5. Family Morse theory

Tuesday, February 26, 2019 1:26 PM

Ref: [Hutchings]

$HM \cong H_{sing}$:

For crit. p, have $\overline{W^u}(p)$ (denoted $\overline{D}(p)$ in [Hutchings]), compactified unstable (which is a manifold with corners itself, but not a submanifold of M): points reached by broken traj. from p.

$$\underset{\overline{W^u}(p)}{\cancel{\overline{\mathcal{D}}(p)}}_k = \coprod_{p=r_0,r_1,\ldots,r_k \text{ distinct}} \mathcal{M}(r_0,r_1)\times\cdots\times\mathcal{M}(r_{k-1},r_k)\times \underset{W^u}{\cancel{\mathcal{D}}}(r_k),$$

$$\partial\overline{\mathcal{D}}(p) = \bigcup_r (-1)^{|p|-|r|-1}\overline{\mathcal{M}}(p,r)\times\overline{\mathcal{D}}(r).$$

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$\overline{W^u}(p)$ is homeomorphic to a closed ball.

ex.

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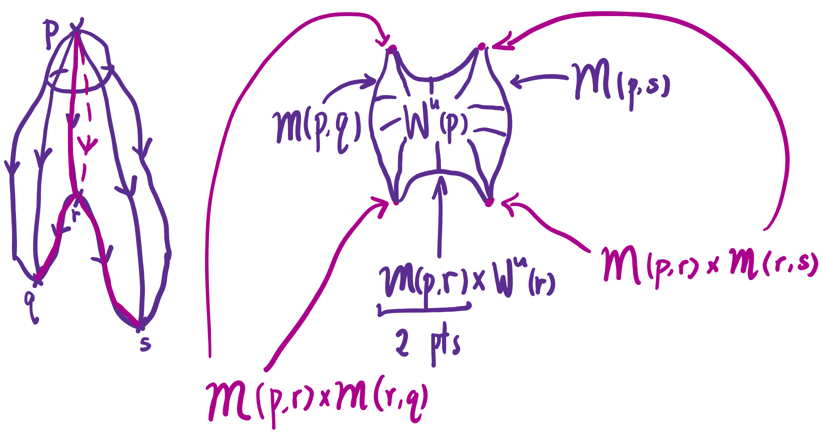


Have $ev: \overline{W^u}(p) \to M$.

Take $CM \to C_{sing}, p \mapsto ev_*[\overline{W^u}(p)]$.

More precisely, need to take (cubical) singular chain to represent $\overline{W^u}(p)$. (Called fundamental chain.)

Represent $\overline{M}(p,q)$ first for $|p|-|q|=1$ (points).

Then represent $\overline{M}(p,q)$ for $|p|-|q|=2$ (intervals), Then represent $\overline{M}(p,q)$ for $|p|-|q|=3$ in such a way tha

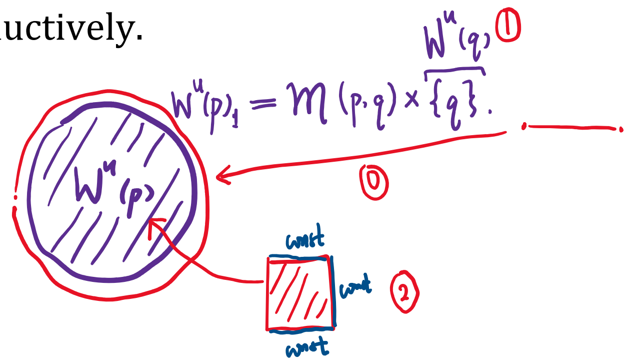
$$\partial m_{p,q}=\sum_r(-1)^{|p|-|r|-1}m_{p,r}\times m_{r,q}\,.$$

Do it inductively for $|p|-|q|$ such that the equation holds.
Now represent $\overline{W^u}(p)$ for $|p|=0$ (points).
Then represent $\overline{W^u}(p)$ for $|p|=1$ (interval), in such a way that

$$\partial d_p=\sum_r(-1)^{|p|-|r|-1}\,m_{p,r}\times d_r\,.$$

Do this inductively.

ex.



