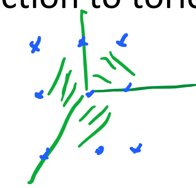
Toric variety from gluing

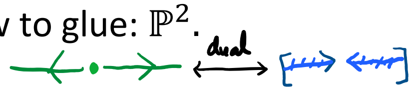
Monday, January 11, 2016 3:02 PM

Standard reference: Fulton - Introduction to toric varieties

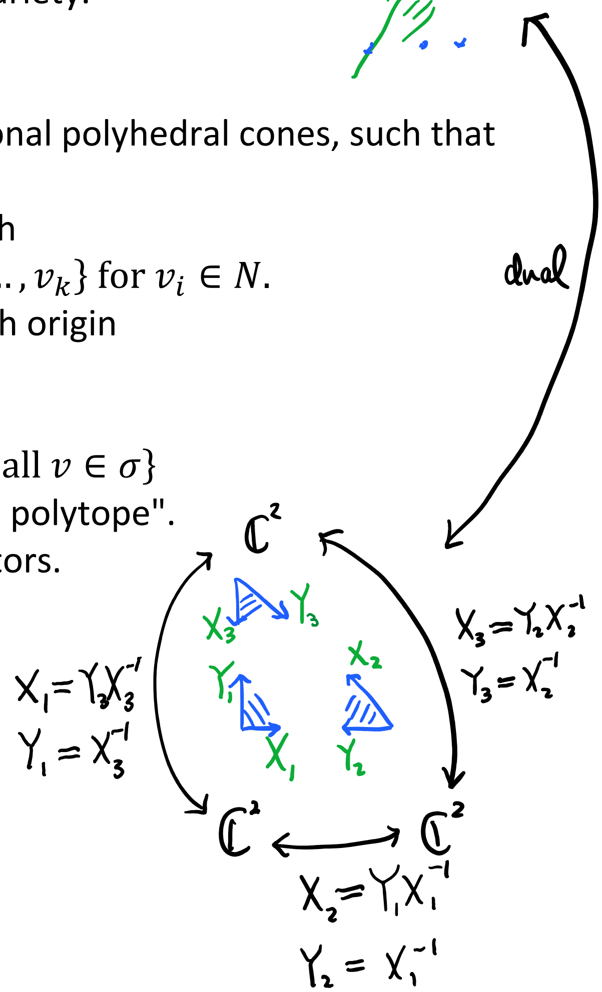


- Start with $\mathbb{P}^2$. Main statement: fan -> variety.
- Lattice $N \cong \mathbb{Z}^n$. $N_{\mathbb{R}} := N \otimes \mathbb{R} \cong \mathbb{R}^n$.

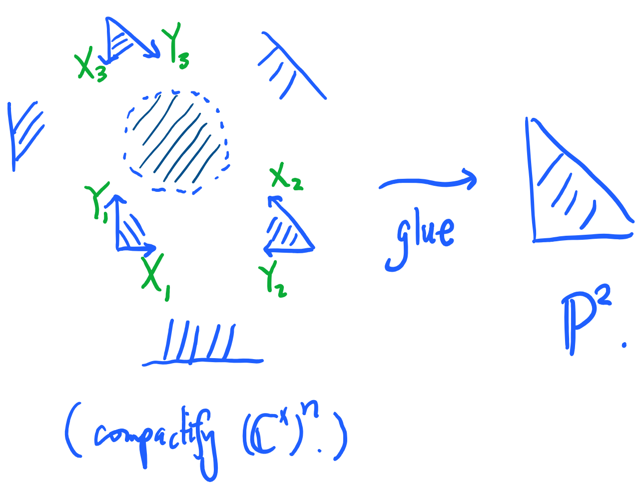
- Fan Σ: collection of strongly convex rational polyhedral cones, such that
  - face of a cone in Σ is still in Σ
  - intersection of cones in Σ is face of each
- Rational polyhedral cone: $\sigma = \mathbb{R}_{\geq 0}\{v_1, \dots, v_k\}$ for $v_i \in N$.
  Strongly convex: contains no line through origin
  Face: $\mathbb{R}_{\geq 0} \cdot S$ for $S \subset \{v_1, \dots, v_k\}$.
- Dual lattice $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$.
- Dual cone $\sigma^\vee = \{v \in M_{\mathbb{R}} : (v, v) \geq 0 \text{ for all } v \in \sigma\}$
- Dual cones patch together to form "dual polytope".
- Simplicial cone: n linearly indep. generators.

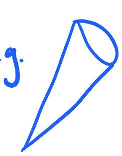
- Standard cone: $\mathbb{Z} \cdot \{v_1, \dots, v_n\} \cong N$.
- $\mathbb{C}^n$ corresponds to standard cone.
- How to glue: $\mathbb{P}^2$.
- $\mathbb{P}^1$ 

- Affine piece: $U_\sigma := \mathrm{Spec}(\mathbb{C}[\sigma^\perp \cap M])$.
- ex. $\sigma = \{0\}$. $U_\sigma = \mathbb{C}^{\times n}$.
  $\sigma = \{e_1\}$. $U_\sigma = \mathbb{C} \times \mathbb{C}^{\times n-1}$.
- $\tau \subset \sigma \Rightarrow U_\tau \subset U_\sigma$.
- ex: $\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^3$.



$$\boxed{\begin{aligned} &\mathbb{P}^2 \ni [x:y:z] \\ &(X_1, Y_1) = \left(\tfrac{x}{z}, \tfrac{y}{z}\right) \\ &(X_2, Y_2) = \left(\tfrac{y}{x}, \tfrac{z}{x}\right) \\ &(X_3, Y_3) = \left(\tfrac{x}{y}, \tfrac{z}{y}\right). \end{aligned}}$$

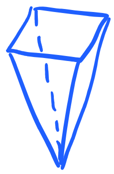


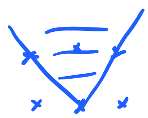
e.g.  not polyhedral cone.

 not strongly convex. Dual cone is lower dimensional

e.g.  not strongly convex. Dual cone is lower dimensional.

e.g.  not fan.

e.g.  not simplicial cone.

e.g.  simplicial but not standard cone.

e.g.  standard cone.

Motivation:

- Simple! Lot of examples. $\mathbb{P}^n$. Projective varieties…
- Intersections between combinatorics, algebraic geometry, tropical geometry, symplectic geometries.
- Provide local models for singularities, eg conifold, orbifold
- Can be used to study non-toric varieties via toric degenerations
- Easiest case of geometric quotient (important in moduli theory)
- Easiest case of SYZ mirror symmetry and Lagrangian Floer theory