4. Properties of Morse homology

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Kunneth formula $HM_*(M \times N) \cong HM_*(M) \otimes HM_*(N)$.

Take $(f+g, X+Y)$ on $M \times N$.

Critical points are $a \otimes a'$ for $a \in Crit(f), a' \in Crit(g)$.

$Ind(a \otimes a') = Ind(a) + Ind(a')$.

$L((a,a'),(b,b')) = L(a,b) \times L(a',b')$.

Hence $\bar{L}((a,a'),(b,b')) = \bar{L}(a,b) \times \bar{L}(a',b')$.

$(\boldsymbol{C_*(f) \otimes C_*(g), \partial_X \otimes Id + Id \otimes \partial_Y}) \cong (\boldsymbol{C_*(f+g), \partial_{X+Y}})$:

∂_{X+Y} increases $Ind(a \otimes a')$ by 1. Output $(b \otimes b')$ has $Ind(b) = Ind(a) + 1, Ind(b') = Ind(a')$ (or the other way). $L(a',b') = \emptyset$ by Smale, unless $a' = b'$.

(Note in this case $\mathbb{R}$ acts trivially, and so $L(a',a') = W^u(a') \cap W^s(a')/\mathbb{R}$ is 0-dim.)

$\bar{L}\big((a,a'),(b,a')\big) = \bar{L}(a,b) \times \{a'\}$, and so $n\big((a,a'),(b,a')\big) = n(a,b)$.

Poincare duality $HM^k(M, \partial_+ M) \cong HM_{n-k}(M, \partial_- M)$.

Morse cohomology defined by $(C_k(f)^*, \partial_X^*)$.

$C_k(f)^*$ can be identified as $C_k(f)$ since already have fixed basis.

$\partial_X^* a^* = \sum n_b b^*$ where $n_b = (\partial_X^* a^*, b) = (a^*, \partial_X b) = n(b,a)$.

$C_k(f)^* = C_{n-k}(-f)$. **Reverse the flow!**

$(C_k(f)^*, \partial_X^*) = (C_{n-k}(-f), \partial_{-X})$.

Thus $HM^k(M, \partial_+ M) \cong HM_{n-k}(M, \partial_- M)$.

NEED M oriented to be over $\mathbb{Z}$: then $W^u(c)$ has induced orientation.

Number of critical points

$\#\text{Crit} = \ker + \text{im} \cong \ker - \text{im}\ (mod\ 2) = \dim HM$

and hence is invariant.

Morse inequality

χ equals to alternating sum of numbers of odd and even critical points.

Moreover $\#\text{Crit}_k \geq \beta_k$.

Prop. HM_0 has rank one for compact connected manifold.

Proof:

$C_0(f) = \text{Span}\,\{a_1, \dots, a_r\}$

where a_1 is the absolute minimum. Want

$Im\ \partial_1 = \text{Span}\,\{a_1 + a_2, \dots, a_1 + a_r\}$.

$\subset$: $\partial_1 c = a_i + a_j = (a_1 + a_i) + (a_1 + a_j)$ (mod 2).

$\supset$: consider A_j, the set of points ending at a_j through a broken trajectory with (possible) broken point to be index 1. A_j are closed.

If $A_i \cap A_j$ non-empty, must have c with $\partial_1 c = a_i + a_j$.

Note: $\bigcup_j A_j$ is the complement N of $W^s(c)$ for $Ind(c) \geq 2$.

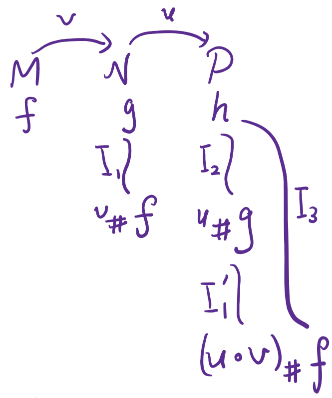
N is still connected.

Hence $A_1 \cap (\bigcup_{j \neq 1} A_j)$ must be non-empty, and hence must have some $A_j \cap A_1 \neq \emptyset$.