(Degenerate cubes are quotient out in the singular complex.)
$\partial\circ ev_*[\overline{W^u}(p)]=ev_*\circ\partial[\overline{W^u}(p)]$:
Note that $m_{p,r}\times d_r$ are sent to 0 under ev_* if $|p|-|r|>1$: ev does not depends on location in $m_{p,r}$.

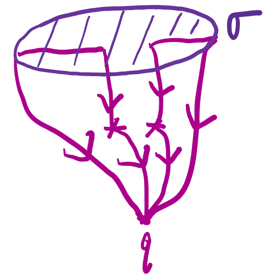
$$\partial\circ ev_*[\overline{W^u}(p)]=\sum_{|r|=|p|-1}ev_*\big(m_{p,r}\times d_r\big)=\sum_{|r|=|p|-1}n_{p,r}\,ev_*[\overline{W^u}(r)]$$
$$=ev_*[\overline{W^u}(\partial p)].$$

$$C_{sing}\to CM\colon\sigma\mapsto\sum_{\mathrm{p}}n(\sigma,p)\cdot p\,.$$

More precisely, need to restrict to C'_{sing}, singular chains which intersect stables transversely. (C'_{sing} has the same homology as C_{sing}.)
$n(\sigma,p)$ is nonzero only when $|p|=|\sigma|$.
Chain map by considering boundary:
$\sum_{\mathrm{p}}n(\sigma,p)\cdot\partial p$ counts broken trajectories from σ to q with $|q|=|\sigma|-1$.
$\sum_q n(\partial\sigma,q)\cdot q$ counts trajectories from $\partial\sigma$ to q.
They add up to boundary of 1d moduli of trajectories from σ to q.
Hence cancel to zero.

(Homotopy of composition to identity is taking 

(Homotopy of composition to identity is taking $\sigma \mapsto \overline{W^u}(\sigma)$)



The family setting.

$\pi: Z \to B$ fiber bundle.

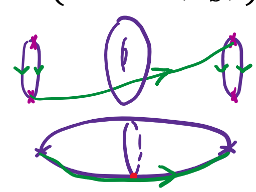
$f: Z \to \mathbb{R}$ understood as family of fiberwise functions.

Admissible: f_b is Morse for b outside a codimension one subvar.

(Unavoidable to have non-Morse fiber when fiber bundle is non-trivial.)

Also fix a connection ∇ (horizontal lift H).

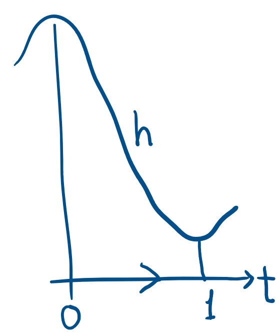
Aim: compute $H(Z)$ via the fibration. Construct $H\big(B, HM(Z_b)\big)$ as E_2 page of spectral sequence converging to $H(Z)$.



Typical example: $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$.

Two approaches: singular or Morse homology on B.

Singular approach:



Easiest case (other than point) is $\boldsymbol{\gamma}: [\mathbf{0}, \mathbf{1}] \to \boldsymbol{B}$.

Consider the vector field

$H(\beta(t)\,\partial_t) + \xi_t$

on $\gamma^* Z$. (Assume f_t is Morse for $t = 0{,}1$.)

$\beta(t)\partial_t = \partial_t h$ as before;

ξ_t is the gradient of f_t on Z_t.

The horizontal lift H is equivalent to a choice of trivialization of $\gamma^* Z$.

(This paper uses fiberwise metric g on $\gamma^* Z$ and negative gradient, rather than pseudo-gradient. g is perturbed to satisfy Smale on $\gamma^* Z$.)

This gives chain map $\Phi: CM\big(f_{\gamma(0)}, g|_{Z_0}\big) \to CM\big(f_{\gamma(1)}, g|_{Z_1}\big)$, which induces isomorphism on homology.

$\boldsymbol{HM(Z_b)}$ **gives a local system over** $\boldsymbol{B}$, meaning a path in B would gives identification of HM at the endpoints.

