Monday, December 28, 2015 7:36 PM

- Lie subalgebra => left-invariant involutive distribution of G
- Lie subalgebra <-> connected Lie subgroup (leaf containing 1)

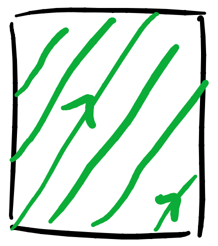
e.g. $\mathbb{R}^n, T^n$.

$\{AJA^t = I\}$ $\| \quad Sp_n(\mathbb{R})$

e.g. $GL_n(\mathbb{R}) \supset SL_n(\mathbb{R}) \supset SO_n(\mathbb{R}) \supset U_{n/2} \supset SU_{n/2}$, $Sp_n(\mathbb{R}) \supset U_{n/2}$, $SO_n(\mathbb{R}) \supset GL_{n/2}(\mathbb{C}) \supset U_{n/2}$

$GL_{n/2}(\mathbb{C}) = \{AJ = JA\} \Leftrightarrow A = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \sim u + iv$

e.g.



T^2 (irrat. slope)

NOT a submfd!

(closed subgp $\Rightarrow$ submfd)

Why subgp: $g, h \in \text{leaf}_1$. preserve distribution which is left G-inv.

$g \cdot h = L_g(h) \in L_g(\text{leaf}_1) = \text{leaf}_g = \text{leaf}_1$

$h^{-1} = (L_h)^{-1} \cdot 1 \in (L_h)^{-1}(\text{leaf}_1) = \text{leaf}_1$.

weakly embedded $\Rightarrow$ sm. map.

- One-parameter subgroup: homomorphism: Real line -> G
- Lie(G) = {1-para subgp} = {1-para subgp of diffeo}
- Eg: e^(tA) in GL or O
- Exp: Lie(G) -> G walk for time 1

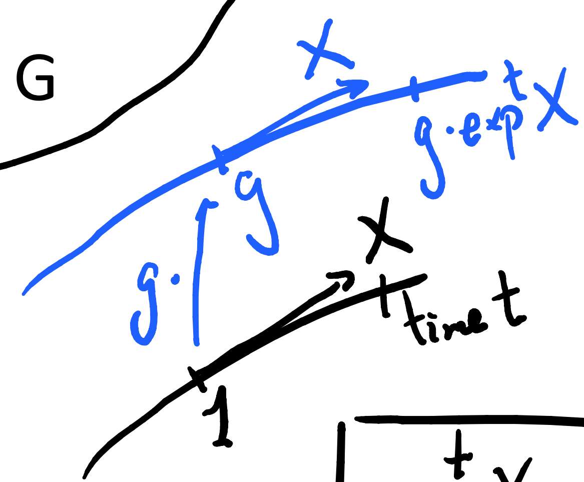
$\boxed{X \longleftrightarrow \exp^t X \longleftrightarrow g \cdot \exp^t X}$

Hence X acts on the right.

$\Leftarrow) \frac{d}{dt}\Big|_{t=0}$

$\Rightarrow)$ Need to show left inv. v.f. is complete. Let $\gamma: (-\varepsilon, \varepsilon) \to G$, $\begin{cases} \gamma(0) = 1 \\ \gamma'(t) = X|_{\gamma(t)} \end{cases}$.

Then $\tilde{\gamma} = g \cdot \gamma$ solves $\begin{cases} \tilde{\gamma}(0) = g \\ \tilde{\gamma}'(t) = X|_{\tilde{\gamma}(t)} \end{cases}$. Take $g = \gamma(\frac{\varepsilon}{2})$. Inductively get $\mathbb{R} \to G$.



For any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}$, $\exp(s+t)X = \exp sX \exp tX$.
For any $X \in \mathfrak{g}$, $(\exp X)^{-1} = \exp(-X)$.
For any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$, $(\exp X)^n = \exp(nX)$.
The differential $(d\exp)_0 : T_0\mathfrak{g} \to T_eG$ is the identity map, under the canonical identifications of both $T_0\mathfrak{g}$ and T_eG with $\mathfrak{g}$ itself.
The exponential map restricts to a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G.

$\gamma(t) \overset{\text{def}}{=} \exp^t X$. $\gamma(at) = \exp^t aX$. $\Leftarrow \frac{d}{dt}\gamma(at) = a\gamma'(at) = aX|_{\gamma(at)}$

$\boxed{\exp^t X = \exp tX.}$ (walk in velocity X for time t $\Leftrightarrow$ walk in velocity tX for time 1.)