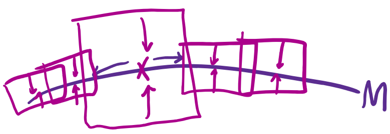
Replace A_1 by $A_1 \cup A_j$ and repeat the argument inductively.
Thus all a_i are homologous to a_1.

Prop. If there is a Morse function with no critical points of index 1, then M is simply connected.
Proof:
Take min point to be the base point. Consider a loop.
$W^s(p)$ for $\text{ind}(p) \geq 2$ has $codim \geq 2$, and so the loop can be made not intersecting with it.
Note that M is disjoint union of all the stables.
Thus the loop has to be entirely contained in stable disc of the min. point. Then obviously contractible.

Functoriality Theorem. Morse homology is a functor from (M, f, X) to graded Abel. groups.
(Morphism $(M, f, X) \to (N, g, Y)$ is simply a map $u: M \to N$.)
Moreover homotopic morphisms are sent to the same morphism of groups.
Proof:
If u is diffeomorphism:
just interpolate $(N, (u^{-1})^* f, u_* X)$ and (N, g, Y) which induces an isomorphism I on homology, independent of choice of interpolation (which are all homotopic to each other).
u induces an isomorphism $u_\#: C_*(f, X) \to C_*((u^{-1})^* f, u_* X)$.
$u_* := I \circ u_\#$.
$(\boldsymbol{u} \circ \boldsymbol{v})_* = \boldsymbol{u}_* \circ \boldsymbol{v}_*$:
$(u \circ v)_* = I_3 \circ (u \circ v)_\# = I_3 \circ u_\# \circ v_\#$.
$u_* \circ v_* = (I_2 \circ u_\#) \circ (I_1 \circ v_\#) = I_2 \circ (u_\# \circ I_1) \circ v_\#$.
u pushes interpolation I_1 is pushed to interpolation I_1'. Thus $u_\# \circ I_1 = I_1' \circ u_\#$. Then
$u_* \circ v_* = I_2 \circ I_1' \circ u_\# \circ u_\# = I_3 \circ u_\# \circ v_\#$
since the interpolations $I_2 \circ I_1'$ and I_3 are homotopic.



If u: $M \to N$ is an embedding:
extend (f, X) on M to $(\tilde{f}, \tilde{X})$ on N. Need $Crit(f) \subset Crit(\tilde{f})$ (with same indices) and $\tilde{\partial} a = \partial a$.
Then have a morphism $u_\#: C_*(f, X) \to C_*(\tilde{f}, \tilde{X})$. Then interpolate from $(\tilde{f}, \tilde{X})$ to (g, Y) as before. $(u \circ v)_* = u_* \circ v_*$ like before.
How to extend:
near critical point of M, take chart and
$\tilde{f} = f(x) + \frac{1}{2}|y|^2$ (don't change index)
$\tilde{X} = (X(x), -y)$.



Then extend by partition of unity in the complement. (Non-critical point of (M, f) remains to be non-critical point of $(N, \tilde{f})$.) Also it is made that $\tilde{X}$ is pointing inward in tubular neighborhood.

There is $\epsilon > 0$ such that $df(X) < -\epsilon$ in the complement of Morse chart in M.
Also $d\tilde{f}(\tilde{X}) \leq 0$ near critical point.
So $d\tilde{f}(\tilde{X}) \leq 0$ for some neighborhood U of $M \subset N$.

In U, $\tilde{f}$ only has (Morse) critical points being that f, and $\tilde{X}$ is a pseudogradient.

Then perturb $\tilde{f}$ outside U such that it becomes Morse, and extend $\tilde{X}$ from U to N using partition of unity such that it is pseudogradient.

Need to perturb $\tilde{X}$ to Smale. Restriction to U already Smale ($L(a,b)$ exactly the same). Small perturbation still have the same set of trajectories. Hence inclusion gives a chain map.

