}^3$ vector field.
(v.f.)

$X \in T_u f$.



Directional derivative : $D_X Y$.

identified as $X \in T_u U \simeq T_u f$

Note : Let c be a curve in the surface f with $c'(0) = X$.

Then $D_X Y = \left.\frac{d}{dt}\right|_{t=0} Y(c(t))$. (chain rule)

independent of choice of c!

Def. 4.3: $Y: U \longrightarrow \mathbb{R}^3$ tangent vector field.

$X \in T_u f$.

project to tangent direction

Covariant derivative $\nabla_X Y \triangleq (D_X Y)^{\text{tang}}$

$$= D_X Y - \langle D_X Y, \underbrace{\nu}_{\text{unit normal}} \rangle \nu.$$

$$= D_X Y - \langle \underbrace{Y, -d\nu \cdot X}_{\mathrm{II}(X,Y)} \rangle \nu$$

$$\therefore D_X Y = \nabla_X Y + \mathrm{II}(X,Y).$$

Note: Need Y to be vector field to define $\overbrace{\nabla_X Y}^{\text{not tensorial}}$.

But $\underbrace{\mathrm{II}(X,Y)}_{}$ is defined for Y being vectors at a point!

tensorial (it is a 'tensor': element in $(T^*)^{\otimes p} \otimes T^{\otimes q}$ for some (p,q).)

$(\mathrm{II} \in \mathrm{Sym}^2(T^* f).)$

Lemma 4.4: $\nabla_X Y$ satisfies:

connection
(1) Linearity in both X, Y over $\mathbb{R}$.
(2) Product rule: $\nabla_X(\varphi \cdot Y) = (X \cdot \varphi) Y + \varphi \nabla_X Y$.

Levi-Civita connection
(3) Preserve inner product:
$$X \cdot (g(Y_1, Y_2)) = g(\nabla_X Y_1, Y_2) + g(Y_1, \nabla_X Y_2).$$
(4). $[X, Y] = \nabla_X Y - \nabla_Y X$.

Pf: follows immediately from corresponding properties for $D_X Y$.

Note: $\nabla_X Y \neq \nabla_Y X$ in general
(where X, Y are vector fields.)

e.g. $\mathbb{R}^2$. $\nabla_{x_1 e_2} e_1 = 0$
$\qquad\qquad\qquad\qquad\qquad \nparallel$
but $\nabla_{e_1}(x_1 e_2) = e_2$.

Lie bracket

Def 4.5: $[X, Y] \triangleq D_X Y - D_Y X$ for X, Y tangent v.f.
$$= \nabla_X Y - \nabla_Y X. \quad (\because \mathrm{II}(X, Y) = \mathrm{II}(Y, X).)$$

Thm 4.6: $\nabla_X Y = \left(\partial_X Y^i + \underbrace{\Gamma^i_j(X)}_{2\times 2 \text{ matrix}} \underbrace{Y^j}_{\text{column vector}}\right) \partial_i .$

(short form: $\nabla = d + \Gamma$)

where $\Gamma^i_j(X) = \Gamma^i_{kj} X^k$,

$$\Gamma^i_{kj} = \frac{1}{2} g^{il} \left(\partial_k g_{jl} + \partial_j g_{kl} - \partial_l g_{kj} \right).$$

Pf: $\nabla_X \underbrace{Y}_{Y^i \partial_i} = (\partial_X Y^i) \partial_i + Y^j \underbrace{(\nabla_X \partial_j)}$

has the form $\Gamma^i_j(X)\partial_i = X^k \underbrace{\Gamma^i_j(\partial_k)}_{\triangleq \Gamma^i_{kj}} \cdot \partial_i$

$(\nabla_{\partial_k} \partial_j = \Gamma^i_{kj} \partial_i)$

$$\Gamma^i_{kj} g_{il} = g(\nabla_{\partial_k} \partial_j, \partial_l) = \partial_k g_{jl} - \underbrace{g(\partial_j, \nabla_{\partial_k} \partial_l)}_{\Gamma^i_{kl} g_{ij}}.$$

$$\Gamma^i_{kj} = g^{il} \partial_k g_{jl} - g^{il} \Gamma^p_{kl} g_{pj}$$

$$\Gamma^i_{jk} = g^{il}\partial_j g_{ki} - g^{il}\Gamma^p_{jl} g_{pk}$$

$$\underbrace{\Gamma^i_{kj} + \Gamma^i_{jk}}_{①\ 2\Gamma^i_{kj}} = g^{il}\left(\partial_k g_{ji} + \partial_j g_{ki} - \underbrace{\left(\Gamma^p_{kl} g_{pj} + \Gamma^p_{jl} g_{pk}\right)}_{②\ \partial_l g_{jk}}\right)$$

① $\Gamma^i_{kj} = \Gamma^i_{jk}$: $\nabla_{\partial_k}(\partial_j) - \nabla_{\partial_j}\partial_k = [\partial_j, \partial_k] = 0.$

② $\partial_l g_{jk} = \partial_l(g(\partial_j, \partial_k)) = \underbrace{g(\Gamma^p_{lj}\partial_p, \partial_k)}_{\Gamma^p_{lj} g_{pk}} + \underbrace{g(\partial_j, \Gamma^p_{lk}\partial_p)}_{\Gamma^p_{lk} g_{pj}}$

① is called torsion-free.

