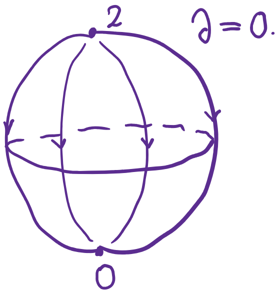
3. Morse complex

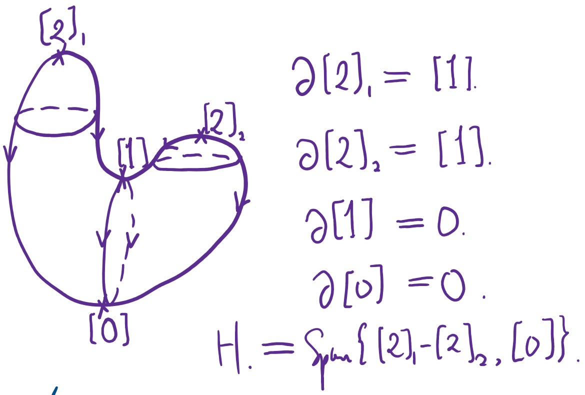
Saturday, December 29, 2018 2:12 PM

The complex.
Take formal span of critical points (over $\mathbb{Z}$ or $\mathbb{Z}_2$) graded by index, and count trajectories (up to reparametrization) between them to define a complex.

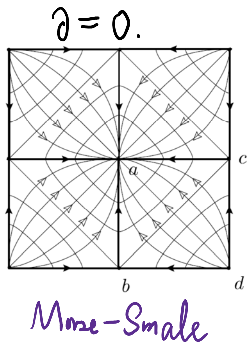
$\partial_X(a) = \sum_{b \in \mathrm{Crit}_{k-1}} n_X(a,b) b$

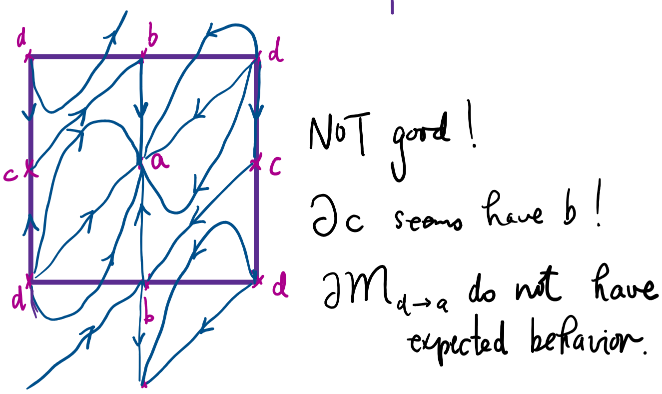
ex.
$\mathbb{S}^2$.



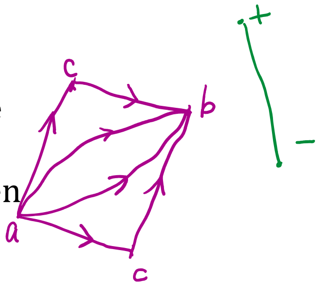


T^2.





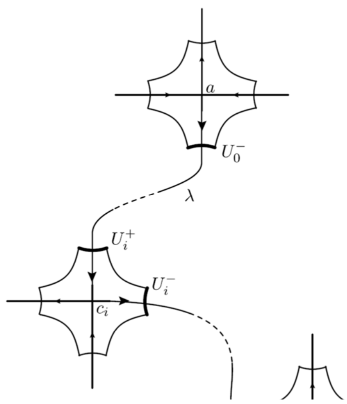
Key is $d^2 = 0$ and so we can take homology.
Fixing a, b with $\deg a = \deg b + 2$, need $\sum_c n(a,c) \cdot n(c,b) = 0$.
LHS is number of broken trajectories from a to b.
Want to prove this is the boundary of a 1d manifold (and hence cancel with each other).
So compactify the space of smooth trajectories $L(a,b)$ by broken trajectories. (The $\mathbb{R}$ action acts freely and so easy to quotient)



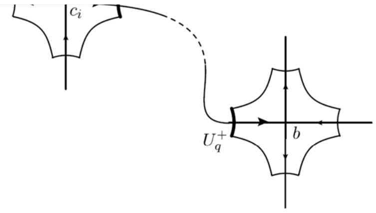
Broken trajectories:

$\overline{\mathcal{L}}(a,b) = \bigcup_{c_i \in \mathrm{Crit}(f)} \mathcal{L}(a, c_1) \times \cdots \times \mathcal{L}(c_{q-1}, b).$