Homotopic morphisms u_t are sent to the same morphism of groups in case u_t are embedding:

u_t gives $[0,1] \times M \to N$, and hence $\tilde{u}: [0,1] \times M \to [0,1] \times N$ which is an embedding. Want: $(u_0)_* = (u_1)_*$.

$$\begin{tikzcd} {[0,1]} \times \cancel{V} M \arrow[r, "\tilde{u}"] & {[0,1]} \times \cancel{W} N \\ \cancel{V} M \arrow[u, "i_0^{\cancel{V} M}"] \arrow[r, "u_0"'] & \cancel{W} N \arrow[u, "i_0^{\cancel{W} N}"'] \end{tikzcd} \qquad \begin{tikzcd} {[0,1]} \times \cancel{V} M \arrow[r, "\tilde{u}"] & {[0,1]} \times \cancel{W} N \\ \cancel{V} M \arrow[u, "i_1^{\cancel{V} M}"] \arrow[r, "u_1"'] & \cancel{W} N \arrow[u, "i_1^{\cancel{W} N}"'] \end{tikzcd}$$

$$\begin{tikzcd} {[0,1]} \times M \arrow[dr] & \\ M \arrow[u, shift left, "i_1"'] \arrow[u, shift right, "i_0"] \arrow[r, bend left, "u_0"] \arrow[r, bend right, "u_1"'] & N \end{tikzcd}$$

NOT embedding! Don't know composition behaves well.

i_0^M induces isomorphism on HM: (extend $\tilde{u}$ a bit to $[-\epsilon, 1+\epsilon]$ to say this)

Take $\tilde{f} = f + \frac{y^2}{2}$ on $[-\epsilon, 1+\epsilon] \times M$. Then obvious $\tilde{f}$ and f have the same Morse complex.

Then $u_{0_*} = \left(i^N_{0\,*}\right)^{-1} \circ \tilde{u}_* \circ i^M_{0\,*}$.

$i_{0_*} = i_{1_*}$. Hence $u_{0_*} = u_{1_*}$.

Take a diffeomorphism ϕ on $[-\epsilon, 1+\epsilon]$ sending 0 to 1. The critical point of $\frac{y^2}{2}$ is sent to the critical pint of $\frac{(y-1)^2}{2}$. (Don't even need interpolation which is not explicit!) Hence $(\phi, Id)_* = Id$.

Thus $\iota_{1_*} = \left((\phi, Id) \circ \iota_0\right)_* = \iota_{0_*}$.

$$\begin{tikzcd} & {[-\varepsilon, 1+\varepsilon]} \times M \\ M \arrow[ur, "\imath_1"] \arrow[dr, "\imath_0"'] & \\ & {[-\varepsilon, 1+\varepsilon]} \times M \arrow[uu, "\imath"', "{(\phi, Id)}"] \end{tikzcd}$$

General $u: M \to N$: enlarge the target to make embedding!

Take embedding $\phi: M \to D^n$. Then $(\phi, u): M \to D \times N$ is embedding, so have $(\phi, u)_*$ to $HM(D \times N)$. Go back to $HM(N)$ by $(i_0)_*^{-1}$.

$u_* := (i_0)_*^{-1} \circ (\phi, u)_*$.

Independent of choice of ϕ:

Say have a different embedding ψ. Have homotopy $i_0 \circ u = (0, u) \sim (\phi, u)$. But u is not an embedding and does not induce map on homology directly. Combine the two choices (ψ, ϕ) and get third choice!

$$\begin{tikzcd} & D^N \times \cancel{W} N \arrow[dr, "l_0"] & \\ \cancel{W} N \arrow[ur, "i_0"] \arrow[dr, "j_0"'] & \cancel{V} M \arrow[l, "u"'] \arrow[u, "{(\varphi, u)}"'] \arrow[d, "{(\psi, u)}"] \arrow[r, "{(\psi, \varphi, u)}"] & D^M \times D^N \times \cancel{W} N \\ & D^M \times \cancel{W} N \arrow[ur, "k_0"'] & \end{tikzcd}$$

The two arrows equal!

Homotopy $l_0 \circ (\phi, u) = (0, \phi, u) \sim (\psi, \phi, u)$ by $(t\psi, \phi, u)$. Thus diagram commutes on homology.