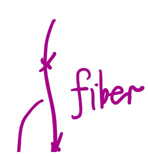
$\boldsymbol{HM(Z_b)}$ **is defined over** $\boldsymbol{b}$ **at which** $\boldsymbol{f_b}$ **is not Morse**:

take a contractible neighborhood around b and define $HM(Z_b)$ as $HM(Z_{b'})$ for b' in this neighborhood.

Any other $HM(Z_{b''})$ are isomorphic by a path $b' \leftrightarrow b''$ inside the neighborhood.

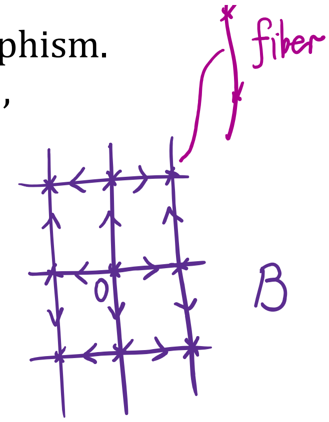
Any two paths are homotopic and hence produce the same isomorphism.

(The path and homotopy are allowed to pass through non-Morse b.



neighborhood.

Any two paths are homotopic and hence produce the same isomorphism. (The path and homotopy are allowed to pass through non-Morse b, hence no monodromy.)

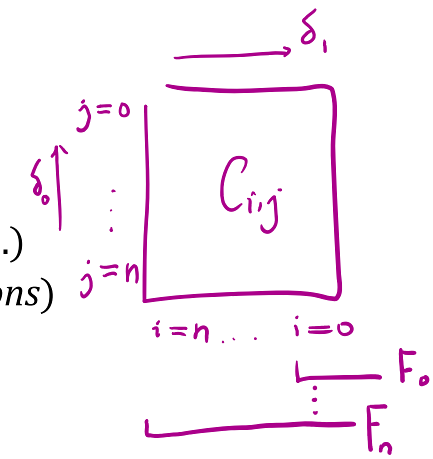


Now consider singular chain $\sigma: [-1,1]^i \to B$.
Have $\sigma^* Z$.
Choose fiberwise metric g on $\sigma^* Z$. Have fiberwise grad $= \xi$.
Gradient vector field on $[-1,1]^i$: $W = -\sum_\mu (x_\mu + 1) x_\mu (x_\mu - 1)\, \partial_\mu$.
$V := H(W) + \xi$ on $\sigma^* Z$.
(σ, g) **admissible** if f_b is Morse for b being zeroes of W, and V is Smale.

Chain: generated by (σ, g, p) where p is a critical point of $f_{\sigma(0)}$. Mod out degenerate (σ, g) (independent of at least one coordinate).
Have bidegree: dim. of cube i (base index), and the fiber index j of p.
Denote by $C_{i,j}$. Total index $i + j$.
δ: counting trajectories from (σ, g, p) to (σ', g', p'), where σ' is a face of σ, g' is the restriction of g to σ', p' is a critical point of $f_{\sigma'(0)}$, the trajectories go from p to q in $\sigma^* Z$.
$\delta: \bigoplus_{i+j=k} C_{i,j} \to \bigoplus_{i+j=k-1} C_{i,j}$.

$$\delta = \sum_{k=0}^{i} \delta_k$$

where $\delta_k: C_{i,j} \to C_{i-k, j+k-1}$. (Codim. k face in base.)
$F_i C_m := \bigoplus_{i' \leq i} C_{i', m-i'}$. (*At least* $m - i$ *fiber directions*)



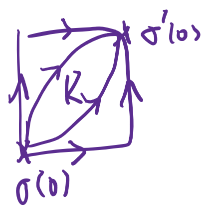
So δ_0 is just fiberwise Morse differential;

$$\delta_1(\sigma, g, p) = \sum_{\sigma' \in F_1(\sigma)} \pm \left(\sigma', g\big|_{\sigma'}, \Phi(p)\right)$$

where Φ is the chain map from $CM(f_{\sigma(0)})$ to $CM(f_{\sigma'(0)})$;

$$\delta_2(\sigma, g, p) = \sum_{\sigma' \in F_2(\sigma)} \pm \left(\sigma', g\big|_{\sigma'}, \mathrm{K}(p)\right)$$

where K is the homotopy.