$\frac{d}{dt}$: $X|_{\exp^t X} = (\exp tX) \cdot X|_1$.

$\Longrightarrow \boxed{\frac{d}{dt} \exp(tX) = X|_{\exp tX}.}$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\Phi_*"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\Phi"'] & H. \end{tikzcd}$$

$\frac{d}{dt}\Phi(\exp tX) = d\Phi_{\exp tX} \cdot X|_{\exp tX} = \Phi_* X|_{\Phi(\exp tX)}$

$\therefore \Phi(\exp tX)$ is the integral curve of $\Phi_* X$.

- Flow by left invariant vf X = right multiply by exp tX
- For $H < G$, $\exp^H = \exp^G|_H$.

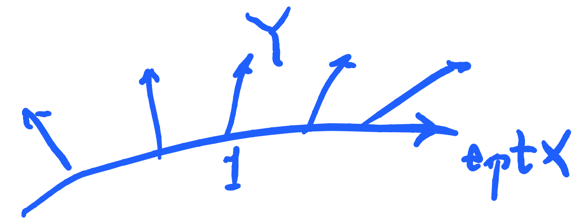
Note:

exp. on conn. Lie gp may not be surj. (Prob. 20.6)

e.g. $SL_2(\mathbb{R}) \ni \begin{pmatrix} -4 & \\ & -1/4 \end{pmatrix}$: no sq. root. real

4 e.v. $\pm 2i, \pm \frac{i}{2}$!

$\{\exp X_1 \dots \exp X_k\}$ is both closed and open



(Assume G is connected.)

- Any g = (exp X_1) ... (exp X_k)
- Abelian <=> [X,Y] = 0 for all X,Y
- Abelian <=> exp (X+Y) = exp(X) exp(Y)
- $(\exp tX)(\exp tY) = \exp(t(X+Y) + \frac{1}{2}t^2[X,Y] + t^3\hat{Z}(t))$.
- Prob. 20.4, 5

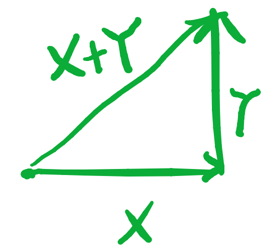
Prob 20.7

$$(\exp tX)(\exp sY) = (\exp sY)(\exp tX) \overset{\frac{d}{ds}}{\Leftrightarrow} (\exp tX)\cdot Y \cdot (\exp(tX))^{-1} = Y \overset{\frac{d}{dt}}{\Leftrightarrow} [X,Y] = 0 \quad \forall X, Y.$$

⇒) trivial.

⇐) Write $g = \exp X_1 \dots \exp X_k$.

Prob 20.8



⇒) More gen.: $X, Y \in \mathrm{Vect}(M)$. $[X,Y] = 0 \Rightarrow$ coord. chart.

$$\frac{d}{dt}(\exp(tX)\cdot \exp(tY)) = (\exp tX)\cdot \underbrace{X|_1 \cdot (\exp tY)}_{(\exp tY)\cdot X|_0 \text{ Abelian}} + (\exp tX)(\exp tY)\cdot Y|_1 = (\exp tX)(\exp tY)\cdot (X+Y)|_1 .$$

$\therefore (\exp tX)\cdot(\exp tY)$ is the 1-parameter subgroup w/ $\left.\frac{d}{dt}\right|_{t=0} = X+Y$.

⇐) $(\exp tX)(\exp sY)$ commutes $\Rightarrow [X,Y] = 0$.

Prob. 20.9 (Baker-Campbell-Hausdorff formula.)

Write $\exp \varphi_t = (\exp tX)(\exp tY)$.

$t=0$: $\quad \varphi_0 = 0$.

$\frac{d}{dt}$: $d\exp_{\varphi_t} \cdot \varphi_t' = (\exp tX)\cdot X \cdot (\exp tY) + (\exp tX)\cdot(\exp tY)\cdot Y$

$\left.\frac{d}{dt}\right|_{t=0}$: $\quad \varphi'(0) = X + Y$.

$(\exp(-\varphi_t)\cdot d\exp_{\varphi_t})\cdot \varphi_t' = \exp(-tY)\cdot X \cdot \exp tY + Y$.

$\left.\frac{d}{dt}\right|_{t=0}$: RHS $= -\mathcal{L}_Y X = [X,Y]$.

LHS $= \varphi''(0)$. ($[X+Y, X+Y] = 0$.)

$\therefore \varphi = t(X+Y) + \frac{t^2}{2}[X,Y] + \dots$