② is called metric-preserving.

∇ is a $\underbrace{\text{torsion-free metric preserving}}_{\text{Levi-Civita}}$ connection.

e.g. Plane in $(\mathbb{R}^3, \langle , \rangle_{std})$. $\Gamma_{ik}^{j} \equiv 0$.

$$\nabla = d.$$

$\therefore$ The Levi-Civita connection is just usual directional derivative.

Cor. 4.8. f: regular surface.

Gauss formula: $\partial_i \partial_j f = \Gamma_{ij}^{k} \partial_k f + h_{ij} \cdot \nu$.

Weingarten equation: $\partial_i \nu = -h_{ij} g^{jk} \partial_k f$

Pf: $\partial_i \partial_j f = D_{\partial_i}(\partial_j) = \nabla_{\partial_i} \partial_j + \mathrm{II}(\partial_i, \partial_j)$

$$= \Gamma_{ij}^{k} \partial_k f + h_{ij} \cdot \nu.$$

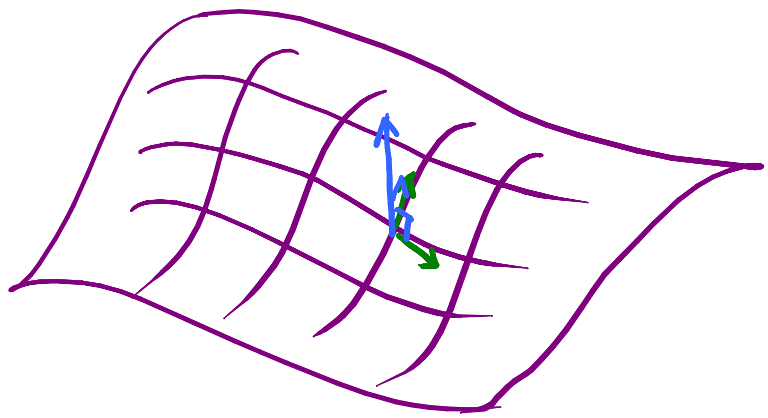
$$g(\underbrace{\partial_i \nu}_{a^k \partial_k}, \partial_j) = -h_{ij}$$

$$a^k g_{kj} = -h_{ij} \Rightarrow a^k = -g^{kj} h_{ij}$$

$$\therefore \partial_i \nu = -g^{kj} h_{ij} \partial_k f. \quad \#$$

Matrix form of $D_X Y = \nabla_X Y + \mathrm{II}(X,Y)$:

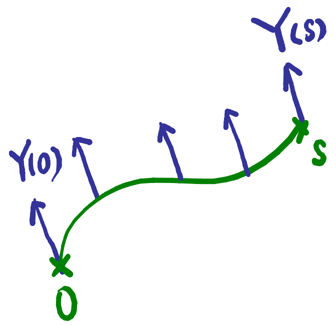
$$\partial_i \cdot \underbrace{\begin{pmatrix}\partial_1\\ \partial_2\\ \nu\end{pmatrix}}_{\text{Gauss frame}} = \begin{pmatrix}\Gamma_{i1}^1 & \Gamma_{i1}^2 & h_{i1}\\ \Gamma_{i2}^1 & \Gamma_{i2}^2 & h_{i2}\\ -h_i^1 & -h_i^2 & 0\end{pmatrix}\begin{pmatrix}\partial_1\\ \partial_2\\ \nu\end{pmatrix}$$



Def 4.9: A vector field Y is parallel if

$$\nabla_X Y = 0 \quad \forall X \in Tf.$$

A vector field $Y(s)$ along a curve $c(s)$ is parallel if $\nabla_{\frac{d}{ds}} Y(s) = 0 \quad \forall s.$



$Y(s)$ is said to be parallel transport of $Y(0)$ along c.

c is a geodesic if $\nabla_{\frac{d}{ds}} c'(s) \equiv 0.$

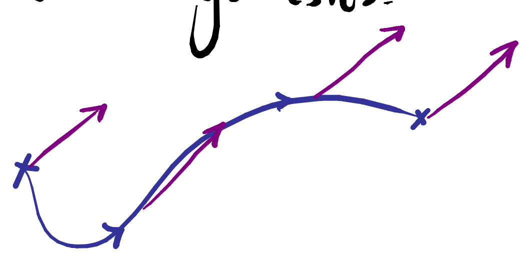
constant

e.g. Euclidean plane.

$$\nabla = d.$$

Constant vector fields are parallel.

Straight lines are geodesics.



Note: (Thm 4.10)
$$\begin{cases} \nabla_{\frac{d}{ds}} Y(s) = \left(Y^{k\,\prime}(s) + \Gamma^k_{ij} \cdot Y^j(s) \cdot (c^i)'(s) \right) \partial_k = 0 \\ Y(0) = Y_0 \end{cases}$$

is a 1^{st} order ODE system.

$\therefore \exists!$ solution.

$\therefore$ We can always parallel transport a vector along a curve.

