The complex.
Take formal span of critical points (over $\mathbb{Z}$ or $\mathbb{Z}_2$) graded by index, and count trajectories (up to reparametrization) between them to define a complex.

$$\partial_X(a) = \sum_{b \in \text{Crit}_{a-1}} n_X(a, b) b$$

ex. $S^2$.

$$\mathcal{L}(a, b)$$

$$\mathcal{H} = \{ \mathcal{L}[2] - \mathcal{L}[2], [0] \}.$$ 

Fixing $a, b$ with $\deg a = \deg b + 2$, need $\sum_c n(a, c) \cdot n(c, b) = 0$.
LHS is number of broken trajectories from $a$ to $b$.
Want to prove this is the boundary of a 1d manifold (and hence cancel with each other).
So compactify the space of smooth trajectories $L(a, b)$ by broken trajectories. (The $\mathbb{R}$ action acts freely and so easy to quotient)

Broken trajectories:

$$\mathcal{L}(a, b) = \bigcup_{c \in \text{Crit}(f)} \mathcal{L}(a, c_1) \times \cdots \times \mathcal{L}(c_{q-1}, b).$$

Topology on $L(a, b)$:
Fix a broken $\lambda = (\lambda_1, ..., \lambda_q)$. Its neighborhood consists of its deformations and smoothings.
Key: every smooth trajectory has $\mathbb{R}$ symmetry. Can reduce to a level set in a Morse chart to see all its small deformations!
Base of open sets: $W(\lambda, U^-, U^+)$
where $U^-$ consists of a neighborhood of the exit point of $\lambda$ in a Morse chart of each broken point; similarly $U^+$ consists of neighborhoods of entry points;
$W(\lambda, U^-, U^+)$ consists of broken trajectories whose broken points are subset of that of $\lambda$, and whose exit and entry points
neighborhoods of entry points; \( W(\lambda, U^-, U^+) \) consists of broken trajectories whose broken points are subset of that of \( \lambda \), and whose exit and entry points are contained in \( U^- \) and \( U^+ \).

**Smooth trajectories must limit to broken trajectories:**

*Key: Find the limit using Morse charts.*

Consider a sequence of smooth \( l_n \in L(a, b) \).
Consider exit and entry points \( l_n^- , l_n^+ \) in Morse charts of \( a, b \).
Have limit \( a^-, b^+ \) by taking subsequence. (Unstable and stable intersect a level set at a sphere which is compact.)
Take the trajectory \( l^1 \) thru \( a^- \), which flows to certain \( c_1 \).
Let \( d^+ \) be the entry point to \( c_1 \). \( l_n \) enters Morse chart of \( c_1 \) at \( d_n^+ \) where \( d_n^+ \rightarrow d^+ \).
If \( c_1 = b \), then \( d^+ = b^+ \). \( l_n \) limits to \( l^1 \) by definition of topology.
Otherwise \( l_n \) (which ends at \( b \)) exits the Morse chart of \( c_1 \) at \( d_n^- \) which has limit \( d^- \).
\( d^- \) lies in the unstable of \( c_1 \): otherwise the trajectory \( l^2 \) of \( d^- \) enters the Morse chart of \( c_1 \) at certain point, and this point must be \( d^+ \) as \( d_n^- \rightarrow d^- \). Contradiction: the trajectory through \( d^+ \) (which is \( l^1 \)) ends at \( c_1 \).
Consider \( l^2 \) thru \( d^- \) emanated from \( c_1 \). \( d^- \) plays the role of \( a^- \) and \( c_1 \) plays the role of \( a \). Do the argument again, and we find a sequence \( l^i \in L(c_{i-1}, c_i) \). \( f(c_i) \geq f(b) \), and hence the sequence must be finite, and the final \( c_q = b \). By definition of the topology \( l_n \) limits to \( (l^1, ..., l^q) \).

**Corollary:** If index difference is one, \( L(a, b) \) is already compact and hence is a finite set. (Cannot break.)

\( \bar{L}(a, b) \) is compact:
Already know that a sequence of smooth \( l_n \in L(a, b) \) has convergent subsequence.
Consider a sequence \( l_n \in \bar{L}(a, b) \).
Have a subsequence in \( L(a, c_1) \times ... \times L(c_{q-1}, b) \subset \bar{L}(a, b) \).
Then just apply the known result to each factor.

**Manifold with boundary structure on \( \bar{L}(a, b) \):**
Just do it for \( \text{Ind}(a) = \text{Ind}(b) + 2 \). It is a 1d manifold with boundary:
Already have manifold structure on \( L(a, b) = W^u(a) \cap W^s(b) \).
Consider \( (\lambda_1, \lambda_2) \in L(a, c) \times L(c, b) \subset \bar{L}(a, b) \).