Composition: Enlarge the target for $v \circ u$ by combining that for the individual u and v.

$$\begin{tikzcd}
V \arrow[r, "u"] \arrow[d, "{(\varphi, u)}"'] & W \arrow[r, "v"] \arrow[dl, "i_0"'] \arrow[d, "{(\psi, v)}"] & Z \arrow[dl, "j_0"] \\
D^N \times W \arrow[d, "{(\mathrm{Id}, (\psi, v))}"'] & D^M \times Z \arrow[dl, "k_0"] & \\
D^{\cancel{M}N} \times D^{\cancel{N}M} \times Z & &
\end{tikzcd}$$

$$\begin{aligned}
v_\star \circ u_\star &= (j_0)_\star^{-1} \circ (\psi, v)_\star \circ (i_0)_\star^{-1} \circ (\varphi, u)_\star \\
&= (j_0)_\star^{-1} \circ (k_0)_\star^{-1} \circ (\psi \circ (\mathrm{Id}, v))_\star \circ (\varphi, u)_\star \\
&= (v \circ u)_\star.
\end{aligned}$$

Homotopic maps induces same morphism on homology:
Can take ϕ embedding, so (ϕ, u_t) are embedding for all $t \in [0,1]$.
Already know $(\phi, u_t)_*$ are all the same. So does $u_* = (i_0)_*^{-1} \circ (\phi, u)_*$.
END OF PROOF OF FUNCTORIALITY

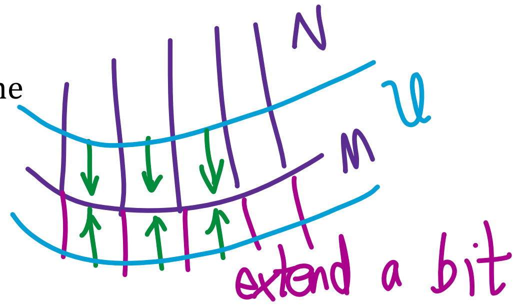
Long exact sequence $HM_k(M) \to HM_k(N) \to HM_k(N, M) \to \cdots$

For embedding $M \to N$, can extend (f, X) to $(\tilde{f}, \tilde{X})$ on N like before such that $\tilde{X}$ is pointing inward to M on ∂U, boundary of tubular neighborhood. So $\partial U = \partial_+ \overline{N - U}$. Also $\tilde{X}$ is perturbed to Smale.

$HM(N, M) \coloneqq HM(\overline{N - U}, \partial U) = HM\left((\tilde{f}, \tilde{X})|_{\overline{N-U}}\right)$.

Have chain maps $CM_k(f, X) \to CM_k(\tilde{f}, \tilde{X}) \to CM_k\left((\tilde{f}, \tilde{X})|_{\overline{N-U}}\right)$. The last arrow is chain map $(\partial_{\tilde{X}}\, p)|_{\overline{N-U}} = \partial_{\tilde{X}|_{\overline{N-U}}}(p|_{\overline{N-U}})$:
if $p \in M$, then both sides are zero. If $p \in \overline{N - U}$, then both sides are counting those trajectories not leaving $\overline{N - U}$.

Then have the long exact sequence. Connecting homo. $\partial: CM_k\left((\tilde{f}, \tilde{X})|_{\overline{N-U}}\right) \to CM_{k-1}(f, X)$ is simply counting trajectories from $N - M$ to M.



When N is manifold with boundary M, similar by extending N a bit to $\tilde{N}$ so that M is a hypersurface in $\tilde{N}$. Then define $HM_k(N, \partial N)$ as $HM_k(\tilde{N}, M)$.

ex. $N = D^n$. $HM_k(D^n, \partial D) = \mathbb{Z}$ for $k = n$ and zero otherwise.
$HM_k(D^n) = \mathbb{Z}$ for $k = 0$ and zero otherwise. So
$HM_0(\mathbb{S}^{n-1}) \cong HM_0(D^n)$;
$HM_n(D^n, \mathbb{S}^{n-1}) \cong HM_{n-1}(\mathbb{S}^{n-1})$.