Compactly supported Mayer-Vietoris sequence and Poincare duality

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- The short exact sequence $0 \to \Omega^\cdot(U_{12}) \xrightarrow{(i_{1*}, -i_{2*})} \Omega^\cdot(U_1) \oplus \Omega^\cdot(U_2) \xrightarrow{j_{1*}+j_{2*}} \Omega^\cdot(M) \to 0.$
- The Mayer-Vietoris sequence (Prob. 18.6)
- Partition of unity to prove the short exact sequence
- Poincare duality (Prob. 18.7)
- Cohomology is a Frobenius algebra $\boxed{(a \cup b, c) = (a, b \cup c)}$
- Euler characteristic = 0 in odd dimensions. (Prob. 18.9)
- Ex. Find a generator of H^1_c(R^2 - 0)

$$\boxed{\begin{array}{c} \Omega^k \xrightarrow{\int \cdot \wedge} (\Omega_c^{n-k})^* \\ H^k \xrightarrow{\sim} (H_c^{n-k})^* \end{array} \quad \text{• well-defined.}}$$

5-Lemma:

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \quad \text{exact} \\ \cong\downarrow & & \cong\downarrow & & \Rightarrow \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \quad \text{exact} \end{array}$$

Pf of Poincaré duality:

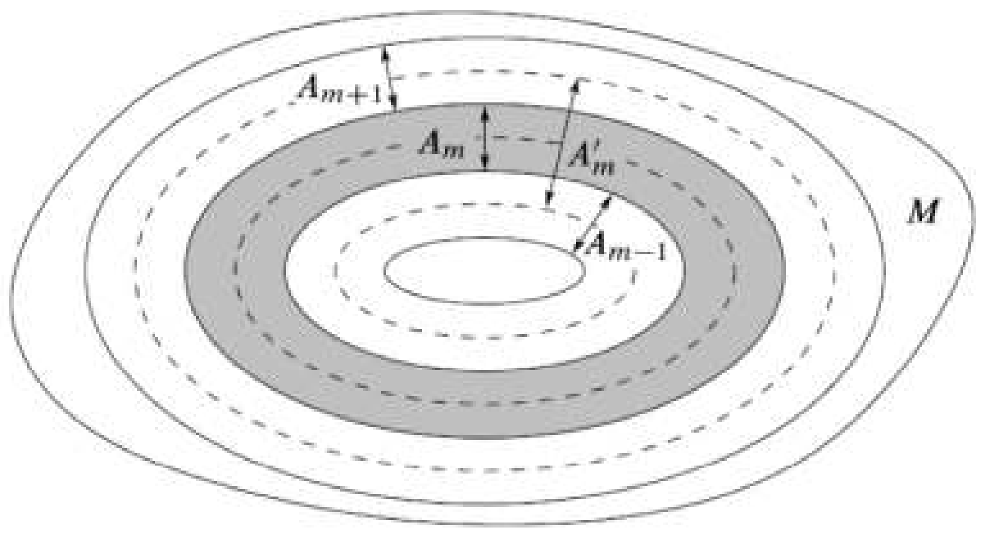
- True for $U \underset{\text{contractible}}{\subseteq} \mathbb{R}^n$. (Poincaré Lemma)

  P.D. true $\forall\, U_{i_1 \dots i_k}$

- If M has finite 'good cover', then P.D. holds for M.

  (Mayer-Vietoris & 5-Lemma)

  $\boxed{\Rightarrow \text{True for compact case if we know P.D. for } U \underset{\text{open}}{\subseteq} \mathbb{R}^n.}$



- If M has a good topo. base, then P.D. holds for M.

  Pf: Take $f: M \to \mathbb{R}$ s.t. $f^{-1}\{(-\infty, c]\}$ cpt. $\forall c$. $A_m = f^{-1}[m, m+1] \subset A'_m = f^{-1}(m-\varepsilon, m+1+\varepsilon)$.

  A_m covered by fin. many basic open sets $U_i^{(m)} \subset A'_m$. $\therefore B_m \triangleq \bigcup_i U_i^{(m)}$ has P.D.

  $\underbrace{\bigcup_{m \text{ odd}}^{\text{disjoint}} B_m}_{B_{odd}}$ & $\underbrace{\bigcup_{m \text{ even}}^{\text{disjoint}} B_m}_{B_{even}}$ also have P.D. $B_{odd} \cap B_{even} = \bigcup_{m \text{ odd}}^{\text{disjoint}} \underbrace{B_m \cap B_{m+1}}_{\text{good cover}: U_i^{(m)} \cap U_j^{(m+1)}}$ also has P.D.

  $\therefore M = B_{odd} \cup B_{even}$ has P.D.

- $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$ has P.D. Good top. base given by $\{\text{balls} \subset U\}$.
- Any M has a topology base of charts $\underbrace{\text{Fin. int.} \underset{\text{open}}{\subseteq} \mathbb{R}^n}_{\text{PD holds}}$. $\therefore$ Good base.