Holomorphic line bundles

Monday, January 18, 2016 11:33 AM

$D = [f_u \in \mathcal{M}^\times(U) : U \underset{\text{open}}{\subseteq} X,\ f_u/f_v \in \mathcal{O}^\times(U\cap V)] \in \Gamma(\mathcal{M}^\times/\mathcal{O}^\times)$.

- Cartier divisor: locally defined by one meromorphic function. ⇒ Weil divisor
- Divisor line bundle $O(D)$ $\quad f_u e_u|_{U\cap V} = f_v e_v|_{U\cap V} \rightsquigarrow$ mero. section $s$ w/ $(s)=D$. $\quad \mathcal{O}(D)|_U = \{h s : (h)+D|_U \geq 0\}$, $\quad h \in \mathcal{M}_U$
- Invariants: Picard group of line bundles
- Holomorphic bundle on $\mathbb{C}^n$ is trivial
- The exact sequence $0 \to M \to \mathrm{Div}_T(X) \to \mathrm{Pic}(X) \to 0$ $\quad \boxed{\text{Note: pole div of } h_1+h_2 \leq \text{sum of that of } h_i}$
- Canonical divisor: $\sum_i D_i = (d\log z_1 \wedge \cdots \wedge d\log z_n)$
- Toric Calabi-Yau: $\sum_i D_i = (z^\nu)$ iff $(\nu, v_i) = 1\ \forall i$. Ex. $O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1)$.
  Holomorphic volume form: $z^\nu\, d\log z_1 \wedge \cdots \wedge d\log z_n$.
- Polytope $P_D := \{\langle v_i, \cdot\rangle \geq -a_i\}$ associated to toric Weil divisor $D = \sum a_i D_i$.
- Global sections: $\Gamma(O(D)) = \{(f) + D \geq 0\} = \mathrm{Span}\{z^\nu : \nu \in P_D \cap M\}$.
- Piecewise lin. fcn. asso. to divisor: $u_\sigma \in M/\sigma^\perp$ defines lin. fcn. on $\sigma$
- Total space of divisor line bundle: $\tilde{\Sigma} := \{\mathbb{R}_{\geq 0} \cdot \{\mathrm{graph}(u_\sigma), (0,-1)\} : \sigma \in \Sigma\}$.
- Globally generated <-> $u$ is convex $\quad$ (Motivate from embedding)
- ample <-> $u$ strictly convex
- Fano <-> reflexive polytope
- Ex. Compute the dimension of the section space of $-K_X$ of $X = \mathbb{P}^2$ and $\mathbb{F}_2$.

$\mathrm{Div}_T(M) \longrightarrow \mathrm{Pic}(X)$. $\quad$ (Gen. var.: $0 \to \Gamma(\mathcal{M}^\times) \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0$.)

surjective: Consider toric charts $U$ to trivialize $\mathcal{L}$.

$e_u|_{(\mathbb{C}^\times)^n} = f_u\, e_{(\mathbb{C}^\times)^n}$ where $f_u$ hol. & nowhere zero on $(\mathbb{C}^\times)^n$

$\Rightarrow f_u$ is a monomial.

$\therefore\ \mathrm{Div}_T(M) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0$.

Ker $= M$: If $\mathcal{L}$ is trivial, $\exists \{s_u$ hol. on $U\}$ s.t. $\{s_u e_u\}$ gives global nowhere-zero section.

$$s_u e_u|_{(\mathbb{C}^\times)^n} = s_u|_{(\mathbb{C}^\times)^n} f_u\, e_{(\mathbb{C}^\times)^n} = s_v|_{(\mathbb{C}^\times)^n} f_v\, e_{(\mathbb{C}^\times)^n}.$$

$\underbrace{\phantom{xx}}_{\text{hol. nowhere zero} \Rightarrow \text{monomial}}$

$\{f_u\}_u \sim \{s_u f_u\}_u$ def. global mero. fcn.

$$\begin{array}{ccccccccc} 0 & \to & M & \to & \mathrm{Div}_T(M) & \to & \mathrm{Pic}(X) & \to & 0 \\ & & \| & & \downarrow\ \text{Weil} & & \wr\!\!\downarrow\ \text{when smooth} & & \\ 0 & \to & M & \to & \mathbb{Z}\langle D_1 \ldots D_m\rangle & \to & \mathbb{Z}\langle D_1 \ldots D_m\rangle/\text{lin. equiv.} & \to & 0 \\ & & & & \wr\| & & \| & & \\ & & & & \mathbb{Z}^m & & H^2(X) & & \\ & & & & \| & & & & \\ & & & & H^2(X,T) & & & & \end{array}$$

Cartier

principal toric divisors

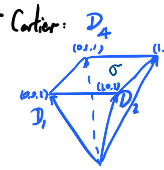
dual sequence: $0 \to H_2(X) \to H_2(X,T) \to H_1(T) \to 0$.

$\qquad H_2(X,T) \cong \mathbb{Z}^m, \quad H_1(T) = N$

toric Cartier div. $\longleftrightarrow \{u_\sigma \in M/\sigma^\perp : \sigma \in \Sigma,\ u_\sigma \mapsto u_\tau$ for $\tau < \sigma\}$ (monomial up to $\mathcal{O}^\times/U_\sigma$) $\longleftrightarrow$ piecewise lin. fcns supp. on $\Sigma$

$\Longrightarrow \sum u_{\langle v_i \rangle}(v_i) \cdot D_i$ toric Weil div.

Weil but not Cartier: conifold.




$\{XZ = YW\} \subset \mathbb{C}^4$.

