Kunneth formula $HM_*(M \times N) \cong HM_*(M) \otimes HM_*(N)$.
Take $(f + g, X + Y)$ on $M \times N$.
Critical points are $a \otimes a'$ for $a \in \text{Crit}(f), a' \in \text{Crit}(g)$.
$\text{Ind}(a \otimes a') = \text{Ind}(a) + \text{Ind}(a')$.
$L((a, a'), (b, b')) = L(a, b) \times L(a', b')$.
Hence $L((a, a'), (b, b')) = L(a, b) \times L(a', b')$.
$(C_*(f) \otimes C_*(g), \partial_X \otimes 1 + 1 \otimes \partial_Y) \cong (C_*(f + g), \partial_{X+Y})$:
$\partial_{X+Y}$ increases $\text{Ind}(a \otimes a')$ by 1. Output $(b \otimes b')$ has $\text{Ind}(b) = \text{Ind}(a) + 1$, $\text{Ind}(b') = \text{Ind}(a')$ (or the other way).
$L(a', b') = \emptyset$ by Smale, unless $a' = b'$.
(Note in this case $\mathbb{R}$ acts trivially, and so $L(a', a') = W^u(a') \cap W^s(a')/\mathbb{R}$ is 0-dim.)
$L((a, a'), (b, a')) = L(a, b) \times \{a'\}$, and so $n((a, a'), (b, a')) = n(a, b)$.

Poincare duality $HM^k(M, \partial_+ M) \cong HM_{n-k}(M, \partial_- M)$.
Morse cohomology defined by $(C_k(f)^*, \partial_X^*)$.
$C_k(f)^*$ can be identified as $C_k(f)$ since already have fixed basis.
$\partial_X^* a^* = \sum n_b b^*$ where $n_b = (\partial_X^* a^*, b) = (a^*, \partial_X b) = n(b, a)$.
$C_k(f)^* = C_{n-k}(-f)$. Reverse the flow!
$(C_k(f)^*, \partial_X^*) = (C_{n-k}(-f), \partial_{-X})$.
Thus $HM^k(M, \partial_+ M) \cong HM_{n-k}(M, \partial_- M)$.

Number of critical points
$\#\text{Crit} = \ker \Box + \text{im} \cong \ker \Box - \text{im} (\text{mod } 2) = \dim HM$
and hence is invariant.

Morse inequality
$\chi$ equals to alternating sum of numbers of odd and even critical points.
Moreover $\#\text{Crit}_k \geq \beta_k$.

Prop. $HM_0$ has rank one for compact connected manifold.
Proof:
$C_0(f) = \text{Span}\{a_1, ..., a_r\}$
where $a_1$ is the absolute minimum. Want
$\text{Im} \partial_1 = \text{Span}\{a_1 + a_2, ..., a_1 + a_r\}$.
$c \subset \partial_1 c = a_i + a_j = (a_i + a_i) + (a_1 + a_j) (\text{mod } 2)$.
$\Rightarrow$ consider $A_j$, the set of points ending at $a_j$ through a broken trajectory with (possible) broken point to be index 1. $A_j$ are closed.
If $A_l \cap A_j$ non-empty, must have $c$ with $\partial_1 c = a_l + a_j$.
Note: $\bigcup A_j$ is the complement $N$ of $W^s(c)$ for $\text{Ind}(c) \geq 2$.
$N$ is still connected.
Hence $A_1 \cap (\bigcup_{j \neq 1} A_j)$ must be non-empty, and hence must have some $A_j \cap A_1 \neq \emptyset$. 
Replace $A_1$ by $A_1 \cup A_j$ and repeat the argument inductively.
Thus all $a_i$ are homologous to $a_1$.

**Prop.** If there is a Morse function with no critical points of index 1, then $M$ is simply connected.

**Proof:**
Take min point to be the base point. Consider a loop $W^s(p)$ for ind$(p) \geq 2$ has codim $\geq 2$, and so the loop can be made not intersecting with it.
Note that $M$ is disjoint union of all the stables.
Thus the loop has to be entirely contained in stable disc of the min. point. Then obviously contractible.

**Functoriality Theorem.** Morse homology is a functor from $(M, f, X)$ to graded Abel. groups.
(Morphism $(M, f, X) \to (N, g, Y)$ is simply a map $u: M \to N$.)
Moreover homotopic morphisms are sent to the same morphism of groups.

**Proof:**
If $u$ is diffeomorphism:
just interpolate $(N, (u^{-1})^* f, u_*X)$ and $(N, g, Y)$ which induces an isomorphism $I$ on homology, independent of choice of interpolation (which are all homotopic to each other).

If $u$ is embedding:
extend $(f, X)$ on $M$ to $(\tilde{f}, \tilde{X})$ on $N$. Need $\text{Crit}(f) \subset \text{Crit}(\tilde{f})$ (with same indices) and $\partial a = \partial a$.
Then have a morphism $u_\#: C_*(f, X) \to C_*(\tilde{f}, \tilde{X})$. Then interpolate from $(\tilde{f}, \tilde{X})$ to $(g, Y)$ as before. $(u \circ v)_* = u_* \circ v_*$ like before.

How to extend:
near critical point of $M$, take chart and
$\tilde{f} = f(x) + \frac{1}{2} |y|^2$ (don’t change index)
$\tilde{X} = (X(x), -y)$.

Then extend by partition of unity in the complement. (Non-critical point of $(M, f)$ remains to be non-critical point of $(N, \tilde{f})$.) Also it is made that $\tilde{X}$ is pointing inward in tubular neighborhood.

