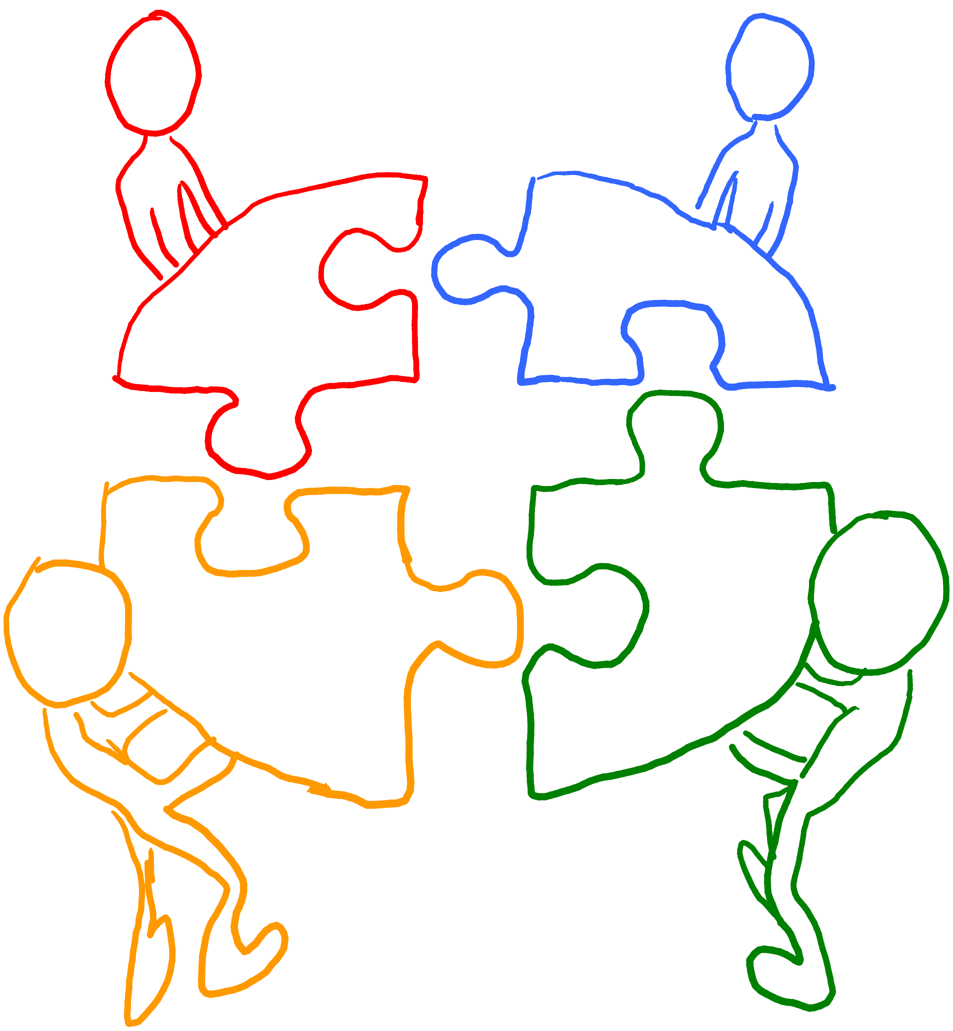
Differential Geometry

IV. Gauss Bonnet formula.

S.C. Lau

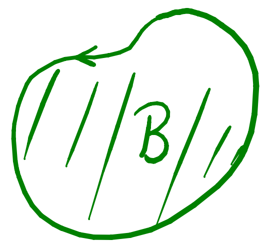
(Ch.4 of [Kühnel].)



Gauss-Bonnet formula.

Thm. 4.38 : Let $B \subset \mathcal{U}$ diffeomorphic to a disk with smooth ∂B.

$$\int_B K \, dA + \int_{\partial B} K_g \, ds = 2\pi.$$



Gauss curvature — area form — geodesic curvature of the loop ∂B — length form

Main technique used in proof:

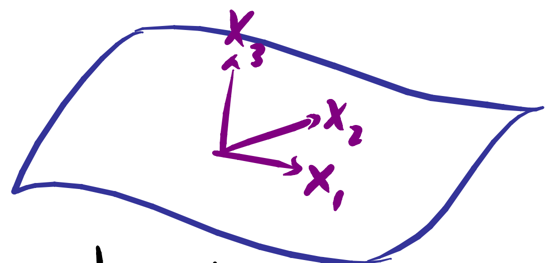
use a (local) moving orthonormal frame instead of coordinate frame to simplify equations.

