

# MATH 563 DIFFERENTIAL GEOMETRY

## Mock Exam

1. Compute the Frenet frame and curvature of the ellipse

$$\{x^2/9 + y^2/4 = 1\} \subset \mathbb{R}^2.$$

Find all the local extremal points of the curvature function.

2. Consider the unit sphere in spherical coordinates  $(\phi, \theta)$ .
  - (a) Compute the area form.
  - (b) Compute the Levi-Civita connection 1-form

$$\Gamma_j^k = \Gamma_{ij}^k dx^i$$

(where  $x^1$  means  $\phi$  and  $x^2$  means  $\theta$ ).

- (c) Find the orthonormal frame  $(X_1, X_2)$  obtained by the Gram-Schmidt process applied on the coordinate frame  $(\partial_\phi, \partial_\theta)$ .
- (d) Compute the connection 1-form  $A$  in the orthonormal frame  $(X_1, X_2)$ , which is a one-form-valued  $(2 \times 2)$ -matrix.
- (e) Compute the curvature 2-form  $\Omega = dA + A \wedge A$  in the orthonormal frame  $(X_1, X_2)$ , which is a two-form-valued  $(2 \times 2)$ -matrix. Hence conclude the Gauss curvature.
- (f) Consider the annulus defined by  $\pi/3 \leq \phi \leq \pi/2$ . What does the Gauss-Bonnet formula say for this annulus? Hence find the difference in the total geodesic curvatures of the two circles  $\phi = \pi/3$  and  $\phi = \pi/2$ .