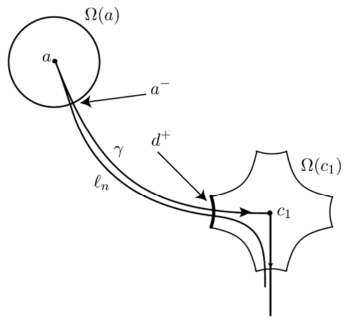
Topology on $\bar{L}(a,b)$:
Fix a broken $\lambda = (\lambda_1, \dots, \lambda_q)$. Its neighborhood consists of its deformations and smoothings.
Key: every smooth trajectory has $\mathbb{R}$ symmetry. Can reduce to a level set in a Morse chart to see all its small deformations!
Base of open sets: $W(\lambda, U^-, U^+)$
where U^- consists of a neighborhood of the exit point of λ in a Morse chart of each broken point; similarly U^+ consists of neighborhoods of entry points;
$W(\lambda, U^-, U^+)$ consists of broken trajectories whose broken points are subset of that of λ, and whose exit and entry points



neighborhoods of entry points;
$W(\lambda, U^-, U^+)$ consists of broken trajectories whose broken points are subset of that of λ, and whose exit and entry points are contained in U^- and U^+.



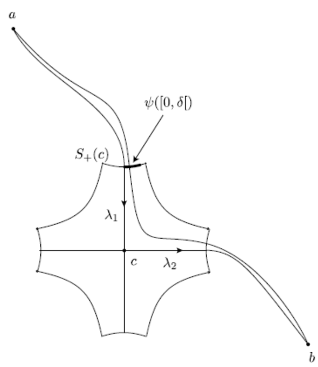
Smooth trajectories must limit to broken trajectories:
Key: Find the limit using Morse charts.
Consider a sequence of smooth $l_n \in L(a,b)$.
Consider exit and entry points l_n^-, l_n^+ in Morse charts of a, b.
Have limit a^-, b^+ by taking subsequence. (Unstable and stable intersect a level set at a sphere which is compact.)
Take the trajectory l^1 thru a^-, which flows to certain c_1.
Let d^+ be the entry point to c_1. l_n enters Morse chart of c_1 at d_n^+ where $d_n^+ \to d^+$.
If $c_1 = b$, then $d^+ = b^+$. l_n limits to l^1 by definition of topology.
Otherwise l_n (which ends at b) exits the Morse chart of c_1 at d_n^- which has limit d^-.
d^- lies in the unstable of c_1: otherwise the trajectory l^2 of d^- enters the Morse chart of c_1 at certain point, and this point must be d^+ as $d_n^- \to d^-$. Contradiction: the trajectory through d^+ (which is l^1) ends at c_1.
Consider l^2 thru d^- emanated from c_1. d^- plays the role of a^- and c_1 plays the role of a. Do the argument again, and we find a sequence $l^i \in L(c_{i-1}, c_i)$. $f(c_i) \geq f(b)$, and hence the sequence must be finite, and the final $c_q = b$. By definition of the topology l_n limits to $(l^1, \dots, l^q)$.

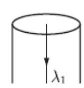


Corollary: If index difference is one, $L(a,b)$ is already compact and hence is a finite set. (Cannot break.)

$\bar{L}(a,b)$ is compact:
Already know that a sequence of smooth $l_n \in L(a,b)$ has convergent subsequence.
Consider a sequence $l_n \in \bar{L}(a,b)$.
Have a subsequence in $L(a,c_1) \times \cdots \times L(c_{q-1},b) \subset \bar{L}(a,b)$.
Then just apply the known result to each factor.

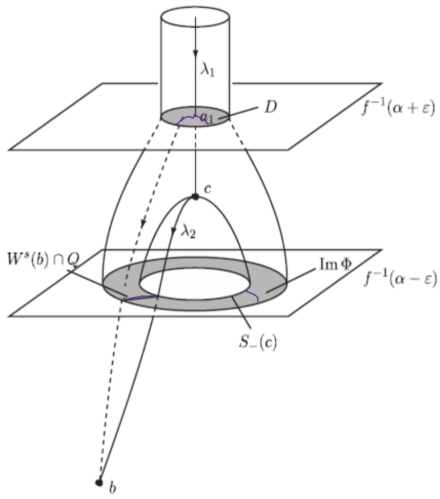
Manifold with boundary structure on $\bar{L}(a,b)$:
Just do it for $\mathrm{Ind}(a) = \mathrm{Ind}(b) + 2$. It is a 1d manifold with boundary:
Already have manifold structure on $L(a,b) = W^u(a) \pitchfork W^s(b)$.
Consider $(\lambda_1, \lambda_2) \in L(a,c) \times L(c,b) \subset \bar{L}(a,b)$.
Need to take a neighborhood and identify with $[0,\delta)$.
Want to embed $[0,\delta)$ into $\bar{L}(a,b)$ (with $0 \mapsto (\lambda_1,\lambda_2)$), and then show that it covers a neighborhood of (λ_1,λ_2).
Again use level set in Morse chart of c to see the trajectories.
Take entry point a_1 of λ_1, and unstable of a in the level set, denoted by $P \cong \mathbb{S}^{i-1} \ni a_1$.
$L(a,c) \subset P$ is finite by transversality. Can take a neighborhood $D \subset P$ of a_1 in the level set such that $\lambda_1 \in L(a,c)$ is the only one