Main ingredient: Stokes thm.

First, rewrite everything in a general frame.

$f: U \longrightarrow \mathbb{R}^3$ regular surface.

Let $\{X_1, X_2, X_3\}$ be a moving frame of $\mathbb{R}^3$ on the surface f.



such that X_1, X_2 are tangent to f, and $X_3 = \nu$.

e.g. $X_1 = \frac{\partial_1 f}{\|\partial_1 f\|}$, $X_2 = \underbrace{J}_{} \cdot X_1$.

rotate by $\frac{\pi}{2}$ around the axis spanned by ν.

This is an oriented orthonomal frame.

For $j = 1, 2, 3$,

$$D_{(\cdot)} X_j : Tf \xrightarrow[\text{linear}]{} \underline{\mathbb{R}^3}.$$

Write $$DX_j = \sum_{i=1}^{3} \omega_j^i X_i$$

where ω_j^i are 1-forms on U.

This means

$$D_Y X_j = \sum_{i=1}^{3} \omega_j^i(Y) X_i \quad \forall Y \in Tf.$$

Similarly

$$\nabla_\cdot X_j : Tf \to \text{linear} \qquad (j = 1, 2)$$

$$= \sum_{i=1}^{2} \underbrace{A_j^i}_{\text{1-forms}} X_i. \qquad \nabla \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} + \underbrace{\underset{\sim}{A}}_{(2\times 2)\ \text{1-form-valued matrix}} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Recall

$$D_{\cdot} X_j = \nabla_{\cdot} X_j + h(X_j, \cdot)\,\nu$$

for $j = 1, 2$.

$$\therefore \begin{cases} A^i_j = \omega^i_j & \text{for } i, j = 1, 2, \\ \omega^3_j = h(X_j, \cdot) = \iota_{X_j} h. \end{cases}$$

$$D_{\cdot} X_3 = d\nu = \omega^1_3 X_1 + \omega^2_3 X_2.$$

In particular $\omega^3_3 = 0$.

Thm. 4.34. (Maurer-Cartan structural equations):

$$d\omega^i_j + \sum_{k=1}^{3} \omega^i_k \wedge \omega^k_j = 0 \quad \text{for } i,j = 1,2,3.$$

(Gauss equation and Codazzi-Mainardi equation)

Pf:

$$dX_j = \sum_{i=1}^{3} \omega^i_j X_i .$$

regarded as 3×1 column vectors

d on both sides:

$$0 = \sum_{i=1}^{3} (d\omega^i_j) X_i - \sum_{i=1}^{3} \omega^i_j \wedge \underbrace{dX_i}_{\sum_{k=1}^{3} \omega^k_i X_k} .$$

$$= \sum_{i=1}^{3} \left(d\omega^i_j - \sum_{l=1}^{3} \omega^l_j \wedge \omega^i_l \right) X_i \quad \#$$

Def. 4.35: $R(X,Y)\cdot X_j = \underbrace{\Omega^i_j}_{\text{2-form on surface } f}(X,Y)\, X_i$.

curvature 2-form

Recall the curvature tensor (skew-symmetric on X,Y)

$$R(X,Y)\cdot Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z$$

$$= \langle LY, Z\rangle \underbrace{LX}_{-D_X X_3 = -\omega^k_3 X_k} - \langle LX, Z\rangle LY.$$

(Ω^i_j) is regarded as a 2×2 matrix.

