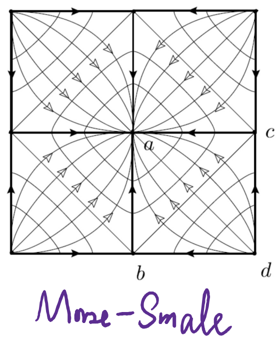
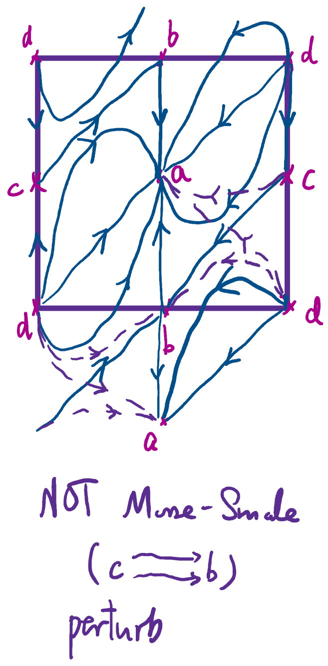
2. Morse-Smale

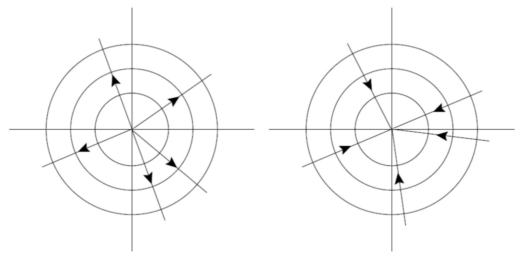
Saturday, December 29, 2018 2:12 PM

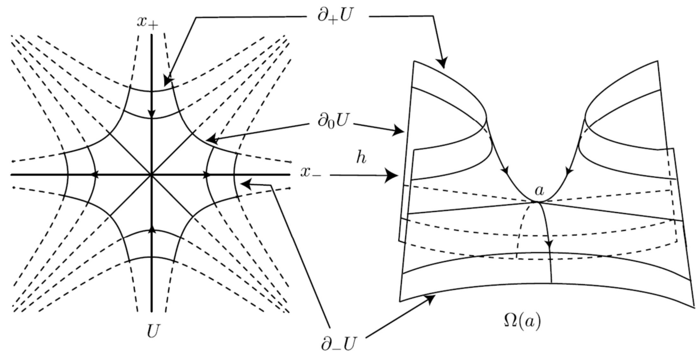




Smale => moduli of trajectories is regular (and hence has correct dim.).
Counter-example: height on vertical torus.

Gradient flow. Need metric.
In Morse charts, if *assuming standard metric* (otherwise may not look perpendicular):

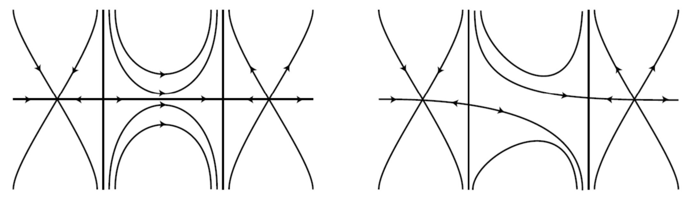




Pseudo-gradient vector field X: (don't use metric on manifold)

- $df(X) \leq 0$, with equality exactly at critical points.
- Equals to negative gradient for the standard metric in Morse chart.

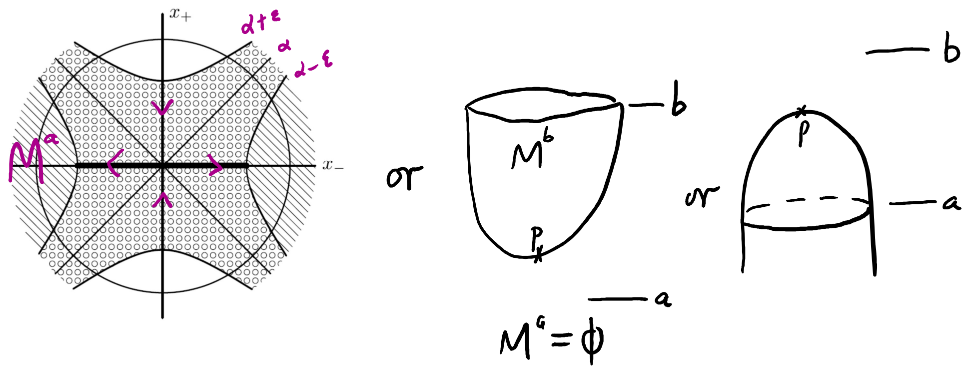
Advantage: can be analyzed easily near Morse chart.
Exists by partition of unity. $\sum \phi_j X_j$.