$\delta^2 = 0$:
As in the usual case. Any subseq. of traj. conv. to a broken one: energy is bounded above, and then use "Gromov compactness". Since
$(f \circ u)'(t) = -|\xi|^2 + \nabla_{W_i} f$,

$$\int_{-\infty}^{\infty} \left|\xi\big(u(t)\big)\right|^2 < f(p) - f(q) + C.$$

Since δ_0 is just fiberwise Morse differential,
$\boldsymbol{E^1_{i,j} = C_i\left(B, HM_j(Z_b)\right)}$.
(Recall that $HM(Z_b)$ whose elements are represented by critical points of fiber at b is a local system over B. It is pulled back to trivial bundle over a singular chain σ, identified as critical points over $0 \in \sigma$.)
Since δ_1 is taking boundary of σ and identify $HM(Z_{\sigma(0)})$ with $HM(Z_{\partial\sigma(0)})$ via the chain map Φ (defining the local system),
$\boldsymbol{E^2_{i,j} = H_i\left(B, HM_j(Z_b)\right)}$.
(More precisely, need to take $\tilde{B} = \{(b, g_b)\}$ and regard (σ, g) as a chain in $\tilde{B}$. $\tilde{B}$ is contractible to B. Also "admissible chains" are generic and so compute usual homology.)

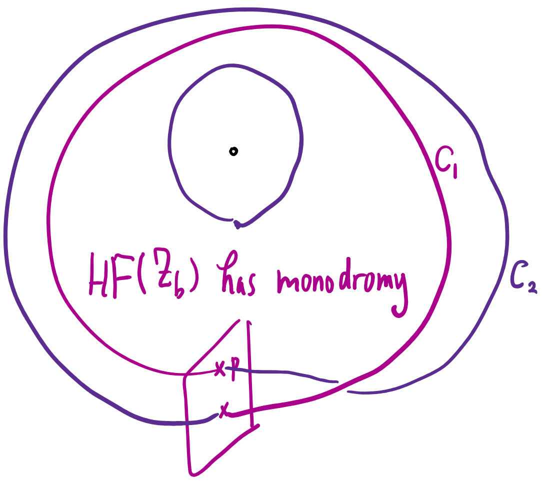
Generic $\boldsymbol{\phi}$: $\boldsymbol{B' \to B}$ induces morphism on spectral sequence:
Assume $\phi^* Z$ is admissible (Morse outside codim=1).
Then have chain map ϕ_* on $\bigoplus C_{i,j}$, preserving filtration, and hence gives a morphism on spectral sequence E^r.

Mayer-Vietoris: $B = U \cup V$.
$0 \to C_*\left(Z\big|_{U\cap V}\right) \to C_*\left(Z\big|_U\right) \oplus C_*\left(Z\big|_V\right) \to C'_*(Z) \to 0$
where $C'_*(Z)$ are the cubes that are contained entirely in U or V.
Thus have the long exact sequence.
For $C'_*(Z)$, ${E^1_{i,j}}' = C'_i\left(B, HM_j(Z_b)\right)$. By subdivision, the inclusion chain map induces isomorphism
${E^2_{i,j}}' = H'_i\left(B, HM_j(Z_b)\right) = H_i\left(B, HM_j(Z_b)\right) = E^2_{i,j}$.
Thus $HF'_*(Z) = HF_*(Z)$.

Prop. If $Z = B \times X$ and f independent of b, then collapse at E^2.
Proof:
Key: Have well-defined sense of up and down. Cannot flow up in fiber direction.
Take ∇ to be trivial.
Consider (σ, g, p) with g independent of b.
Then vertical projection of flow is just the flow of fiberwise f. Hence $\delta_k = 0$ for $k \geq 2$.
This is still true for g that is close enough to one that constant in b.

(Smale)
By subdivision, the inclusion chain map gives isomorphism between C' formed by these (σ, g, p) and C.