denoted by $P \cong \mathbb{S}^{c-1} \ni a_1$.

$L(a,c) \subset P$ is finite by transversality. Can take a neighborhood $D \subset P$ of a_1 in the level set such that $\lambda_1 \in L(a,c)$ is the only one intersecting D.



$L(a,b) \subset P$ is an open curve. However don't know what it exactly look like in D yet. (Want to say it is an open curve in D with boundary point a_1.)

The open curve is $(D - \{a_1\}) \pitchfork W^s(b)$. However cannot talk about a_1. Go to a lower level to see better.

Flow $D - \{a_1\}$ to a lower level (than c) in the Morse chart. It is an open annulus A with inner boundary being $S_-(c)$.

$W^s(b) \cap \big(A \cup S_-(c)\big) = (L(a,b)$ seen in $A) \cup L(c,b)$ is a finite union of curves with boundary $W^s(b) \cap S_-(c) = L(c,b)$ contaning λ_2.

Thus we can pick the curve with boundary λ_2 and identify with $[0,\delta)$.

The image of $[0,\delta)$ *covers an open neighborhood of* (λ_1,λ_2):

Take a sequence $\bar{L}(a,b) \ni l_n \to (\lambda_1,\lambda_2)$. Want to show it gradually falls in image of $[0,\delta)$.

If infinitely many l_n are smooth, consider their intersections with the lower level set, which lie in the open curve and tend to its boundary λ_2. Hence gradually fall in $[0,\delta)$.

Otherwise can assume they all have broken point c. Just finitely many due to index reason. Since they tend to (λ_1,λ_2), gradually all of them are (λ_1,λ_2) (corresponding to the point $0 \in [0,\delta)$).

Thus $\bar{L}(a,b)$ is either a circle or an interval.

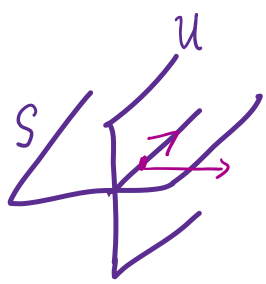
Orientation of moduli

(DON'T NEED M to be oriented.)

Choose orientation on $W^s(c)$ for all c. Then have co-orientation on $W^u(c)$, and hence on

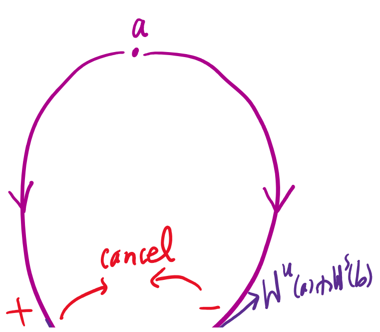
$L(a,b) \cong W^u(a) \pitchfork W^s(b) \pitchfork f^{-1}\{r\}$

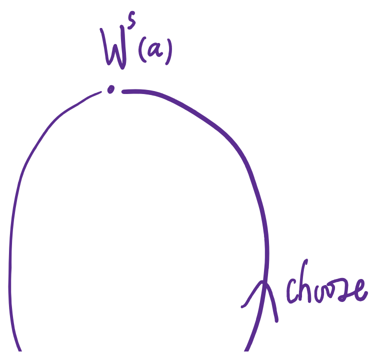
(where co-orientation of $f^{-1}\{r\}$ is given by pseudo-grad).

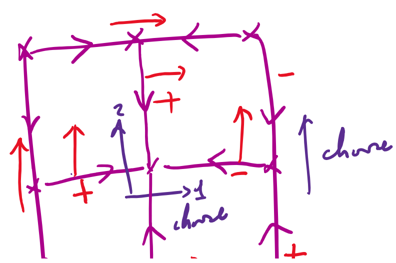


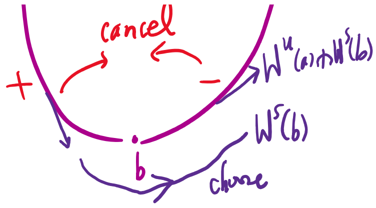
***S* ⋔ *U* is oriented if *S* is oriented and *U* is co-oriented**: take a basis B of $S \cap U$, extended it to that of S by attaching oriented basis of $N(U) \cong S/(S \cap U)$. B is oriented if the extended basis is oriented in S.