Cor. 4.11 : Parallel transport preserves length of a vector.

Pf : $$\frac{d}{ds}\left(\langle Y(s), Y(s) \rangle \right) = \langle \underbrace{\nabla_{\frac{d}{ds}} Y(s)}_{0 \text{ if } Y \text{ is parallel}}, Y(s) \rangle \cdot 2$$

$\therefore \|Y\|$ is constant. #

The geodesic equation is

$$0 = \nabla_{\frac{d}{ds}} c'(s) = \left((c^k)''(s) + \Gamma^k_{ij} (c^i)'(s) (c^j)'(s) \right) \partial_k .$$

2nd order ODE

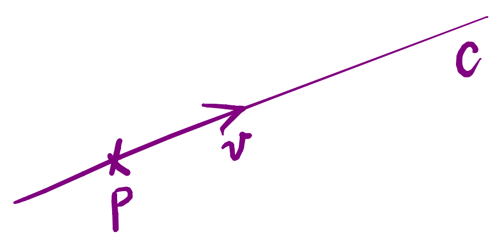
With initial condition $\begin{cases} c(0) = p \\ c'(0) = v \end{cases}$

$\exists !$ solution around $s = 0$. Thus

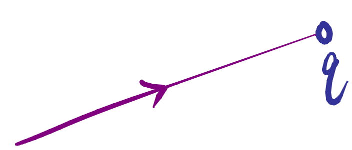
Thm 4.12: Given $p \in \mathcal{U}$ and $v \in T_p f$,

$\exists !$ geodesic $c(s)$ for $s \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ with

$$\begin{cases} c(0) = p \\ c'(0) = v. \end{cases}$$

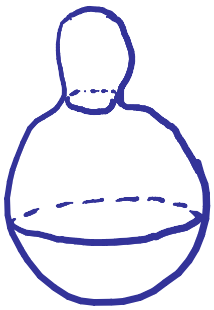


Note: It may not exist globally. e.g.



$\mathbb{R}^2 - \{q\}$.

e.g. $f = (r(t)\cos\varphi,\ r(t)\sin\varphi,\ h(t))$ surface of rotation.



$\{\varphi = \text{const } \varphi_0\}$ are geodesics (up to reparametrization):

Normal at (t,φ) is $(-h'(t)\cos\varphi, -h'(t)\sin\varphi, r'(t))$

which lies in the same plane

$$(-\sin\varphi_0, \cos\varphi_0, 0)^{\perp} + f(t_0,\varphi_0)$$

for $\varphi = \varphi_0$.

$\{\varphi = \text{const } \varphi_0\}$ is the intersection between f and this plane. #

ex. 1. Show that all geodesics on a circular cylinder

$$f(u, v) = (\cos u, \sin u, v)$$

are either Euclidean lines, circles, or helices. What do the geodesics on a circular cone look like?

2. Show that the geodesics on the surface of the sphere are precisely the great circles.

ex. 3. Suppose we are given a curve c on a surface element, which passes through a fixed point p. Show that the geodesic curvature $\kappa_g(p)$ of c coincides with the curvature $\kappa(p)$ of the plane curve which is obtained as the orthogonal projection of c in the tangent plane at p.

4. Show that (locally) a curve on a surface element is uniquely determined by the geodesic curvature as a function of the arc length, if one prescribes a point $c(0)$ and the direction $c'(0)$. Compare this with the plane case discussed in Section 2B as well as the case $\kappa_g = 0$ in 4.12.

5. Show that a Frenet curve on a surface element is a geodesic if and only if the unit normal to the surface coincides with the principal normal of the curve (at least up to sign).

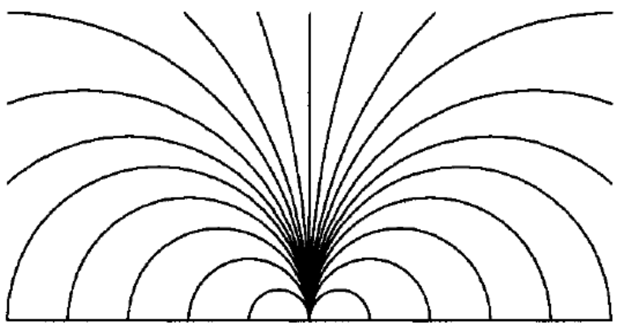


Figure 4.9. Geodesics in the Poincaré upper half-plane

11. The *Poincaré upper half-plane* is defined as the set $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ endowed with an abstractly given first fundamental form (or metric) $(g_{ij}) = \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Although this metric is not induced by a surface f in $\mathbb{R}^3$, one can nevertheless calculate the Christoffel symbols and the geodesics[14] as quantities of the intrinsic geometry, see Figure 4.9. Hint: The geodesics are the curves with constant x as well as the half-circles whose centers lie on the x-axis. Introduce appropriate polar coordinates.

12. Calculate the Gaussian curvature of the Poincaré upper half plane (along the lines of 4.26 (ii))

13. Show that for $z = x + iy \in \mathbb{C}$ all transformations

$$z \mapsto \frac{az+b}{cz+d}, \qquad a, b, c, d \in I\!R,\ ad - bc > 0,$$

are *isometries* of the Poincaré upper half-plane, i.e., preserve the abstract first fundamental form g_{ij} above.