$$\Omega = dA + A \wedge A:$$

Recall Gauss equation

$$R(X,Y)\cdot Z = \langle LY, Z\rangle \underbrace{LX}_{-D_X X_3 = -\omega_3^k X_k} - \langle LX, Z\rangle LY.$$

$$R(X,Y)\cdot X_j = -\underbrace{h(X_j,Y)}_{\omega_j^3(Y)}\, \omega_3^k(X)\cdot X_k \left(\sum_{k=1}^{2}\right) + \underbrace{h(X_j,X)}_{\omega_j^3(X)}\, \omega_3^k(Y)\, X_k$$

$$= -\Big(\underbrace{\omega_3^k \wedge \omega_j^3}_{-d\omega_j^k - \omega_1^k\wedge\omega_j^1 - \omega_2^k \wedge \omega_j^2}(X,Y)\Big) X_k$$

$$= \Big(d\omega_j^k + \sum_{i=1}^{2} \omega_i^k \wedge \omega_j^i\Big) X_k.$$

$$= \big(dA_j^k + A_i^k \wedge A_j^i\big) X_k. \ \#$$

Recall $K = \langle \mathcal{R}(X,Y)\cdot Y, X\rangle$ also called sectional curvature of the section $\mathrm{Span}\{X,Y\}$.

Gauss curvature when $\{X,Y\}$ is an orthonormal basis.

From now on, take $\{X_1, X_2\}$ to be an oriented orthonormal basis.

$$K = \langle \mathcal{R}(X_1,X_2)X_2, X_1\rangle$$
$$= \Omega^1_2(X_1,X_2) = \Omega_{12}(X_1,X_2).$$

Since we use orthonormal frame,
$g_{ij} = \langle X_i, X_j\rangle = \delta_{ij}$,
so $\exists$ no difference between upper & lower indices.

Write the dual frame of $\{X_1, X_2\}$

as $\{\underbrace{\varphi^1, \varphi^2}_{\text{1-forms}}\}$.

Then $\Omega_{12} = K \underbrace{\varphi^1 \wedge \varphi^2}_{\text{area form as } \{X_1, X_2\} \text{ orthonormal}}$.

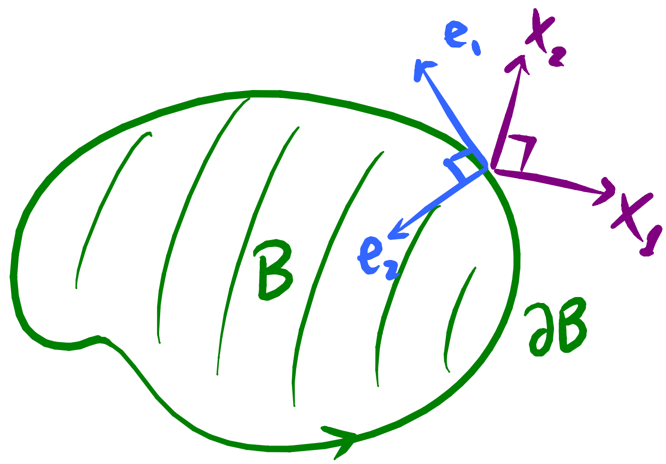
$$\left(\begin{aligned} A_{ij} &= \langle \nabla X_i, X_j \rangle \\ &= -\langle X_i, \nabla X_j \rangle. \end{aligned} \right)$$

Since $\{X_1, X_2\}$ orthonormal,

$\{A_i^j\}$ is skew-symmetric : $\begin{pmatrix} 0 & A_2^1 \\ -A_2^1 & 0 \end{pmatrix}$.

$$\Omega = dA + A \wedge A = \begin{pmatrix} 0 & dA_2^1 \\ -dA_2^1 & 0 \end{pmatrix}.$$

Pf of Gauss-Bonnet theorem:



Take the o.o. frame $\{e_1, e_2\}$ along $c = \partial B$:

$$e_1 = \frac{dc}{ds} \quad (s : \text{arc-length parameter})$$

$$e_2 = J \cdot e_1 = \nu \times e_1.$$

Write $e_1 = (\cos\theta) X_1 + (\sin\theta) X_2$

$$e_2 = -(\sin\theta) X_1 + (\cos\theta) X_2$$

where θ is a function on ∂B.