Pseudo-gradient flow. Stable and unstable submanifolds W^s and W^u. (Both are open discs.) **Examples.**
$\dim W^u(a) = \operatorname{codim} W^s(a) = \operatorname{Ind}(a).$

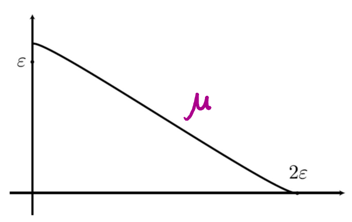
Sublevel sets $M^a = \{f \leq a\}$. **Examples. Morse function gives a cell decomposition.**
$M^a \cong M^b$ if no critical value in between. (Use the flow to retract.)

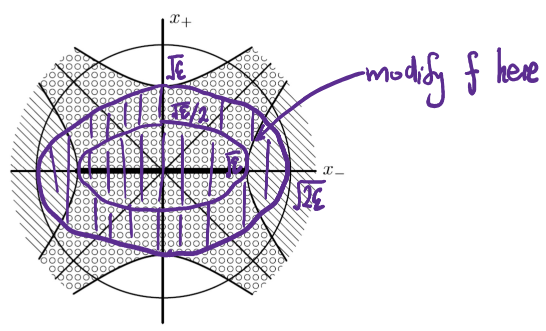
Crossing critical value α: suppose only one critical point p in between.
Then M^b is homotopic to M^a with the unstable submanifold of p attached.

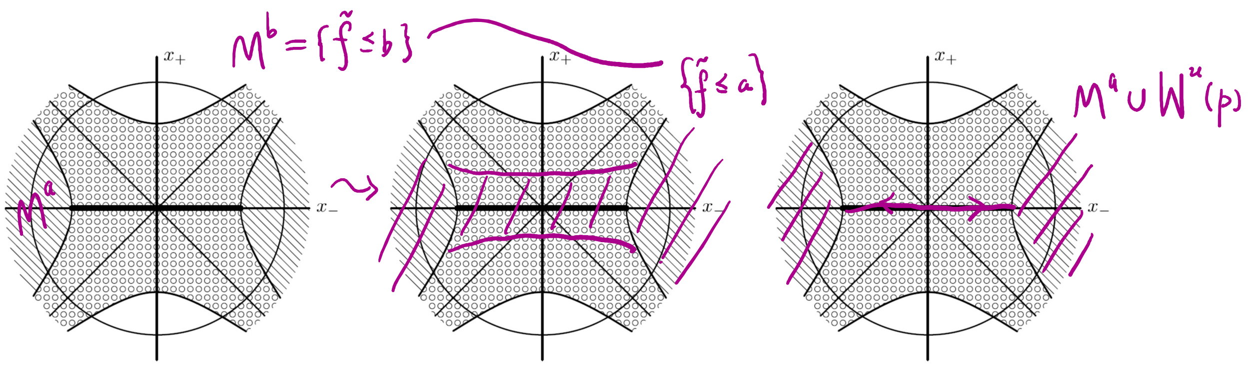


Proof:
Use Morse chart at p.
Trouble: M^a and M^b have parts lying outside of the chart.
Expect to have retract for this part.
Trick: modify the Morse function to $\tilde{f}$ such that
$\{\tilde{f} \leq b\} = M^b$; $\tilde{f}$ has the same critical points as f;
$\{\tilde{f} \leq a\}$ is contained in union of M^a and the Morse chart of f;
$\tilde{f}$ has no critical value in between a and b.
Then M^b retracts to $\{\tilde{f} \leq a\}$, which equals to M^a outside the chart and is explicit inside the chart. Then the final retract can be constructed by hand in the chart.