*Need to take a neighborhood and identify with [0, \( \delta \)].*
Want to embed [0, \( \delta \)] into \( \bar{L}(a, b) \) (with 0 \( \mapsto (\lambda_1, \lambda_2) \)), and then show that it covers a neighborhood of \( (\lambda_1, \lambda_2) \).
Again use level set in Morse chart of \( c \) to see the trajectories.
Take entry point \( a_1 \) of \( \lambda_1 \), and unstable of \( a \) in the level set, denoted by \( P \cong S^{d-1} \subset a_1 \).
\( L(a, c) \subset P \) is finite by transversality. Can take a neighborhood \( D \subset P \) of \( a_1 \), in the level set such that \( \lambda, \in L(a, c) \) is the only one
denoted by $P \cong S^r \ni \exists a_1$.
$L(a, c) \subset P$ is finite by transversality. Can take a neighborhood $D \subset P$ of $a_1$ in the level set such that $\lambda_1 \in L(a, c)$ is the only one intersecting $D$.
$L(a, b) \subset P$ is an open curve. However don’t know what it exactly look like in $D$ yet. (Want to say it is an open curve in $D$ with boundary point $a_1$.)
The open curve is $(D - \{a_1\}) \pitchfork W^s(b)$. However cannot talk about $a_1$. Go to a lower level to see better.
Flow $D - \{a_1\}$ to a lower level (than $c$) in the Morse chart. It is an open annulus $A$ with inner boundary being $S_-(c)$.
$W^s(b) \cap (A \cup S_-(c)) = (L(a, b) \text{ seen in } A) \cup L(c, b)$ is a finite union of curves with boundary $W^s(b) \cap S_-(c) = L(c, b)$ containing $\lambda_2$.
Thus we can pick the curve with boundary $\lambda_2$ and identify with $[0, \delta]$.

The image of $[0, \delta)$ covers an open neighborhood of $(\lambda_1, \lambda_2)$:
Take a sequence $L(a, b) \ni l_n \to (\lambda_1, \lambda_2)$. Want to show it gradually falls in image of $[0, \delta)$.
If infinitely many $l_n$ are smooth, consider their intersections with the lower level set, which lie in the open curve and tend to its boundary $\lambda_2$. Hence gradually fall in $[0, \delta)$.
Otherwise can assume they all have broken point $c$. Just finitely many due to index reason. Since they tend to $(\lambda_1, \lambda_2)$, gradually all of them are $(\lambda_1, \lambda_2)$ (corresponding to the point $0 \in [0, \delta)$).

Thus $L(a, b)$ is either a circle or an interval.

**Orientation of moduli**
(DON’T NEED $M$ to be oriented.)
Choose orientation on $W^s(c)$ for all $c$. Then have co-orientation on $W^u(c)$, and hence on $L(a, b) \cong W^u(a) \pitchfork W^s(b) \pitchfork f^{-1}\{r\}$
(where co-orientation of $f^{-1}\{r\}$ is given by pseudo-grad).
$S \pitchfork U$ is oriented if $S$ is oriented and $U$ is co-oriented: take a basis $B$ of $S \cap U$, extended it to that of $S$ by attaching oriented basis of $N(U) \cong S/(S \cap U)$. $B$ is oriented if the extended basis is oriented in $S$.
Then have correct signs.
Morse homology as an invariant
Want to show it is independent of choice of $f$ and $X$.
For $(f_0, X_0)$ and $(f_1, X_1)$, easy to find a homotopy $F$ between them. Need to show that it induces a morphism of the two chain complexes, which is compatible with concatenation of homotopies.

Given a homotopy $F: M \times [0,1] \to \mathbb{R}$ has $F_s = f_0$ for $s \leq \frac{1}{3}$ and $F_s = f_1$ for $s \geq \frac{2}{3}$. Can extend $F$ to $s \in \mathbb{R}$ trivially.
Want to make it Morse with only critical points being $\text{Crit}(f_0) \times \{0\}$ and $\text{Crit}(f_1) \times \{1\}$.
Take $\tilde{F} = F(x, s) + g(s)$, where $g'(0) = g'(1) = 0$, $\partial_s F + g' < 0$ for $s \in (0,1)$.
Note that flow direction must be from $s = 0$ to $s = 1$.
$\text{Ind}_{\tilde{F}}(a, 0) = \text{Ind}_{f_0}(a) + 1$, $\text{Ind}_{\tilde{F}}(b, 1) = \text{Ind}_{f_1}(b)$.
Use partition of unity to construct $X$ which equals to $X_0 - \text{grad } g$ for $s < \frac{1}{3}$ and $X_1 - \text{grad } g$ for $s > \frac{2}{3}$.
Can perturb a little bit to Smale $\tilde{X}$. It is still transverse to $M \times \left\{ -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3} \right\}$. Moreover since $X|_{\left[ -\frac{1}{3}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{4}{3} \right]}$ is Smale, a small perturbation $\tilde{X}$ has critical points and flow lines that can be identified with that of $X$ when restricted to $\left[ -\frac{1}{3}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{4}{3} \right]$.