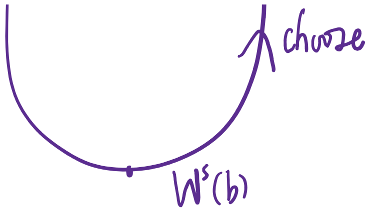
Then have correct signs.

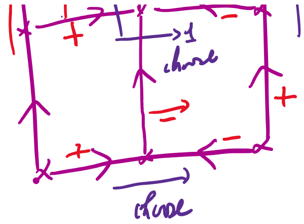










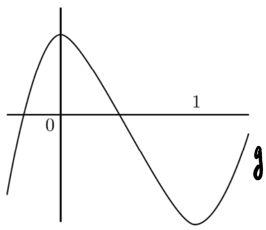


Morse homology as an invariant
Want to show it is independent of choice of f and X.
For (f_0, X_0) and (f_1, X_1), easy to find a homotopy F between them. Need to show that it induces a morphism of the two chain complexes, which is compatible with concatenation of homotopies.

Given a homotopy $F: M \times [0,1] \to \mathbb{R}$ has $F_s = f_0$ for $s \le \frac{1}{3}$ and $F_s = f_1$ for $s \ge \frac{2}{3}$. Can extend F to $s \in \mathbb{R}$ trivially.

Want to make it Morse with only critical points being $\mathrm{Crit}(f_0) \times \{0\}$ and $\mathrm{Crit}(f_1) \times \{1\}$.

Take $\tilde{F} = F(x,s) + g(s)$, where $g'(0) = g'(1) = 0$, $\partial_s F + g' < 0$ for $s \in (0,1)$.



Note that flow direction must be from $s = 0$ to $s = 1$.

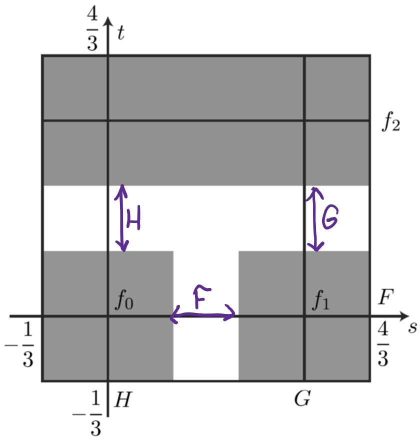
$Ind_{\tilde{F}}(a,0) = Ind_{f_0}(a) + 1, \quad Ind_{\tilde{F}}(b,1) = Ind_{f_1}(b)$.

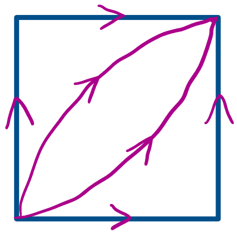
Use partition of unity to construct X which equals to $X_0 - \mathrm{grad}\, g$ for $s < \frac{1}{3}$ and $X_1 - \mathrm{grad}\, g$ for $s > \frac{2}{3}$.

Can perturb a little bit to Smale $\tilde{X}$. It is still transverse to $M \times \left\{-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}\right\}$. Moreover since $X|_{\left[-\frac{1}{3},\frac{1}{3}\right]\cup\left[\frac{2}{3},\frac{4}{3}\right]}$ is Smale, a small perturbation $\tilde{X}$ has critical points and flow lines that can be identified with that of X when restricted to $\left[-\frac{1}{3}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{4}{3}\right]$.

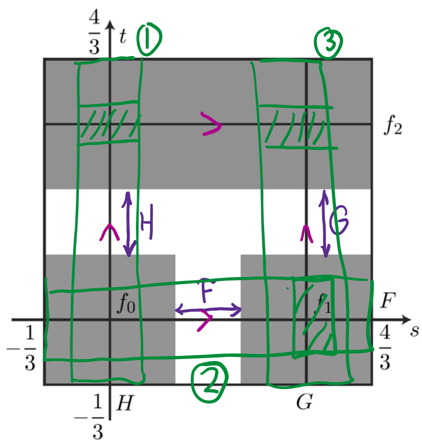
Now consider Morse complex of $\tilde{X}$. Two kinds of trajectories:
$(a_1, 0) \to (a_2, 0)$ or $(b_1, 0) \to (b_2, 0)$, and $(a, 0) \to (b, 1)$.
Can be written as
$\partial = \begin{pmatrix} \partial_{X_0} & 0 \\ \Phi & \partial_{X_1} \end{pmatrix}$ where $\Phi: C^*(M, f, X_0) \to C^*(M, f, X_1)$.
(Note that Φ has degree zero.)
By $\partial^2 = 0$, have
$\Phi \circ \partial_{X_0} + \partial_{X_1} \circ \Phi = 0$.
Hence Φ descends to morphism on homology.
(Indeed Φ on chain level depends on choice of perturbations in the construction)

