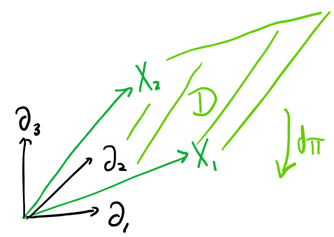
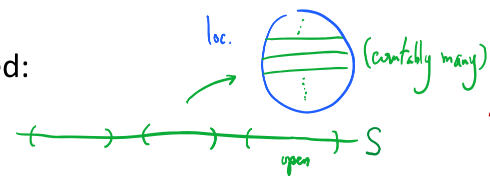
Frobenius theorem and foliations

Monday, December 28, 2015 7:33 PM

- **Frobenius theorem**: involutive <=> integrable. <= is trivial.
- Already known: $V_1, \ldots V_k$ lin. indep. and $[V_i, V_j] = 0\ \forall i, j$ => has coord. $(s_1, \ldots, s_k)$ such that $V_i = \partial_{s_i}$.
  Because the flows commute: $\rho_i^{-t} \cdot \rho_j^s \cdot \rho_i^t(p) = \rho_j^s(p)$. $\left(\rho_i^{-t} \cdot V\left(\rho_i^t(p)\right) = V(p).\right)$
- **Key:** Distribution D closed under [ ] => can choose local frame V_i of D such that $[V_i, V_j] = 0$.
  Take local coord. x_i and local frame $\{X_1, \ldots, X_k\} \subset D$.
  By lin. change, can assume $D \pitchfork \langle \partial_{k+1}, \ldots, \partial_n \rangle$.
  $\pi \coloneqq (x_1, \ldots, x_k): U \to \mathbb{R}^k$.
  $d\pi|_D : D \cong T\mathbb{R}^k$ bundle iso.
  Define $V_i \coloneqq (d\pi|_D)^{-1}(\partial_i)$.
  $d\pi[V_i, V_j] = [\partial_i, \partial_j] = 0$.
  $[V_i, V_j] \in D$ and $d\pi|_D$ iso $\Rightarrow [V_i, V_j] = 0$.
  Then by integrating along V_i starting from $(0, \ldots, 0, x'_{k+1}, \ldots, x'_n) \in U$,
  has coordinates $s_1, \ldots, s_k, x'_{k+1}, \ldots, x'_n$ such that $\{x'_{k+1} = \text{const}, \ldots, x'_n = \text{const}\}$
  defines integral of D.



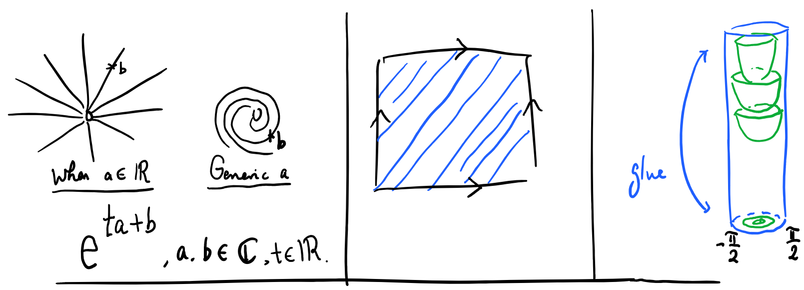
- Integral manifold S is weakly embedded:
  any smooth $N \to M$ with image in S is
  a smooth map $N \to S$.



$S \hookrightarrow M$ is not homeo. to its image.

- Integrable distribution <-> foliation: collection of disjoint connected immersed submanifolds whose union is the whole space, and can take local coordinates such that defined by $x_{k+1} = \text{const}, \ldots, x_n = \text{const}$.

ex



$z \coloneqq \sec y + c$ for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then take surface of revolution.

- Surjective submersion (fibration without singular fiber) is a special case of a foliation.
- Prob. 19.3, 4, 5, 10

Prob. 19.3: ω nowhere-0 1-form.

$\exists$ loc. int. factor μ s.t. $\mu\omega = df \iff d\omega \wedge \omega = 0$.

$\Rightarrow)$ $\underbrace{d(\mu\omega)}_{0}\wedge\omega = \underset{\substack{\vee \\ 0}}{\mu}\, d\omega\wedge\omega$.

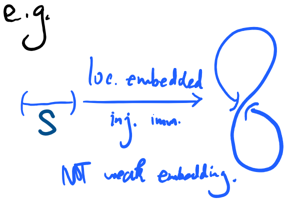
$\Leftarrow)$ $d\omega = \eta\wedge\omega$. (extend ω to a loc. frame.)

$$d(\mu\omega) = d\mu\wedge\omega + \mu\, d\omega \overset{\text{want}}{=\!=} 0.$$

$$-(d\log\mu)\wedge\omega \overset{\text{want}}{=\!=} d\omega = \eta\wedge\omega.$$

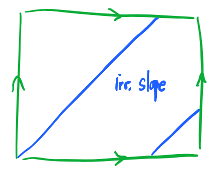
Ker ω Integrable $\Rightarrow$ coord. s.t. $\mu\omega = \underbrace{dx_n}_{\text{nowhere zero}}$.

e.g.



There exists smooth $(-\epsilon,\epsilon) \to \mathbb{R}^2$ with image in S, but not smooth as a map $(-\epsilon,\epsilon) \to S$.

e.g.



not embedded

It is weakly embedded