DIFFERENTIAL GEOMETRY HOMEWORK 11

LECTURER: SIU-CHEONG LAU

(1) Compute
$$\int_D e^{x^2+y^2}\, dx \wedge dy$$
by changing to polar coordinates, where $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is the unit disc.

(2) Prove that on $\mathbb{R}^2 - \{0\}$,
$$\omega = \frac{-y\, dx + x\, dy}{x^2 + y^2}$$
belongs to $\mathrm{Ker}(d)$ but not $\mathrm{Im}(d)$. (Hint: consider $\oint \omega$).

(3) The area form of the $(n-1)$-dimensional sphere
$$S = \left\{ \sum_{i=1}^n x_i^2 = R^2 \right\} \subset \mathbb{R}^n$$
of radius R is given by
$$\frac{1}{R} \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \Big|_S .$$
(This is obtained by contracting the volume form $dx_1 \wedge \ldots \wedge dx_n$ by the outward unit normal $\frac{1}{R}\sum_i x_i \partial_{x_i}$ of the sphere.) Use the Fundamental Theorem of Calculus to explain that
$$\text{Volume of } B = \frac{R}{n} \cdot \text{Area of } S$$
where $B \subset \mathbb{R}^n$ is an n-dimensional ball of radius R. (For instance, the volume of the 3-dimensional ball is $4\pi R^3/3$ while the area of the 2-dimensional sphere is $4\pi R^2$.)

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