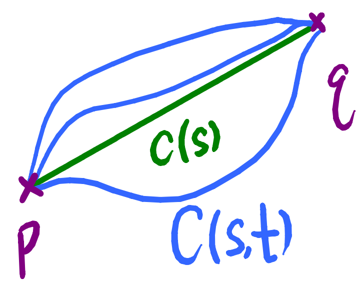
Thm. 4.13:

If c is the shortest path joining two given points, then c is a geodesic.

Pf: Consider $c_t(s) \triangleq C(s,t)$: family of paths

$$C(s,0) = c(s),\quad C(0,t) = p,\quad C(1,t) = q.$$

(c is parametrized by arc-length, but $c_t(s)$ may not for $t \neq 0$.)



$$L(t) \triangleq \int_0^L \left\| \frac{d}{ds} c_t(s) \right\| ds$$

c is minimal

$$0 \overset{\downarrow}{=} \frac{d}{dt}\Big|_{t=0} L(t) = \int_0^L \langle \underbrace{\partial_s c_0(s)}_{c'(s)}, \underbrace{\partial_t\big|_{t=0} \partial_s C(s,t)}_{\underbrace{\partial_s}_{\text{same as } \nabla_s \text{ when dot with } c'(s)} \partial_t|_{t=0} c_t(s)} \rangle \Big/ \underbrace{\|\partial_s c_0(s)\|}_{1} \, ds$$

(c_0 is arc-length parametrized)

$$= \int_0^L \langle c'(s), \nabla_s \partial_t\big|_{t=0} c_t(s) \rangle \, ds$$

$$= \Big[\langle c'(s), \underbrace{\partial_t\big|_{t=0} c_t(s)\rangle}_{\text{0 at end-points}} \Big]_{s=0}^{s=L} - \int_0^L \langle \nabla_s c'(s), \partial_t\big|_{t=0} c_t(s)\rangle \, ds$$

$$0 = -\int_0^L \langle \nabla_s c'(s), \partial_t\big|_{t=0} c_t(s)\rangle \, ds \quad \text{for any variation } c_t(s).$$

$$\therefore \quad \nabla_s c'(s) = 0, \text{ i.e. } c \text{ is a geodesic.} \ \#$$

ex. 8. Show that for a Tchebychev grid (cf. Exercise 6 in Chapter 3) the curvature is given by $K = -\frac{\partial^2 \vartheta}{\partial u_1 \partial u_2} / \sin \vartheta$.

> 6. Let $f\colon [0, A] \times [0, B] \to \mathbb{R}^3$ be a parametrized surface element. Show that the following conditions (i) and (ii) are equivalent:
> (i) For each rectangle $R = [u_1, u_1 + a] \times [u_2, u_2 + b] \subset U$, the opposite sides of $f(R)$ are of equal length.
> (ii) One has $\frac{\partial g_{11}}{\partial u_2} = \frac{\partial g_{22}}{\partial u_1} = 0$ in all of U.
> The coordinate grid (or two-parameter family of curves) formed by the u_1 and the u_2 lines is called a *Tchebychev grid*. ~~Show~~

14. Let $\lambda(x)$ be a positive differentiable function. For an abstract surface of rotation with metric $ds^2 = dx^2 + \lambda^2(x)dy^2$ ("warped product metric"), calculate the Christoffel symbols and show that the x-lines are geodesics parametrized by arc length. What do the rest of the geodesics look like?

15. Determine all functions λ in Exercise 14 such that the Gaussian curvature of this abstract surface of rotation is -1. Hint: Look at 4.28.

Integrability condition.

Given g and h,

want to solve back the surface f s.t.

$$\begin{cases} \partial_i \partial_j f = \Gamma_{ij}^k \partial_k f + h_{ij} \cdot \nu & \text{(Gauss formula)} \\ \partial_i \nu = -h_{ij} g^{jk} \partial_k f. & \text{(Weingarten equation)} \end{cases}$$

necessary condition:

$$0 = \underbrace{\partial_k(\partial_i \partial_j f)}_{(\partial_k \Gamma_{ij}^{\ell}) \underbrace{\partial_\ell f}_{\partial_\ell} + \Gamma_{ij}^{\ell} \underbrace{\partial_\ell \partial_k f}_{\underbrace{\nabla_\ell \partial_k}_{\Gamma_{\ell k}^p \partial_p} + h_{\ell k} \nu} + (\partial_k h_{ij}) \cdot \nu + h_{ij} \underbrace{(\partial_k \nu)}_{-h_{kp} g^{pq} \partial_q}} - \partial_j(\partial_i \partial_k f) \leftarrow \text{express this in basis } \{\partial_1, \partial_2, \nu\}$$

Coefficient of ∂_ℓ :

Gauss equation

$$\underbrace{\left(\partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik}\right) + \left(\Gamma^r_{ij}\Gamma^\ell_{rk} - \Gamma^r_{ik}\Gamma^\ell_{rj}\right)}_{R_{kj}{}^{\ell}{}_{i} \text{ curvature tensor}} - \left(h_{ij}h^\ell_k - h_{ik}h^\ell_j\right) = 0.$$

no f appears in these equations.

