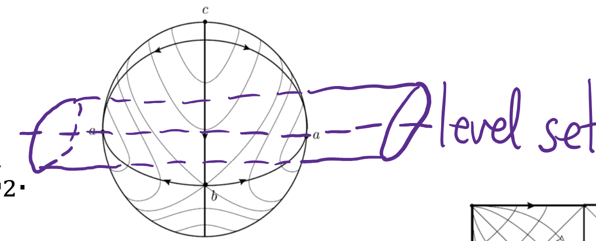
1. Morse functions

Saturday, December 29, 2018 2:08 PM

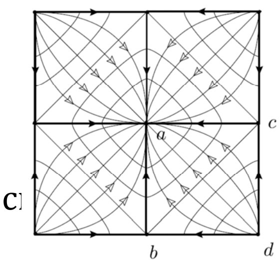
Audin; Damian

Non-degenerate critical points.



Examples.

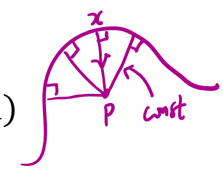
Sphere. Torus. $\mathbb{CP}^n$. $\mathbb{RP}^2$: y^2+2z^2 quotient by $\mathbb{Z}_2$.

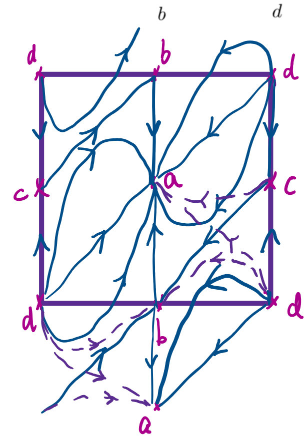


Distance function squared f_p on submanifold S from a generic point p in $\mathbb{R}^n$.

Proof:

- Critical points x iff $(x-p)\perp S$.
- Hessian of f_p at crit. x degenerate (like constant in the second order) iff p is c value of the function
  $N_S \to \mathbb{R}^n: (x,v)\mapsto x+v$.
  (Note: same dim, non-deg. along v direction)
- Sards Theorem.





Morse Lemma. Let c be a critical point.

$$f = f(c) - \sum_{j=1}^{i} x_j^2 + \sum_{i+1}^{n} x_j^2 .$$

Proof: suppose $c=0, f(0)=0$.

Induction on dimension.

$n=1$:

$f = h\,x^2\big(1+o(x)\big)$ $(h\neq 0.)$

$$= \operatorname{sgn}(h)\cdot\left(\sqrt{|h|\big(1+o(x)\big)}\cdot x\right)^2 .$$

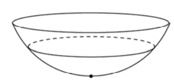


Fig. 1.1 A minimum

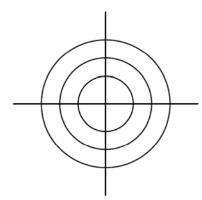


Fig. 1.2 Critical point of index 0

Consider level set and gradients.

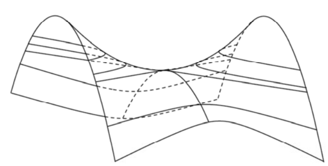


Fig. 1.3 A saddle point

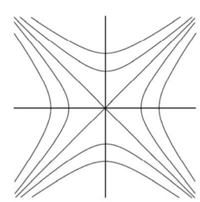


Fig. 1.4 Critical point of index 1

Write $(x,y)\in\mathbb{R}\times\mathbb{R}^{n-1}$.

$$f = k_y + g_y\cdot x + \operatorname{sgn}(h)\cdot\left(x\sqrt{|h_y|\Big(1+o_y(x)\Big)}\right)^2 .$$

Want $g_y=0$, then done. $\big(k_y(0)=g_y(0)=0.\big)$

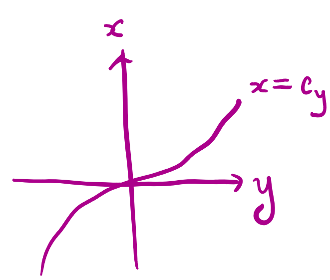
$g_y=\partial_x f(0,y)$ may not 0. *Can expand at $x=c_y$ instead of $x=0$!*

So want to solve $(\partial_x f)\big(c_y,y\big)=0$ for c_y.

That is, regarding the zero set of $\partial_x f$ as a graph over y.

Implicit function needs $\partial_x^2 f(0,0)\neq 0$.

Diagonalize the Hessian by linear change in the beginning to ensure this.



Consider $f\big(\tilde{x}+c_y,y\big)$.

$$f\big(\tilde{x}+c_y,y\big) = k_y + g_y\cdot\tilde{x} + \operatorname{sgn}(h)\cdot\left(\tilde{x}\sqrt{h_y\Big(1+o_y(\tilde{x})\Big)}\right)^2 .$$

$$\begin{pmatrix}\partial_x^2 f & 0\\ 0 & *\end{pmatrix} \text{ non.-deg}$$

$g_y = (\partial_{\tilde{x}} f)(0,y) = \partial_x f\big(c_y,y\big) = 0.$ Done.

Index of a critical point. In above examples.

Why Morse theory?

- **Computable way to study topology of manifold (ex homogeneous space G/P). (Finite)**
- **Classical part of Hamiltonian/Lagrangian Floer theory and mirror symmetry.**