Frobenius theorem and foliations

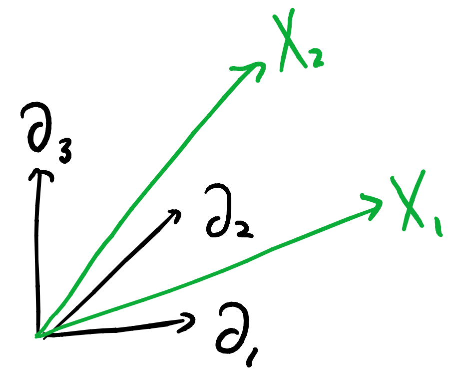
Monday, December 28, 2015 7:33 PM

- Frobenius theorem: involutive <=> integrable. <= is trivial.
- Already known: $V_1, \dots, V_k$ l.i. comm. $\Rightarrow \exists$ coord. s_i s.t. $V_i = \partial_{s_i}$. ($\rho_j^s \cdot \rho_i^t(p) = \rho_i^t \rho_j^s(p)$ as $[V_i, V_j] = 0$)
- Key: D closed under [ ] => can choose local frame {V_i} such that [V_i,V_j]=0

take loc. coord. x_i, loc. frame $D = \langle X_1 \dots X_k \rangle$ s.t. $D \pitchfork \langle \partial_{k+1} \dots \partial_n \rangle$.

$$\pi : U \xrightarrow[(x_1 \dots x_k)]{} \mathbb{R}^k.$$

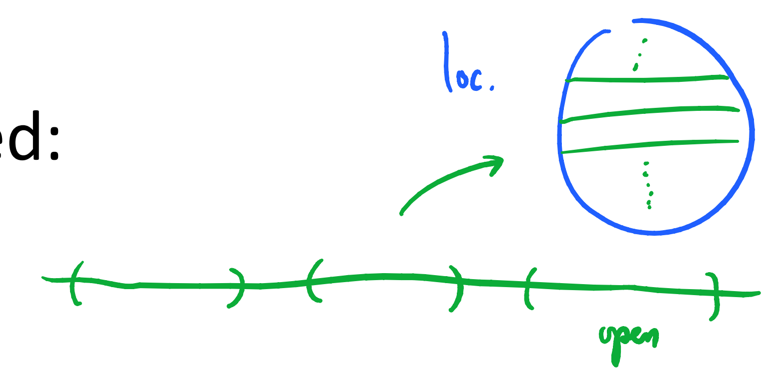
$$d\pi|_D : D \xrightarrow[\text{bdl map}]{\sim} T\mathbb{R}^k$$



Define V_i by $V_i \longmapsto \partial_i$.

$$d\pi(\underbrace{[V_i, V_j]}_{\in D}) = [\partial_i, \partial_j] = 0 \Rightarrow [V_i, V_j] = 0.$$

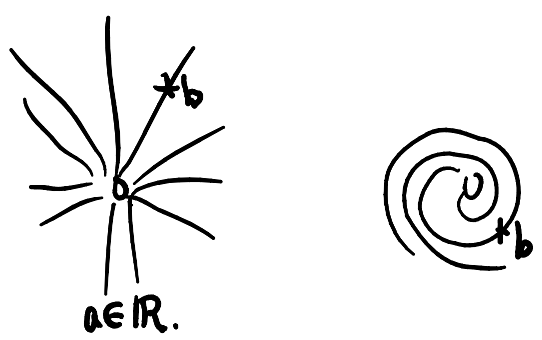
- Integral manifold S is weakly embedded: any smooth N->M with image in S is a smooth map N->S



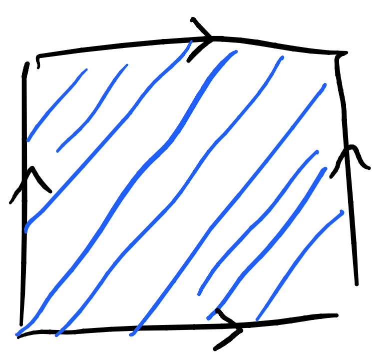
(M ⊂ $\mathbb{R}^N$ has countably many disj. open sets.)