Coefficient of ν :

Codazzi-Mainardi equation

$$\left(\Gamma^\ell_{ij} h_{\ell k} - \Gamma^\ell_{ik} h_{\ell j}\right) + \left(\partial_k h_{ij} - \partial_j h_{ik}\right) = 0.$$

Similarly

$$0 = \underbrace{\partial_\ell \partial_i \nu}_{-(\partial_\ell h_i^k)\,\partial_k f \,-\, h_i^k\, \underbrace{\partial_\ell \partial_k f}_{\underbrace{\nabla_\ell \partial_k}_{\Gamma^p_{\ell k}\partial_p} + h_{\ell k}\nu}} - \partial_i \partial_\ell \nu$$

Coefficient of ν is 0.

Coefficient of ∂_k:

$$0 = \partial_\ell \underbrace{h_i^k}_{g^{kq}h_{qi}} + \underbrace{h_i^p}_{g^{pq}} \Gamma^k_{\ell p} - (\text{switch } (i,\ell))$$

$$\overbrace{g^{kq}h_{qi} \quad g^{pq} h_{qi}\Gamma^k_{\ell p}}$$

$$h_{qi}\underbrace{\left(\partial_\ell g^{kq} + g^{pq}\Gamma^k_{\ell p}\right)}_{(*)\ -g^{kj}\Gamma^q_{j\ell}} + g^{kq}\partial_\ell h_{qi} = g^{kq}\left(-h_{ji}\Gamma^j_{q\ell} + \partial_\ell h_{qi}\right)$$

$$\therefore -h_{pi}\Gamma^p_{q\ell} + \partial_\ell h_{qi} = -h_{p\ell}\Gamma^p_{qi} + \partial_i h_{q\ell}.$$

Same as Codazzi-Mainardi equation

$$\left(\Gamma^\ell_{ij}h_{\ell k} - \Gamma^\ell_{ik}h_{\ell j}\right) + \left(\partial_k h_{ij} - \partial_j h_{ik}\right) = 0.$$

Derive (*): $0 = \partial_\ell(g^{kq} g_{qi}) = (\partial_\ell g^{kq})\, g_{qi} + g^{kq}\partial_\ell g_{qi}$

$$\therefore \quad \partial_\ell g^{kq} = -g^{qi} g^{kp}\partial_\ell g_{pi}.$$

$$-g^{qj} g^{kp}\partial_\ell g_{pj} + g^{jq}\underbrace{\Gamma^{k}_{\ell j}}_{\frac{1}{2}g^{kp}(\partial_\ell g_{pj}+\partial_j g_{p\ell}-\partial_p g_{\ell j})} = g^{kp} g^{jq}\cdot\left(-\tfrac{1}{2}\right)(\partial_p g_{\ell j} + \partial_\ell g_{pj} - \partial_j g_{p\ell})$$

$$= -g^{kp}\,\Gamma^{q}_{p\ell} \quad \#$$

e.x.

16. Is there a surface element in $\mathbb{R}^3$ with $(g_{ij}(u,v)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(h_{ij}(u,v)) = \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}$?

17. Is there a surface element in $\mathbb{R}^3$ with $(g_{ij}(u,v)) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}$ and $(h_{ij}(u,v)) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}$?

23. Prove that the equations of of Gauss and Codazzi-Mainardi in 4.15 are equivalent to the following two equations:
(a) $R_{ijkl} := \sum_s g_{is} R^s_{jkl} = h_{ik}h_{jl} - h_{il}h_{jk}$,
(b) $\nabla_i h^j_k = \nabla_k h^j_i$.
Here $\nabla_i h^j_k$ denotes the jth component of the tangential vector

$$\left(\nabla_{\frac{\partial f}{\partial u^i}} L\right)\left(\frac{\partial f}{\partial u^k}\right) := \nabla_{\frac{\partial f}{\partial u^i}}\left(L\left(\frac{\partial f}{\partial u^k}\right)\right) - L\left(\nabla_{\frac{\partial f}{\partial u^i}}\frac{\partial f}{\partial u^k}\right)$$

in local coordinates $u^1, \ldots, u^n$. (Compare the remark in 4.19.) As a consequence we obtain once again the Theorema Egregium in the form

$$K = \mathrm{Det}(h_{ij})/\mathrm{Det}(g_{ij}) = R_{1212}/\mathrm{Det}(g_{ij}).$$

Theorema Egregium (Gauss):

Gaussian curvature K depends only on g.

Pf: $K = \underbrace{\det(h_{ij})}_{h_{11}h_{12} - h_{12}^2} / \det(g_{ij})$.

$$\left(\partial_k \Gamma_{ij}^{\ell} - \partial_j \Gamma_{ik}^{\ell}\right) + \left(\Gamma_{ij}^{r} \Gamma_{rk}^{\ell} - \Gamma_{ik}^{r} \Gamma_{rj}^{\ell}\right) - \underbrace{\left(h_{ij} h_{k}^{\ell} - h_{ik} h_{j}^{\ell}\right)}_{g^{\ell p}(h_{ij}h_{pk} - h_{ik}h_{pj})} = 0.$$

$$h_{ij} h_{pk} - h_{ik} h_{pj} = g_{\ell p}\left(\left(\partial_k \Gamma_{ij}^{\ell} - \partial_j \Gamma_{ik}^{\ell}\right) + \left(\Gamma_{ij}^{r} \Gamma_{rk}^{\ell} - \Gamma_{ik}^{r} \Gamma_{rj}^{\ell}\right)\right).$$

Put $i,j=1$, $p,k=2$, LHS $= \det(h_{ij})$.