There is $\epsilon > 0$ such that $df(X) < -\epsilon$ in the complement of Morse chart in $M$.
Also $d\tilde{f}(\tilde{X}) \leq 0$ near critical point.
So $d\tilde{f}(\tilde{X}) \leq 0$ for some neighborhood $U$ of $M \subset N$. 
In $U$, $\tilde{f}$ only has (Morse) critical points being that $f$, and $\tilde{X}$ is a pseudogradient.

Then perturb $\tilde{f}$ outside $U$ such that it becomes Morse, and extend $\tilde{X}$ from $U$ to $N$ using partition of unity such that it is pseudogradient.

Need to perturb $\tilde{X}$ to Smale. Restriction to $U$ already Smale ($L(a,b)$ exactly the same). Small perturbation still have the same set of trajectories. Hence inclusion gives a chain map.

**Homotopic morphisms $u_t$ are sent to the same morphism of groups**

in case $u_t$ are embedding:

$u_t$ gives $[0,1] \times M \to N$, and hence $\tilde{u}: [0,1] \times [0,1] \times N$ which is an embedding. Want: $(u_0)_* = (u_1)_*$.

$i_0^M$ induces isomorphism on $HM$: (extend $\tilde{u}$ a bit to $[-\epsilon, 1 + \epsilon]$ to say this)

Take $\tilde{f} = f + \frac{y^2}{2}$ on $[-\epsilon, 1 + \epsilon] \times M$. Then obvious $\tilde{f}$ and $f$ have the same Morse complex.

Then $u_{0*} = (i_0^N)^{-1} \circ \tilde{u}_* \circ i_0^M$.

$i_{0*} = i_{1*}$. Hence $u_{0*} = u_{1*}$.

Take a diffeomorphism $\phi$ on $[-\epsilon, 1 + \epsilon]$ sending 0 to 1. The critical point of $\frac{y^2}{2}$ is sent to the critical pint of $\frac{(y-1)^2}{2}$. (Don't even need interpolation which is not explicit!) Hence $\phi(I) = I$.

Thus $u_* = ((\phi, I))_* = I$.

General $u: M \to N$: enlarge the target to make embedding!

Take embedding $\phi: M \to D^n$. Then $(\phi, u): M \to D \times N$ is embedding, so have $(\phi, u)_*$ to $HM(D \times N)$. Go back to $HM(N)$ by $(i_0)^{-1}$.

$u_* := (i_0)^{-1} \circ (\phi, u)_*$.

**Independent of choice of $\phi$:**

Say have a different embedding $\psi$. Have homotopy $i_0 \circ u = (0, u) \sim (\phi, u)$. But $u$ is not an embedding and does not induce map on homology directly. Combine the two choices $(\psi, \phi)$ and get third choice!

Homotopy $l_0 \circ (\phi, u) = (0, \phi, u) \sim (\psi, \phi, u)$ by $(t\psi, \phi, u)$. Thus diagram commutes on homology.

**Composition:** Enlarge the target for $v \circ u$ by combining that for the
individual $u$ and $v$.

Homotopic maps induces same morphism on homology:
Can take $\phi$ embedding, so $(\phi, u_t)$ are embedding for all $t \in [0,1]$. Already know $(\phi, u_t)_*$ are all the same. So does $u_* = (i_0)^{-1} \circ (\phi, u)_*$.

END OF PROOF OF Functoriality

**Long exact sequence** $HM_k(M) \to HM_k(N) \to HM_k(N, M) \to \cdots$

For embedding $M \to N$, can extend $(f, X)$ to $(\tilde{f}, \tilde{X})$ on $N$ like before such that $\tilde{X}$ is pointing inward to $M$ on $\partial U$, boundary of tubular neighborhood. So $\partial U = \partial N - U$. Also $\tilde{X}$ is perturbed to Smale.

$HM(N, M) := HM(N - U, \partial U) = HM \left( (\tilde{f}, \tilde{X})|_{\tilde{N} - \tilde{U}} \right)$.

Have chain maps $CM_k(f, X) \to CM_k(\tilde{f}, \tilde{X}) \to CM_k \left( (\tilde{f}, \tilde{X})|_{\tilde{N} - \tilde{U}} \right)$. The last arrow is chain map $(\partial_{\tilde{X}} p)|_{\tilde{N} - \tilde{U}} = \partial_{\tilde{X}}|_{\tilde{N} - \tilde{U}} (p|_{\tilde{N} - \tilde{U}})$:

if $p \in M$, then both sides are zero. If $p \in \tilde{N} - \tilde{U}$, then both sides are counting those trajectories not leaving $\tilde{N} - \tilde{U}$.

Then have the long exact sequence. Connecting homo. $\partial: CM_k \left( (\tilde{f}, \tilde{X})|_{\tilde{N} - \tilde{U}} \right) \to CM_{k-1}(f, X)$ is simply counting trajectories from $\tilde{N} - \tilde{U}$ to $M$.

When $N$ is manifold with boundary $M$, similar by extending $N$ a bit to $\tilde{N}$ so that $M$ is a hypersurface in $\tilde{N}$. Then define $HM_k(N, \partial N)$ as $HM_k(\tilde{N}, M)$.

**ex.** $N = D^n$. $HM_k(D^n, \partial D) = \mathbb{Z}$ for $k = n$ and zero otherwise.

$HM_k(D^n) = \mathbb{Z}$ for $k = 0$ and zero otherwise. So

$HM_0(S^{n-1}) \cong HM_0(D^n)$;

$HM_n(D^n, S^{n-1}) \cong HM_{n-1}(S^{n-1})$. 