For homotopies $F: (f_0, X_0) \sim (f_1, X_1)$, $G: (f_1, X_1) \sim (f_2, X_2)$, $H: (f_0, X_0) \sim (f_2, X_2)$ (that are identities on the two ends $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$), take homotopy between $F \circ G$ and H, which induces identification between $\Phi_{F \circ G}$ and Φ_H on homologies.





Construct a map $K: M \times \left[-\frac{1}{3}, \frac{4}{3}\right] \times \left[-\frac{1}{3}, \frac{4}{3}\right]$ as shown. Again can modify
$\widetilde{K} = K(x, s, t) + g(s) + g(t)$
where $\partial_s K + g'(s) < 0$ and $\partial_t K + g'(t) < 0$ for $s, t \in [0,1]$
(and g is like above with $g'(0) = g'(1) = 0$).
$\widetilde{K}$ has critical points $(a, 0,0), (b, 1,0), (c, 0,1), (c, 1,1)$ where
$Ind(a, 0,0) = Ind(a) + 2, \quad Ind(b, 1,0) = Ind(b) + 1,$
$Ind\ (c, 0,1) = Ind(c) + 1, \quad Ind(c, 1,1) = Ind(c).$
Use partition of unity to construct X (pseudo-gradient of $\widetilde{K}$) agreeing with



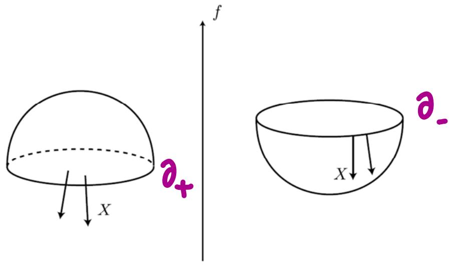
$X_{H+g(t)} - \text{grad } g(s)$ on $s \in \left[-\frac{1}{3}, \frac{1}{3}\right]$,
$X_{G+g(t)} - \text{grad } g(s)$ on $s \in \left[\frac{2}{3}, \frac{4}{3}\right]$.
$X_{F+g(s)} - \text{grad } g(t)$ on $t \in \left[-\frac{1}{3}, \frac{1}{3}\right]$,
$X_{f_2} - \text{grad } g(s) - \text{grad } g(t)$ on $t \in \left[\frac{2}{3}, \frac{4}{3}\right]$.
Then perturb to Smale. Already Smale in shaded region.
First perturb to $\tilde{X}_{G+g(t)}$ and $\tilde{X}_{H+g(t)}$ (near f_2, and use partition of unity to glue with the original). Note the variables s and t are still separated.
Then perturb to $\tilde{X}_{F+g(s)}$ (near f_1).
Then already Smale in the four strips. Also no flow from (0,1) to (1,0). Finally perturb to Smale for whole domain.
Trajectories in the four strips have one-one correspondence with the original ones.
Morse complex of $\tilde{X}$:

$$\partial = \begin{pmatrix} \partial_{X_0} & 0 & 0 & 0 \\ \Phi_F & \partial_{X_1} & 0 & 0 \\ \Phi_H & 0 & \partial_{X_2} & 0 \\ S & \Phi_G & Id & \partial_{X_2} \end{pmatrix}$$

$\Phi^G \circ \Phi^F - \Phi^H = S \circ \partial_{X_0} + \partial_{X_2} \circ S.$
S gives a homotopy.

Then take $(f_2, X_2) = (f_0, X_0)$, H to be identity. Easy to see $\Phi^H = \text{Id}$. Hence $\Phi^G = \left(\Phi^F\right)^{-1}$ on homology.

Morse homology defined similar for manifold with boundary.
Need to choose which components belong to ∂_+ or ∂_-. The pseudo-gradient is required to be outward on ∂_+ and inward on ∂_- (this pose a condition for f).



Left: $H_* = \mathbb{Z}[-n]$. Right: $\mathbb{Z}$.
Call it $H_*(M, \partial_+ M)$. (RHS: $\partial_+ = \emptyset$. So it is $H_*(M)$.)