RHS only depends on g. #

Note: H does not only depends on g.

e.g. $H(\text{cylinder}) = H(\text{plane})$, but cylinder $=$ plane.

Thm 4.18: (Coordinate-free form of the two equations)

(i) Gauss equation ($L = -d\nu$ Weingarten operator)

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = h(Y,Z) LX - h(X,Z) LY.$$

(ii) Codazzi-Mainardi equation

$$\nabla_X (LY) - \nabla_Y (LX) - L([X,Y]) = 0.$$

also written as

$$(\nabla_X L)\cdot Y = (\nabla_Y L)\cdot X$$

where $(\nabla_X \cdot L)\cdot Y \triangleq \nabla_X (L\cdot Y) - L(\nabla_X Y)$.

Pf: $D_X \underbrace{D_Y Z}_{\nabla_Y Z + h(Y,Z)\nu} - D_Y D_X Z = D_{[X,Y]} Z.$

Pr_T: $\left(\nabla_X \nabla_Y Z + h(Y,Z) D_X \nu\right) - \left(\nabla_Y \nabla_X Z + h(X,Z) D_Y \nu\right)$

$= \nabla_{[X,Y]} Z. \qquad \Rightarrow$ (i).

pr: $\left(\underbrace{h(X,\nabla_Y Z)}_{-\langle \nabla_Y(LX), Z\rangle} + \underbrace{X\cdot h(Y,Z)}_{\substack{\langle X\cdot\nu,\, Y\cdot Z\cdot f\rangle \\ +\langle \nu,\, X\cdot Y\cdot Z\cdot f\rangle}}\right) - \left(h(Y,\nabla_X Z) + Y\cdot h(X,Z)\right)$

$= -\langle Z, \underbrace{D_{[X,Y]}\nu}_{L([X,Y])}\rangle.$

#

$$\underbrace{\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z}_{R(X,Y)\cdot Z} = h(Y,Z)\,LX - h(X,Z)\,LY$$

curvature tensor

is tensorial, that is, the inputs X,Y,Z are vectors at a point and the output is a vector (and it is multilinear.)

$(R \in (T^*)^{\otimes 3} \otimes T.)$

Also $\cdot\ R(X,Y) = -R(Y,X).$

$\cdot\ \langle R(X,Y)Z, W\rangle = -\langle R(X,Y)\cdot W, Z\rangle.$

$= h(Y,Z)\,h(X,W) - h(X,Z)\,h(Y,W)$

Cor. 4.20. $K = \langle R(X,Y)Y, X\rangle$ also called sectional curvature

where $\{X, Y\}$ is orthonormal basis.

Gauss curvature

Pf: $\text{RHS} \underset{\substack{\text{Gauss} \\ \text{equation}}}{=} h(Y,Y)h(X,X) - h(X,Y)h(Y,X)$

$= \det h = K$ #

because $\det g = 1$

for orthonormal basis.

23. Prove that the equations of of Gauss and Codazzi-Mainardi in 4.15 are equivalent to the following two equations:

(a) $R_{ijkl} := \sum_s g_{is} R^s_{jkl} = h_{ik}h_{jl} - h_{il}h_{jk}$,

(b) $\nabla_i h^j_k = \nabla_k h^j_i$.

Here $\nabla_i h^j_k$ denotes the jth component of the tangential vector

$$\left(\nabla_{\frac{\partial f}{\partial u^i}} L\right)\left(\frac{\partial f}{\partial u^k}\right) := \nabla_{\frac{\partial f}{\partial u^i}} \left(L\left(\frac{\partial f}{\partial u^k}\right)\right) - L\left(\nabla_{\frac{\partial f}{\partial u^i}} \frac{\partial f}{\partial u^k}\right)$$

in local coordinates $u^1, \ldots, u^n$. (Compare the remark in 4.19.) As a consequence we obtain once again the Theorema Egregium in the form

$$K = \mathrm{Det}(h_{ij})/\mathrm{Det}(g_{ij}) = R_{1212}/\mathrm{Det}(g_{ij}).$$

Invariance under rigid motion.

Lemma 4.23: $f, \tilde{f} : \mathcal{U} \xrightarrow[\text{regular}]{} \mathbb{R}^{n+1}$.

$$g_{ij} \equiv \tilde{g}_{ij} \text{ and } h_{ij} \equiv \tilde{h}_{ij}$$

$$\Leftrightarrow \tilde{f} = B \circ f \text{ for some}$$

$$B = \underbrace{A}_{\substack{\text{constant} \\ \text{matrix}}}(\cdot) + \underbrace{b}_{\substack{\text{constant} \\ \text{vector}}} : \mathbb{R}^{n+1} \circlearrowleft$$

Pf: $\Leftarrow$ is trivial by $\dfrac{\partial \tilde{f}}{\partial u^i} = A \cdot \dfrac{\partial f}{\partial u^i}$.