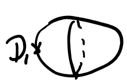
Impossible to define a toric irred. div. $D_i$ by only 1 eqn. $\leftarrow$ Weil but not Cartier

For a lin. fcn. $u$ on $\sigma$,
$u(v_1) = u(v_2) = u(v_3) = 0 \Rightarrow u(v_4) = 0$.

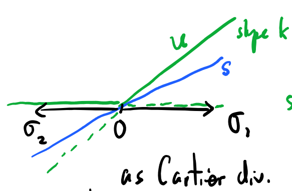
$\mathcal{O}_{\mathbb{P}^1}(kD_1)$

$\|$

e.g. $\mathcal{O}_{\mathbb{P}^1}(k)$.




$z_1^k e_1 = e_2$.



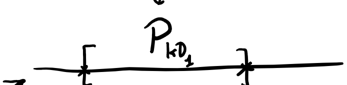

convex if $k \geq 0$.

strictly convex if $k > 0$.

as Cartier div.

global sections: $z^s$. $z^s \cdot z^{u(\sigma)} e_\sigma$ hol.

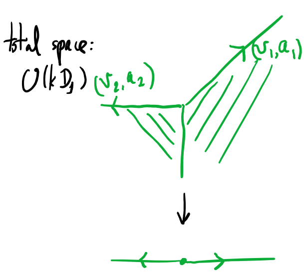
$\Rightarrow -s \leq u$.




$= \phi$ when $k < 0$.

$-k = -u(\sigma_1)$ $\qquad 0 = -u(\sigma_2)$ $\qquad$ as Weil div.

total space: $\mathcal{O}(kD_1)$




$z^{(0,-1)} = 1$ on $N_\mathbb{C}/N \simeq (\mathbb{C}^\times)^{n+1}$ gives a mero. section $s$ with $(s) = -\sum_i ((0,-1),(v_i,a_i)) D_i$
$\qquad = \sum_i a_i D_i$.

$\uparrow$ fiber coord. over $N_\mathbb{C}/N$

$\boxed{\begin{array}{l}\mathcal{O}(D) \text{ global gen.} \\ \Leftrightarrow u \text{ convex}\end{array}}$ : $\forall \sigma$ (max. cone), $\exists s \in M$ s.t. $\begin{cases} (s, v_i) \geq -a_i \ \ \forall i & (\text{i.e. } z^s \in \Gamma(\mathcal{O}(D))) \\ (s, v_i) = -a_i \ \text{ for } v_i \in \sigma. & (\text{i.e. } z^s \neq 0 \text{ on } U_\sigma.) \end{cases}$

$\updownarrow \begin{cases} (-s, v_i) \leq a_i & \forall i \\ (-s, v_i) = a_i & \text{for } v_i \in \sigma. \end{cases}$

multiplicity of $D_i$ in $D$

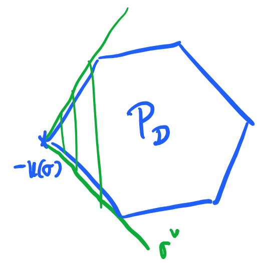
$\hookrightarrow a_i = (u(\sigma), v_i) \ \forall v_i \in \sigma$ by def. $\Rightarrow -s = u(\sigma) \in M$. $(\sigma^\perp = 0.)$

$a_i = (u(\langle v_i \rangle), v_i) \ \forall i$ by def.

$\underset{(u(\sigma), v_i)}{\geq} \Longleftrightarrow$ convex. ($u \geq$ lin. part over every max. cone.)

$\boxed{\text{Kodaira embedding theorem: } L \text{ has metric } h \text{ such that } Ric(h) \text{ defines a Käh. metric} \Rightarrow X \xhookrightarrow[|L|]{} \mathbb{P}^N}$

$\mathcal{O}(D)$ $\underbrace{\text{very ample}}_{\text{is embedding}}$ ('a lot of sections') $\Leftrightarrow \begin{cases} u \text{ is strictly convex} \\ \forall \text{ max cone } \sigma, \{u - (-u(\sigma)) : u \in P_D \cap M\} \text{ generates } \sigma^\vee \cap M. \end{cases}$




Pf: $X_\Sigma \xrightarrow{[z^v : v \in P_D]} \mathbb{P}^N$. (globally gen. $\Rightarrow$ well-def.)

$u$ strictly convex $\Leftrightarrow (\underset{\text{max cone}}{u(\sigma)}, v_i) < a_i \ \forall v_i \notin \sigma$

i.e. $z^{-u(\sigma)} = 0$ on $X_\Sigma - U_\sigma$

(Hence $p \in U_\sigma$ & $q \in X - U_\sigma$ have different values)

$$\begin{array}{ccc} \therefore\ U_\sigma & \xrightarrow{\left(\frac{z^u}{z^{-u(\sigma)}} : u \in P_D\right)} & \mathbb{C}^N = \{z^{-u(\sigma)} \neq 0\} \\ \cap & & \cap \\ X_\Sigma & \longrightarrow & \mathbb{P}^N \end{array}$$

$(z^{u+u(\sigma)})$ is embedding

$\Updownarrow$

$\{u + u(\sigma) : u \in P_D\}$ generates $\sigma^\vee \cap M$. $(\mathbb{C}[z_1, \dots, z_N] \twoheadrightarrow \mathbb{C}[\sigma^\vee \cap M].)$

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$\mathcal{O}(D)$ ample $\Leftrightarrow$ $u$ is strictly convex.

($\Leftrightarrow \mathcal{O}(kD)$ is very ample for $k \gg 0$.)

Pf: $\underbrace{\sigma^\vee \cap M}_{\text{fin. gen.}}$ gen. by $\underbrace{(k \cdot P_D)}_{P_{kD}} \cap M + u(\sigma)$.