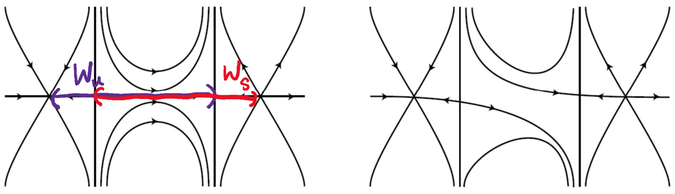
$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \notin \Omega(a) \\ \alpha - \|x_-\|^2 + \|x_+\|^2 - \mu\left(\|x_-\|^2 + 2\|x_+\|^2\right) & \text{if } x = h(x_-, x_+). \end{cases}$$





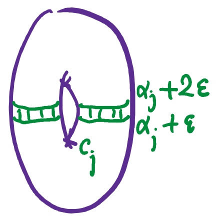


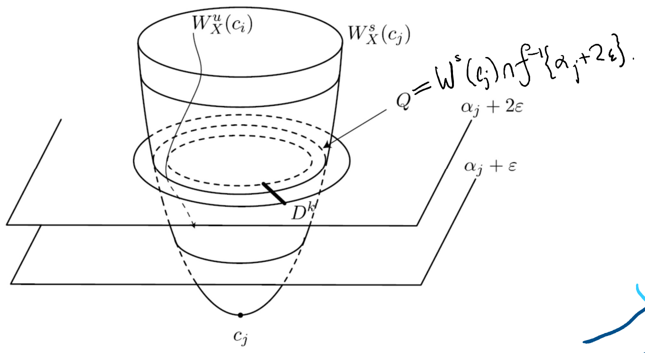
Smale condition: unstable and stable intersect **transversely**.
$\dim(W^u(a) \cap W^s(b)) = \mathrm{Ind}(a) - \mathrm{Ind}(b).$
HARDEST ISSUE in Floer theory.
Remark: has $\mathbb{R}$ action on $M(a, b)$. Use this to reduce to Morse chart!
Want to perturb Morse f and pseudo-gradient X to obtain Smale.

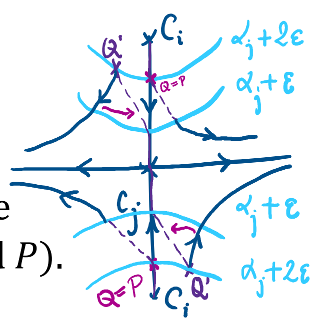


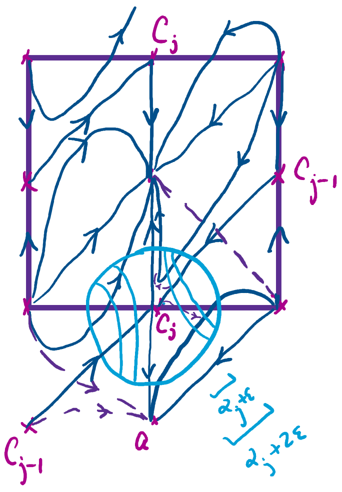
First can perturb f such that all critical points have distinct values:
take $f + h$, where
h is constant in disjoint Morse charts,
$f + h$ has distinct values at different critical points,
$|dh| < \frac{\epsilon}{2}$ where ϵ satisfies $df(X) < -\epsilon$ outside Morse charts.

Then perturb X:
Just do perturbation in complement of neighborhoods of critical points.
Induction on critical points c_j, from abs. max. to abs. min., to get
$W^s(c_j) \pitchfork W^u(c_i)$ for all i.
$j = 1$ is trivial.
Suppose already have it before j. Then modify X only in
$f^{-1}([\alpha_j + \epsilon, \alpha_j + 2\epsilon])$ (in Morse chart).
(Then transversality above c_j is not affected.)









Consider the stable and an unstable
in the level set $f^{-1}\{\alpha_j + 2\epsilon\}$ (Q and P).
If not transversal, perturb X
such that stable becomes Q'.

$P = W^u(c_i) \cap f^{-1}\{\alpha_j+2\varepsilon\}$ NOT $\pitchfork$ Q
$\boxed{\text{Perturb } Q \text{ to } Q'.}$

Reformulate transversality of the
intersection as regularity of function value:
$f^{-1}([\alpha_j + \epsilon, \alpha_j + 2\epsilon]) \cong D^k \times Q \times I.$
(D^k is the normal disc in the level set $f^{-1}\{\alpha_j + 2\epsilon\}$.)
Consider $g: P \to D^k$. $g^{-1}\{0\} = P \cap Q$.
Want $0 \in D$ to be a regular value. But it may not.
Sards: has dense set of regular values.
For each unstable has such a dense set. Take w in
the intersection.
Perturb X such that $Q' = \{(w, q, 1): q \in Q\}$.
$P \cap Q' = g^{-1}\{w\}$ transverse for any P. Done.