$\Rightarrow$) Define $A(u) : \mathbb{R}^{n+1} \circlearrowleft$

$$\begin{cases} \dfrac{\partial f}{\partial u^i} \longmapsto \dfrac{\partial \tilde{f}}{\partial u^i} & \forall i = 1, 2. \\ \nu \longmapsto \tilde{\nu} \end{cases}$$

Want: $A(u)$ is constant.

(cont.)

$$\underset{\tilde f_{,ij}}{\underbrace{\frac{\partial}{\partial u^i}}}\Big(\underbrace{A(u)\cdot\frac{\partial f}{\partial u^j}}_{\frac{\partial \tilde f}{\partial u^j}}\Big) = A_{,i} f_{,j} + A \underbrace{f_{,ij}}_{\Gamma^k_{ij} f_{,k} + h_{ij}\nu}$$

$\Vert$ Gauss formula

$$\tilde\Gamma^k_{ij}\tilde f_{,k} + \tilde h_{ij}\tilde\nu = \Gamma^k_{ij} A\cdot f_{,k} + h_{ij} A\cdot\nu$$

$$\therefore\ A_{,i} f_{,j} = 0.$$

$$\frac{\partial}{\partial u^i}\big(A(u)\cdot\nu\big) = A_{,i}\nu + A\cdot\underbrace{\nu_{,i}}_{-h_{ij}g^{jk}f_{,k}}$$

$$\tilde\nu_{,i} = -\tilde h_{ij}\tilde g^{jk}\tilde f_{,k} = -h_{ij}g^{jk}A\cdot f_{,k}$$

$$\therefore\ A_{,i}\cdot\nu = 0.$$

$\Downarrow$

Then $A_{,i} = 0 \Rightarrow A(u) \equiv A$ const.

(cont.) $\therefore\ \tilde{f}_{,i} = \underbrace{A}_{\text{const}} \cdot f_{,i}$.

$$\Rightarrow \tilde{f} = A \cdot f + b. \quad \#$$

Fundamental thm : [Bonnet]

$\mathcal{U} \underset{\text{open}}{\subset} \mathbb{R}^2$.

$(g_{ij}), (h_{ij})$ are given matrix-valued fcns on $\mathcal{U}$, such that
g_{ij} is positive-definite everywhere and

$$\left(\partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik}\right) + \left(\Gamma^r_{ij}\Gamma^\ell_{rk} - \Gamma^r_{ik}\Gamma^\ell_{rj}\right) - \left(h_{ij}h^\ell_k - h_{ik}h^\ell_j\right) = 0.$$

$$\left(\Gamma^\ell_{ij} h_{\ell k} - \Gamma^\ell_{ik} h_{\ell j}\right) + \left(\partial_k h_{ij} - \partial_j h_{ik}\right) = 0.$$

Given $\underline{u} \in \mathcal{U}$, $\underline{p} \in \mathbb{R}^3$, $\underline{X}_1, \underline{X}_2, \underline{\nu} \in \mathbb{R}^3$ oriented basis

$\langle \underline{X}_i, \underline{X}_j \rangle = g_{ij}(u)$, $\langle \underline{\nu}, \underline{X}_i \rangle = 0$ and $\| \underline{\nu} \| = 1$,

$\exists\, \underset{\substack{\cup \\ \underline{u}}}{\mathcal{V}} \subset \mathcal{U}$ and unique $f : \mathcal{V} \xrightarrow[\text{regular}]{} \mathbb{R}^3$ such that

$$\begin{cases} f(\underline{u}) = \underline{p}, \\ f_{,i}(\underline{u}) = \underline{X}_i, \\ \nu(\underline{u}) = \underline{\nu}, \\ \text{The } 1^{\text{st}} \text{ \& } 2^{\text{nd}} \text{ fundamental forms are } g_{ij} \text{ \& } h_{ij}. \end{cases}$$

Pf: Solve PDE system

$$\begin{cases} \partial_i \partial_j f = \Gamma_{ij}^k \partial_k f + h_{ij} \cdot \nu & \text{(Gauss formula)} \\ \partial_i \nu = -h_{ij} g^{jk} \partial_k f. & \text{(Weingarten equation)} \end{cases}$$

rewritten as

$$(*) \begin{cases} \partial_i X_j = \Gamma_{ij}^k X_k + h_{ij} \nu \\ \partial_i \nu = -h_{ij} g^{jk} X_k \end{cases} \quad \text{for } (X_1, X_2, \nu), \quad (1^{st} \text{ order PDE})$$

and

$$(**) \quad \partial_j f = X_j \quad (j = 1, 2) \quad \text{for } f. \quad (1^{st} \text{ order PDE})$$

Due to the integrability conditions given,

$\exists !$ local solution (X_1, X_2, ν) around $\underline{u}$ to $(*)$ with given initial conditions $(\underline{X}_1, \underline{X}_2, \underline{\nu})$.