$$\int_{\partial B} d\theta = 2\pi.$$

$$\|$$

$$\int_0^L \frac{d\theta}{ds}\, ds$$

$$\| \longleftarrow$$

$$\langle e_1, X_1 \rangle = \cos\theta.$$

$$\frac{d}{ds}\langle e_1, X_1 \rangle = -(\sin\theta)\,\theta'$$

$$-\int_0^L \frac{1}{\sin\theta}\cdot \underbrace{\frac{d}{ds}\langle e_1, X_1 \rangle}_{\langle \nabla_{e_1} e_1,\, \underbrace{X_1}_{(\cos\theta)e_1-(\sin\theta)e_2} \rangle + \langle \underbrace{e_1}_{(\cos\theta)X_1+(\sin\theta)X_2},\, \nabla_{e_1} X_1 \rangle}\, ds$$

$$\langle \nabla_{e_1} e_1, e_1 \rangle = 0$$

$$\langle \nabla_{e_1} X_1, X_1 \rangle = 0$$

as $\|e_1\| = \|X_1\| = 1$

$$= \int_0^L \langle \nabla_{e_1} e_1, e_2 \rangle\, ds - \int_0^L \langle X_2, \nabla_{e_1} X_1 \rangle\, ds$$

Recall $\langle \nabla_{e_1} e_1, e_2 \rangle = K_g$ geodesic curvature.

$$\int_0^L \langle X_2, \overbrace{\nabla_{e_1} X_1}^{A_1^1(e_1)X_1 + A_1^2(e_1)X_2} \rangle \, ds = \int_0^L A_1^2(e_1)\, ds$$

$$= \int_{\partial B} A_1^2$$

Stokes thm

$$= \int_B dA_1^2$$

orthonormal basis

$$= \int_B \Omega_1^2 = -\int_B K \underbrace{\varphi^1 \wedge \varphi^2}_{\text{area form}}.$$

$$= -\int_B K \, dA. \quad \#$$

Note: Recall $\nu : U \longrightarrow S^2$ Gauss map.

$$K = \det(-d\nu) \underset{(\dim.=2)}{=} \det(d\nu).$$

$\therefore\ d\nu(X_1) \times d\nu(X_2) = K\nu.$ ($\{X_1, X_2\}$ o.n.b. of Tf.)

Write $d\nu \cdot X_i = h_i^j \partial_j$. Then

$$(d\nu \cdot X_1) \times (d\nu \cdot X_2) = \underbrace{\det(h_i^j)}_{K}\ \underbrace{X_1 \times X_2}_{\nu}$$

$\nu^*(\text{area form on } S^2) = K\,dA$:

$$(\text{area form of } S^2)(d\nu(X_1), d\nu(X_2)) = K.$$

$$\therefore \int_B K\,dA = \int_{\nu(B)} (\text{area form of } S^2)$$

is the (signed) area of image of B (with multiplicity) under ν.

3. Let $S \subset R^3$ be a regular surface homeomorphic to a sphere. Let $\Gamma \subset S$ be a simple closed geodesic in S, and let A and B be the regions of S which have Γ as a common boundary. Let $N: S \longrightarrow S^2$ be the Gauss map of S. Prove that $N(A)$ and $N(B)$ have the same area.

[do Carmo]

9. Let (M, g) be a two-dimensional Riemannian manifold, and let $\Delta \subset M$ be a geodesic triangle which is the boundary of a simply connected domain. Show that the parallel translation along this boundary (traced through once) is a rotation in the tangent plane. Calculate the angle of rotation in terms of quantities which only depend on the interior of Δ. Hint: Gauss-Bonnet formula.

[Ch.5 Künnel]

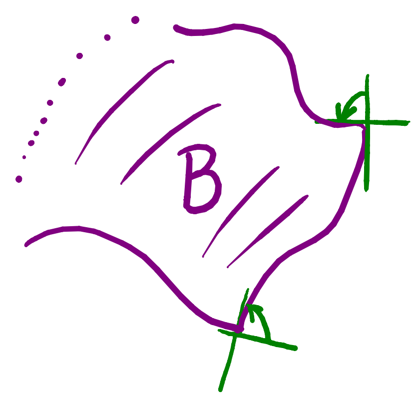
10. Show that the holonomy group of the standard two-sphere S^2 really contains all the rotations. Hint: Consider curves which are constructed piecewise from great circles.

