EXERCISE IN EINSTEIN NOTATIONS

LECTURER: SIU-CHEONG LAU

(1) Let V be a vector space and $\partial_i : i = 1, \ldots, n$ be a basis of this vector space. (In the course, V is the tangent space of a regular parametrized surface f at a point, and ∂_i are the vectors $\partial_i f$.) Write the following in terms of Einstein notation and also matrix notation using the basis ∂_i.
  (a) A vector v. (The coefficients are denoted as v^j.)
  (b) $L(v)$ for a linear endomorphism $L : V \to V$. (The matrix coefficients of L are denoted as L_i^j.)
  (c) The equation $L(v) = u$ for $u \in V$. (The coefficients of u are denoted as u^j.)
  (d) The composition $P(L(v))$ where $P : V \to W$ be another linear map, and W is a vector space with a basis $\{e_1, \ldots, e_m\}$. (The matrix coefficients of P are denoted as P_k^l.)
  (e) $h(v, u)$ for a bilinear form h. (The matrix coefficients of h are denoted as h_{ij}. The coefficients of u are denoted as u^j.)
  (f) $h(L(v), u)$.

(2) Now suppose $\phi = (t_1(s_1, s_2), t_2(s_1, s_2))$ be a diffeomorphism $S \to T$, where V and U are domains in $\mathbb{R}^2$. Denote its inverse by $\phi^{-1} = (s_1(t_1, t_2), s_2(t_1, t_2))$. Let $f : U \to \mathbb{R}^3$ be a regular parametrized surface.
  (a) What is the differential $d\,(\phi^{-1})$? Deduce the relation with $d\phi$ by using $\phi(\phi^{-1}(t_1, t_2)) = (t_1, t_2)$ using chain rule, in both matrix and Einstein notations.
  (b) Let $v = (v^1, v^2)$ be a tangent vector at a point in U. Write the directional derivative $\partial_v f$ in Einstein notation.
  (c) Write down the chain rule for $d(f \circ \phi)$ in both marix form and Einstein notation.
  (d) Let $g_{ij} = \left\langle \frac{\partial f}{\partial t^i}, \frac{\partial f}{\partial t^j} \right\rangle$. let $g(v, w) = \langle v, w \rangle$ for two tangent vectors v, w of the surface at the same point. Write down $g(v, w)$ in both matrix and Einstein notations.
  (e) Write down the projection of $\partial_i \partial_j f$ (for $i, j = 1, 2$) to the tangent plane using Einstein notation (where the coefficients in $\partial_k f$ are denoted as Γ_{ij}^k).

Department of Mathematics and Statistics, Boston University
E-mail address: lau@math.bu.edu