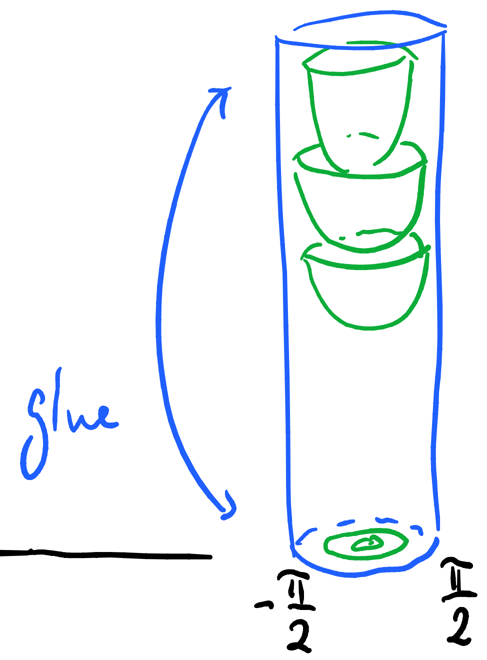
- Integrable distribution <-> foliation: collection of disjoint connected immersed submanifolds whose union is the whole space, and can take local coordinates such that defined by x_{k+1}=...=x_n=0

e.g.



e^{ta+b}, $a, b \in \mathbb{C}$.





$z = \sec y + c$ for $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

- Prob. 19.3, 4, 5, 10

Prob. 19.3: ω nonzero-0 1-form.

$\exists$ loc. int. factor μ s.t. $\mu\omega = df \Leftrightarrow d\omega \wedge \omega = 0$.

$\Rightarrow$) $\underbrace{d(\mu\omega)}_{0} \wedge \omega = \underset{\substack{\vee \\ 0}}{\mu}\, d\omega \wedge \omega$.

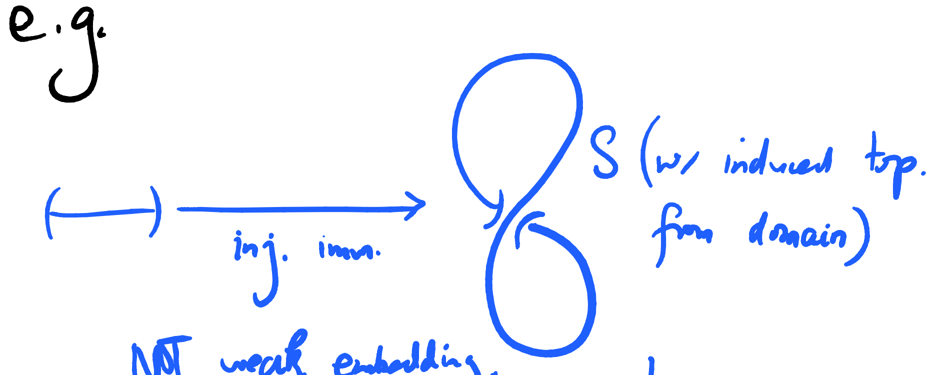
$\Leftarrow$) $d\omega = \eta \wedge \omega$. (extend ω to a loc. frame.)

$d(\mu\omega) = d\mu \wedge \omega + \mu\, d\omega \overset{\text{want}}{=} 0$.

$-(d\log\mu) \wedge \omega \overset{\text{want}}{=} d\omega = \eta \wedge \omega$.

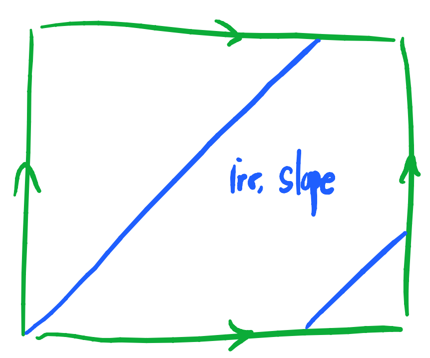
Ker ω Integrable $\Rightarrow$ coord. st $\mu\omega = \underbrace{dx_n}_{\text{nonzero}}$.

e.g.



(any sm. N→M w/ Im ⊂ S is sm. N→S)

e.g.



not embedded

It is weakly embedded