Claim: $\langle \nu, \nu \rangle = 1$, $\langle \nu, X_i \rangle = 0$, $\langle X_i, X_j \rangle = g_{ij}$.

Already know they hold at $\underline{u}$.

Idea: Show that both LHS and RHS satisfy the same first order PDE. Then they have the same initial values $\underset{\text{uniqueness}}{\Longrightarrow}$ they are the same.

$$\frac{\partial}{\partial u^i}(\langle \nu, \nu \rangle) = 2\langle \underbrace{\partial_i \nu}_{-h_{ij} g^{jk} X_k}, \nu \rangle = -h_{ij} g^{jk} \langle X_k, \nu \rangle.$$

$$\frac{\partial}{\partial u^j}(\langle \nu, X_i \rangle) = \langle \underbrace{\partial_j \nu}_{-h_{jl} g^{lk} X_k}, X_i \rangle + \langle \nu, \underbrace{\partial_j X_i}_{\Gamma_{ji}^k X_k + h_{ji} \nu} \rangle$$

$$= -h_{jl} g^{lk} \langle X_k, X_i \rangle + \Gamma_{ji}^k \langle X_k, \nu \rangle + h_{ji} \langle \nu, \nu \rangle.$$

(cont.) $\partial_k(\langle X_i, X_j\rangle) = \langle \underbrace{\partial_k X_i}_{\Gamma^l_{ki} X_l + h_{ki}\nu}, X_j\rangle + \langle X_i, \partial_k X_j\rangle$

$$= \Gamma^l_{ki}\langle X_l, X_j\rangle + \Gamma^l_{kj}\langle X_l, X_i\rangle + h_{ki}\langle \nu, X_j\rangle + h_{kj}\langle \nu, X_i\rangle.$$

RHS satisfies the same PDE system.

(e.g. LHS $= g_{ij,k}$;

RHS $= \underbrace{\Gamma^l_{ki}\, g_{lj}}_{\frac{1}{2}(g_{kj,i} + g_{ij,k} - g_{ki,j})} + \Gamma^l_{kj}\, g_{li} = g_{ij,k}$.

$\therefore$ LHS $=$ RHS.)

This proves the claim. #

(cont.) Now solve for

$$(**) \quad \partial_j f = X_j \quad (j=1,2) \quad \text{for } f.$$

Integrability condition is

$$\underbrace{\partial_i X_j}_{\Gamma_{ij}^k X_k + h_{ij}\nu} - \partial_j X_i = 0.$$

$\therefore \exists!$ soln with $f(\underline{u}) = p$. #

g_{ij} & h_{ij} are 1st & 2nd fundamental forms:

$$g_{ij} \underset{\text{(claim before)}}{=} \langle X_i, X_j \rangle = \langle \partial_i f, \partial_j f \rangle. \quad \#$$

$$\partial_i \nu = -h_{ij} g^{jk} X_k$$

$$\Rightarrow \langle -\partial_i \nu, \underbrace{X_\ell}_{\partial_\ell f} \rangle = h_{i\ell} \quad \#$$

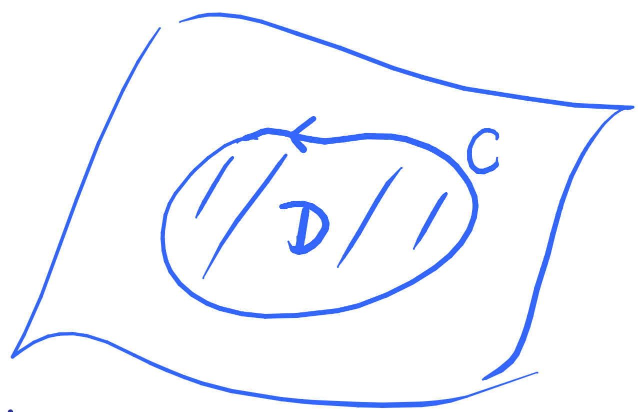
16. Is there a surface element in $I\!R^3$ with $(g_{ij}(u,v)) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $(h_{ij}(u,v)) = \begin{pmatrix}0 & 0\\ 0 & u\end{pmatrix}$?

17. Is there a surface element in $I\!R^3$ with $(g_{ij}(u,v)) = \begin{pmatrix}1 & 0\\ 0 & \cos^2 u\end{pmatrix}$ and $(h_{ij}(u,v)) = \begin{pmatrix}1 & 0\\ 0 & \sin^2 u\end{pmatrix}$?

We will talk about Gauss-Bonnet theorem:

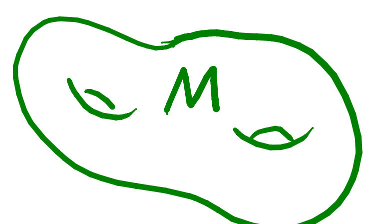
↳ the most important theorem relating geometry and topology.

$$\int_D K\, dA + \int_C k_g\, ds = 2\pi.$$



'$\int$ geometry = topology'

(globally: $\int_M K\, dA = 2\pi \underbrace{\chi(M)}_{\text{Euler characteristic}}$.)



We will introduce tensor calculus & differential forms first, which is useful for doing integration and Stokes theorem.