EXERCISE IN EINSTEIN NOTATIONS

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(1) Let $V$ be a vector space and $\partial_i : i = 1, \ldots, n$ be a basis of this vector space. (In the course, $V$ is the tangent space of a regular parametrized surface $f$ at a point, and $\partial_i$ are the vectors $\partial_if$.) Write the following in terms of Einstein notation and also matrix notation using the basis $\partial_i$.

(a) A vector $v$. (The coefficients are denoted as $v^j$.)

(b) $L(v)$ for a linear endomorphism $L : V \to V$. (The matrix coefficients of $L$ are denoted as $L^j_i$.)

(c) The equation $L(v) = u$ for $u \in V$. (The coefficients of $u$ are denoted as $u^j$.)

(d) The composition $P(L(v))$ where $P : V \to W$ be another linear map, and $W$ is a vector space with a basis $\{e_1, \ldots, e_m\}$. (The matrix coefficients of $P$ are denoted as $P^k_l$.)

(e) $h(v, u)$ for a bilinear form $h$. (The matrix coefficients of $h$ are denoted as $h_{ij}$. The coefficients of $u$ are denoted as $u^j$.)

(f) $h(L(v), u)$.

(2) Now suppose $\phi = (t_1(s_1, s_2), t_2(s_1, s_2))$ be a diffeomorphism $S \to T$, where $V$ and $U$ are domains in $\mathbb{R}^2$. Denote its inverse by $\phi^{-1} = (s_1(t_1, t_2), s_2(t_1, t_2))$. Let $f : U \to \mathbb{R}^3$ be a regular parametrized surface.

(a) What is the differential $d(\phi^{-1})$? Deduce the relation with $d\phi$ by using $\phi(\phi^{-1}(t_1, t_2)) = (t_1, t_2)$ using chain rule, in both matrix and Einstein notations.

(b) Let $v = (v^1, v^2)$ be a tangent vector at a point in $U$. Write the directional derivative $\partial_vf$ in Einstein notation.

(c) Write down the chain rule for $d(f \circ \phi)$ in both marix form and Einstein notation.

(d) Let $g_{ij} = \left\langle \frac{\partial f}{\partial t_i}, \frac{\partial f}{\partial t_j} \right\rangle$. Let $g(v, w) = \left\langle v, w \right\rangle$ for two tangent vectors $v, w$ of the surface at the same point. Write down $g(v, w)$ in both matrix and Einstein notations.

(e) Write down the projection of $\partial_i\partial_jf$ (for $i, j = 1, 2$) to the tangent plane using Einstein notation (where the coefficients in $\partial_i f$ are denoted as $\Gamma^k_{ij}$).