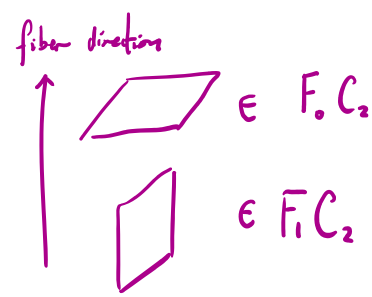
Prop. If f_b is Morse for all b and have g_b Smale for all b, then collapse at E^2.
Note: the local system $HM(Z_b)$ has monodromy when not simply connected. So $\boldsymbol{H_i(B, HM(Z_b)) \neq H_i(B) \otimes HM(Z_b)}$!



$p \sim q \in H_0(B, HM(F))$
(For instance when $q = -p$ then have torsion $2p \sim 0$.)
$C_1 + C_2 \in H_1(B, HM(F))$.

Proof: similar to above. First consider constant σ.
Then $\delta = \delta_0 + \delta_1$.
For those sufficiently close to this, still true since Smale.
By subdivision, the inclusion chain map from such chains to C gives isomorphism on homologies.

Leray-Serre for singular:
$F_i C_k$: $\pi \circ \sigma: [-1,1]^k \to B$ is independent of at least $k - i$ directions.
(Meaning at least $k - i$ fiber directions.)



Computes $H_*(Z)$.

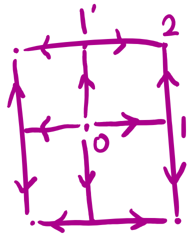
Thm. E^k pages for sing and Morse are isomorphic for $k \geq 2$.
Proof:
Need $C_{i,j}^{Morse} \to F_i C_{i+j}^{sing}$.
For $(\sigma, g, p) \in C_{i,j}^{Morse}$, consider $\overline{W^u(p)}$ for $\sigma^* Z$ like before.

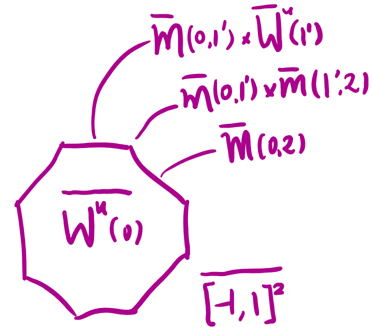
$W^u \cancel{\overline{\mathcal{D}}}(p) \xrightarrow{e} \sigma^* Z$

$\overline{M}(0,1') \times \overline{W}^u(1')$

$$
\begin{tikzcd}
W^u\overline{\cancel{\mathcal{D}}(p)} \arrow[r, "e"] \arrow[d, "\bar{\pi}"'] & \sigma^* Z \arrow[d] \\
\overline{[-1,1]^i} \arrow[r, "e"] & {[-1,1]^i}
\end{tikzcd}
$$

$\overline{[-1,1]^i}$ is truncated cube.





$\overline{W^u(p)}$ has j fiber directions, and so $\sigma_* ev_*\left[\overline{W^u(p)}\right] \in F_i C_{i+j}^{sing}$. Chain map as before.

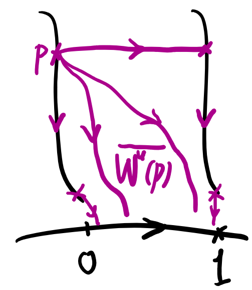
This induces isomorphism on E^2 pages

$H_i\left(B, HM_j(Z_b)\right) \to H_i\left(B, H_j(Z_b)\right)$:

It is indeed changing coefficients $HM_j(Z_b) \cong H_j(Z_b)$.

Determined by image intersecting $\sigma(0)$.

$ev\left(\overline{W^u(p)}\right) \cap Z_{\sigma(0)} = ev\left(\left(\overline{W^u_{f|_{\sigma(0)}}(p)}\right)\right)$.



$$\partial_{sing.}\left[\overline{W^u(p)}\right] = \left[\overline{W^u}\left((\delta_0+\delta_1+\dots)\,p\right)\right].$$

Morse approach

Morse on base B such that f_x is Morse for crit. $x \in B$.

Fix metric on B Smale. Negative gradient W on B.

Again $V = \xi + H(W)$.

Zeros are (x, p) for $x \in Crit(f^B), p \in Crit(f_x)$.

Again have $C_{i,j}$ and $\delta = \sum \delta_k$

i is base index, k is base index drop.

$E^2 = HM_i\left(B, HM(Z_b)\right)$.

Equivalent to the previous one…