DIFFERENTIAL GEOMETRY HOMEWORK 3

LECTURER: SIU-CHEONG LAU

(1) Given a regular plane curve $\gamma(t)$ not passing through the origin, the polar moving frame is defined as $(f_1(t), f_2(t))$, where
$$f_1 = \frac{\gamma(t)}{\|\gamma(t)\|}$$
and f_2 is obtained by rotating f_1 by $\pi/2$. Such a frame is useful in describing planet motion. Let (r, θ) be the polar coordinates, and so $\gamma(t) = (r(t)\cos(\theta(t)), r(t)\sin(\theta(t)))$.
 (a) Derive the frame equation
$$(f_1\ f_2)' = (f_1\ f_2) \cdot \begin{pmatrix} 0 & -\theta' \\ \theta' & 0 \end{pmatrix}$$
 (b) Show that
$$\gamma'(t) = r'(t) f_1(t) + r(t)\theta'(t) f_2(t).$$
 (c) Suppose $\gamma''(t) = -C f_1(t)$ where C is a positive constant. (It is the motion of a planet under gravitation from the origin.) Show that the cross product $\gamma \times \gamma'$, and hence $r^2\theta'$, is constant.
 (d) Let $A(t)$ be the area of the sector swapped by the line segment with endpoints 0 and $\gamma(\tau)$ for $\tau \in [0, t]$. Using the above, show that $A(t) = kt$ for some $k > 0$.

(2) Suppose c is a regular curve contained in the unit sphere $x^2 + y^2 + z^2 = 1$. Derive a formula for the curvature and torsion in terms of the geodesic curvature
$$j(s) = \langle c''(s), c(s) \times c'(s) \rangle.$$
(Hint: express c'', c''' in terms of the moving frame $(c', c \times c', c)$.)

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