11. Determine the holonomy group of the hyperbolic plane as a surface in three-dimensional Minkowski space (cf. 3.44). Here the covariant derivative is to be taken as in Euclidean space, that is, with tangent components which are directional derivatives.

Piecewise smooth version of Gauss-Bonnet thm:

Let $B \subset \mathcal{U}$ diffeomorphic to a disk with piecewise smooth ∂B. Then

$$\int_B K \, dA + \int_{\partial B} K_g \, ds + \sum_i (\text{exterior angles at non-smooth points}) = 2\pi.$$



Pf is basically the same,

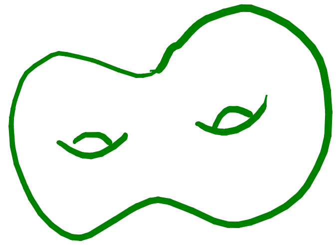
using $\displaystyle\sum_i \int_{\gamma_i} d\theta + \sum_i (\text{exterior angles at non-smooth points}) = 2\pi$

(instead of $\int_{\partial B} d\theta = 0$ before.)

(Also A_1^2 is smooth, so can perturb ∂B to smooth loop to compute $\int_{\partial B} A_1^2$ by Stokes thm.)

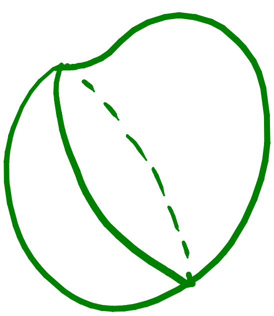
Global version of Gauss-Bonnet thm:

S compact oriented regular surface
with Riemannian metric.

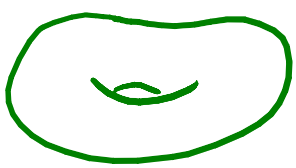


$$\frac{1}{2\pi}\int_S K\,dA = \underbrace{\chi(S)}$$

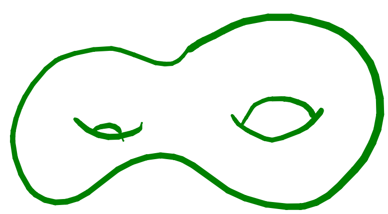
Euler characteristic:
Take a triangulation of S.
$\chi(S) = \#\text{vertices} - \#\text{edges} + \#\text{faces}$.



$\chi = 2.$



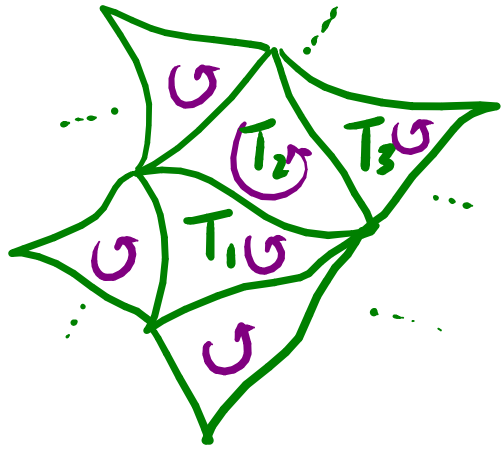
$\chi = 0.$



$\chi = -2.$

$\chi = 2 - 2g$ where g is the genus, that is
'number of holes'.

Pf: Pick a (finite) triangulation of S.



For each triangle T_i:

$$\int_{T_i} K\,dA + \int_{\partial T_i} K_g\,ds + \sum_{j=1}^{3} \underbrace{(\text{ext } \angle_j)}_{\pi - (\text{interior } \angle_j)} = 2\pi.$$

$$\int_{T_i} K\,dA + \underbrace{\int_{\partial T_i} K_g\,ds}_{\text{cancel under } \Sigma_i \text{ due to orientation}} + \sum_{j=1}^{3} (\text{int. } \angle_j) = 2\pi - 3\pi.$$

$$\sum_i : \int_S K\,dA + 2\pi(\#\text{vertices}) = 2\pi(\#\text{faces}) - 2\pi(\#\text{edges}).$$

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