Now consider Morse complex of $\tilde{X}$. Two kinds of trajectories: $(a_1, 0) \to (a_2, 0)$ or $(b_1, 0) \to (b_2, 0)$, and $(a, 0) \to (b, 1)$.
Can be written as $\partial = \begin{pmatrix} \partial_{x_0} & 0 \\ \Phi & \partial_{x_1} \end{pmatrix}$ where $\Phi: C^*(M, f, X_0) \to C^*(M, f, X_1)$.
(Note that $\Phi$ has degree zero.)
By $\partial^2 = 0$, have $\Phi \circ \partial_{x_0} + \partial_{x_1} \circ \Phi = 0$.
Hence $\Phi$ descends to morphism on homology.
(Indeed $\Phi$ on chain level depends on choice of perturbations in the construction)

For homotopies $F: (f_0, X_0) \sim (f_1, X_1)$, $G: (f_1, X_1) \sim (f_2, X_2)$, $H: (f_0, X_0) \sim (f_2, X_2)$ (that are identities on the two ends $\left[ 0, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, 1 \right]$), take homotopy between $F \circ G$ and $H$, which induces identification between $\Phi_{F \circ G}$ and $\Phi_H$ on homologies.
Construct a map $K: M \times \left[-\frac{1}{3}, \frac{4}{3}\right] \times \left[-\frac{1}{3}, \frac{4}{3}\right]$ as shown. Again can modify $\tilde{K} = K(x,s,t) + g(s) + g(t)$ where $\partial_x K + g'(s) < 0$ and $\partial_t K + g'(t) < 0$ for $s, t \in [0,1]$ (and $g$ is like above with $g'(0) = g'(1) = 0$).

$\tilde{K}$ has critical points $(a, 0,0), (b, 1,0), (c, 0,1), (c, 1,1)$ where $\text{Ind}(a,0,0) = \text{Ind}(a) + 2$, $\text{Ind}(b,1,0) = \text{Ind}(b) + 1$, $\text{Ind}(c,0,1) = \text{Ind}(c) + 1$, $\text{Ind}(c,1,1) = \text{Ind}(c)$.

Use partition of unity to construct $X$ (pseudo-gradient of $\tilde{K}$) agreeing with

$X_{H+g(t)} - \text{grad } g(s)$ on $s \in \left[-\frac{1}{3}, \frac{1}{3}\right]$,

$X_{G+g(t)} - \text{grad } g(s)$ on $s \in \left[\frac{2}{3}, \frac{4}{3}\right]$,

$X_{F+g(s)} - \text{grad } g(t)$ on $t \in \left[-\frac{1}{3}, \frac{1}{3}\right]$,

$X_{f_2} - \text{grad } g(s) - \text{grad } g(t)$ on $t \in \left[\frac{2}{3}, \frac{4}{3}\right]$.

Then perturb to Smale. Already Smale in shaded region. First perturb to $\tilde{X}_{G+g(t)}$ and $\tilde{X}_{H+g(t)}$ (near $f_2$, and use partition of unity to glue with the original). Note the variables $s$ and $t$ are still separated.

Then perturb to $\tilde{X}_{F+g(s)}$ (near $f_1$). Then already Smale in the four strips. Also no flow from $(0,1)$ to $(1,0)$. Finally perturb to Smale for whole domain.

Trajectories in the four strips have one-one correspondence with the original ones.

Morse complex of $\tilde{X}$:

$$\partial = \begin{pmatrix}
\partial_{X_0} & 0 & 0 \\
\Phi_F & \partial_{X_1} & 0 \\
\Phi_H & 0 & \partial_{X_2} \\
S & \Phi_G & \text{Id}
\end{pmatrix}$$

$\Phi^G \circ \Phi^F - \Phi^H = S \circ \partial_{X_0} + \partial_{X_2} \circ S$.

$S$ gives a homotopy.

Then take $(f_2, X_2) = (f_0, X_0)$, $H$ to be identity. Easy to see $\Phi^H = \text{Id}$. Hence $\Phi^G = (\Phi^F)^{-1}$ on homology.
Morse homology defined similar for manifold with boundary. Need to choose which components belong to $\partial_+$ or $\partial_-. The pseudo-gradient is required to be outward on $\partial_+$ and inward on $\partial_- \text{ (this pose a condition for } f)$. 

Left: $H_* = \mathbb{Z}[-n]$. Right: $\mathbb{Z}$. Call it $H_*(M, \partial_+ M)$. (RHS: $\partial_+ = \emptyset$. So it is $H_*(M)$.)