(1) Given a regular plane curve $\gamma(t)$ not passing through the origin, the polar moving frame is defined as $(f_1(t), f_2(t))$, where

$$f_1 = \frac{\gamma(t)}{\|\gamma(t)\|}$$

and $f_2$ is obtained by rotating $f_1$ by $\pi/2$. Such a frame is useful in describing planet motion. Let $(r, \theta)$ be the polar coordinates, and so $\gamma(t) = (r(t) \cos(\theta(t)), r(t) \sin(\theta(t)))$.

(a) Derive the frame equation

$$(f_1 \ f_2)' = (f_1 \ f_2) \cdot \begin{pmatrix} 0 & -\theta' \\ \theta' & 0 \end{pmatrix}$$

(b) Show that $\gamma'(t) = r'(t)f_1(t) + r(t)\theta'(t)f_2(t)$.

(c) Suppose $\gamma''(t) = -Cf_1(t)$ where $C$ is a positive constant. (It is the motion of a planet under gravitation from the origin.) Show that the cross product $\gamma \times \gamma'$, and hence $r^2\theta'$, is constant.

(d) Let $A(t)$ be the area of the sector swapped by the line segment with endpoints $0$ and $\gamma(\tau)$ for $\tau \in [0, t]$. Using the above, show that $A(t) = kt$ for some $k > 0$.

(2) Suppose $c$ is a regular curve contained in the unit sphere $x^2 + y^2 + z^2 = 1$. Derive a formula for the curvature and torsion in terms of the geodesic curvature

$$j(s) = \langle c''(s), c(s) \times c'(s) \rangle.$$  

(Hint: express $c''$, $c'''$ in terms of the moving frame $(c', c \times c', c)$.)