EXERCISE ON DIFFERENTIAL FORMS

LECTURER: SIU-CHEONG LAU

(1) Consider the two vectors $v_1 = (-2, 1)$ and $v_2 = (2, 1)$ in $\mathbb{R}^2$. Compute $v_1 \wedge v_2$ in terms of the basic vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$. What is the relation with $\det(v_1\ v_2)$? Is the basis (v_1, v_2) positively oriented?

(2) Similarly, consider three vectors $v_i = \sum_{j=1}^{3} a_i^j e_j$ where $\{e_1, e_2, e_3\}$ is the standard basis and $a_i^j \in \mathbb{R}$. Compute $v_1 \wedge v_2 \wedge v_3$ in terms of $e_1 \wedge e_2 \wedge e_3$. What is the relation with $\det(v_1\ v_2\ v_3)$?

(3) Consider $\eta = x\,dx$. Compute $\eta(y^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y})$ at $(x, y) = (2, 1)$.

(4) Consider $\omega = x\,dx \wedge dy$. Compute $\omega(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$ at $(x, y) = (2, 1)$.

(5) Compute $(xdx + ydy) \wedge (-ydx + xdy)$.

(6) Now consider $f(u, v) = (x(u, v), y(u, v))$. Write down dx and dy in term of du, dv. Then compute $dx \wedge dy$. (You should get the Jacobian of f).

(7) Compute $d(xdy - ydx)$. How about

$$d\left(\frac{x}{x^2 + y^2}dy - \frac{y}{x^2 + y^2}dx\right)?$$

($\frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx$ is known as the angle form on the plane.)

(8) Express $v \otimes v \otimes v$ in terms of the basis $\{e_1, e_2\}$ for $v = 2e_1 - 3e_2$.

(9) Check that $V^* \otimes W \to \mathrm{Hom}(V, W)$ defined by $(\nu \otimes w)(v) := \nu(v)w$ is an isomorphism.

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