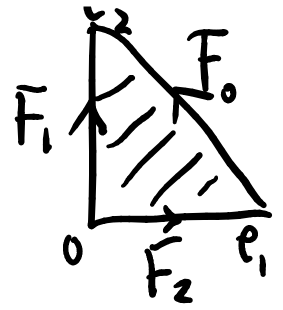
Singular homology and cohomology

Sunday, December 27, 2015 8:55 PM

- Motivation: a more intuitive way to define homology. Poincare dual.
- The singular chain complex (simplex, chain, face, boundary map, $\partial^2 = 0$)
- Eg. Circle in R^2-0 as a singular cycle
- Push forward and fuctoriality ($f: M \to N$ commutes with ∂.)
- Homotopy
- The cochains and coboundaries $C^\cdot = \mathrm{Hom}(C_\cdot, \mathbb{R})$.
- Mayer-Vietoris sequences
- Can replace continuous by smooth chains (Whitney approximation -> smooth map homotopic to each continuous map; get chain map)

standard simplex Δ^p



$$\left\{ \begin{matrix} x_i \geq 0 \\ \sum_i x_i \leq 1 \end{matrix} \right\} = \mathrm{Conv}\{e_0, e_1 \dots e_p\}, \quad e_0 = 0$$

face $F_i : \Delta^{p-1} \to \Delta^p$

$(e_0 \dots e_{p-1}) \mapsto (e_0 \dots \hat{e}_i \dots e_p)$.

$\partial \Delta^p \triangleq \sum_{i=0}^{p} (-1)^i F_i$. $\quad F_i = (e_0 \dots \hat{e}_i \dots e_p)$

$\partial^2 \Delta^p = \sum_{i \neq j} \left((-1)^{i+j-1} + (-1)^{j+i}\right)(e_0 \dots \hat{e}_i \dots \hat{e}_j \dots e_p)$

For $\sigma : \Delta \to M$,

$\partial \sigma = \sigma \circ \partial \Delta^p$.

$$0 \to C_\cdot(U \cap V) \xrightarrow{(\imath_u, -\imath_v)} C_\cdot(U) \oplus C_\cdot(V) \xrightarrow{r_u + r_v} C_\cdot(M) \to 0$$

adjoint

(like $(\Omega_c^\cdot)^*$)

$$0 \to C^\cdot(M) \xrightarrow{(r_u^*, r_v^*)} C^\cdot(U) \oplus C^\cdot(V) \xrightarrow{\imath_u^* - \imath_v^*} C^\cdot(U \cap V) \to 0.$$

(like $\Omega^\cdot$)

From homotopy to chain homotopy

$H : M \times I \to N$.

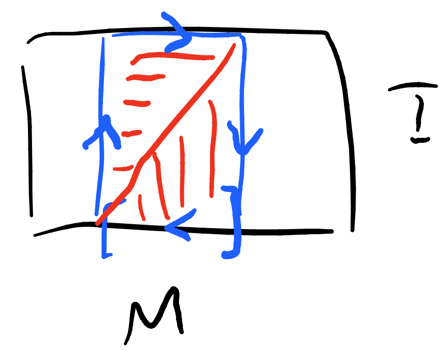
$$\begin{tikzcd}
\cdots \arrow[r] & C_\cdot(M) \arrow[r, "\partial"] \arrow[d] \arrow[dr, phantom, "h"] & C_{\cdot-1}(M) \arrow[r] \arrow[d, "f, g"] \arrow[dl, "h"'] & \cdots \\
\cdots \arrow[r] & C_\cdot(N) \arrow[r, "\partial"] & C_{\cdot-1}(N) \arrow[r] & \cdots
\end{tikzcd}$$

$g - f = \partial h + h \partial$.

$$C_p(M) \xrightarrow{\tilde{h}} C_{p+1}(M \times I) \xrightarrow{H_*} C_{p+1}(N), \quad h = H_* \circ \tilde{h}$$

$\tilde{h}\partial + \partial \tilde{h} = (\imath_1)_* - (\imath_0)_*$.

$\tilde{h}(\Delta) = (-1)^n \Delta \times I$.



Subdivide $\Delta \times I$ into simplices.