

- Subgroup + closed => submanifold. (a priori only weakly embedded)
- Eg. SL, SO and so on are all defined by equations => They are embedded Lie groups. Irrational slope in torus is not!
- $(\exp tX)(\exp tY) = \exp(t(X+Y) + t^2 Z(t))$

for some $Z: (-\epsilon, \epsilon) \rightarrow \mathfrak{g}$.

$$\therefore \lim_{n \rightarrow \infty} \left(\exp \frac{t}{n} X \right) \left(\exp \frac{t}{n} Y \right)^n = \exp t(X+Y).$$

Pf of closed subgp thm: \Downarrow closed

$$\mathfrak{h} \triangleq \{ X : \exp tX \in H \quad \forall t \} \text{ is a v.s.}$$

To get chart of H:

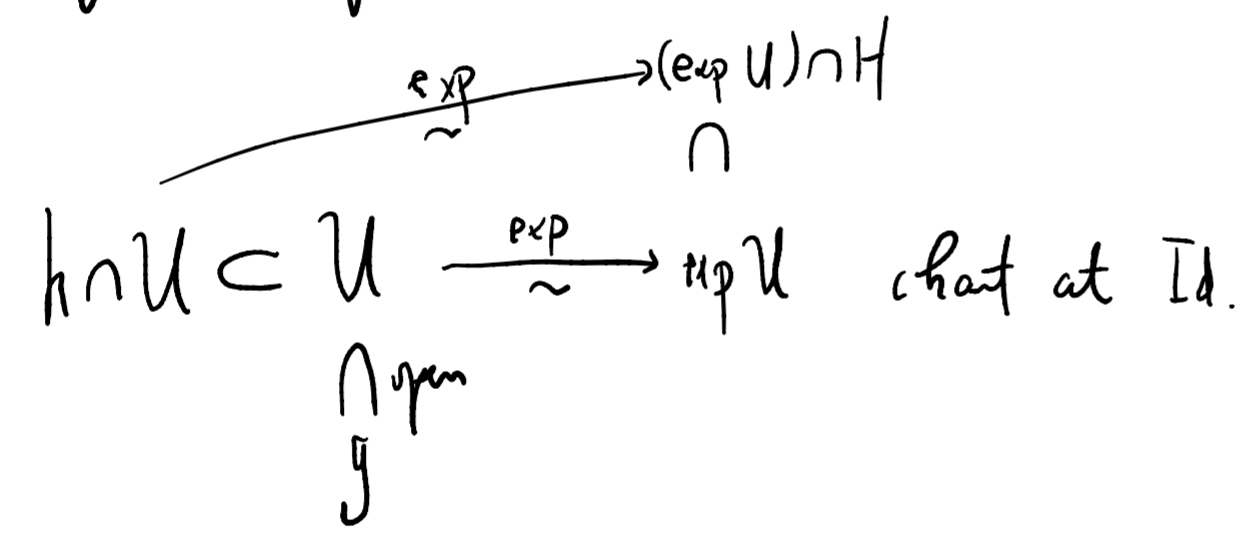


chart at h: $h \cdot \exp(\cdot)$.

- Every Lie subgroup is a dense subset of an embedded Lie subgroup (Prob. 20.10)

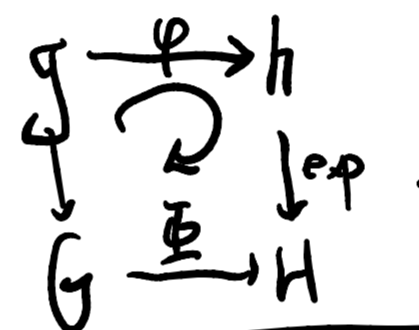
- Any continuous homo $G \rightarrow H$ is smooth (Prob. 20.11)

1. any cont. homo. $\mathbb{R} \xrightarrow{\gamma} H$ is sm.: need to show $\gamma(t) = \exp tX$ for some X . Same as pf of closed gp thm.

$\gamma(t) \in \exp(\mathfrak{u})$ where $\exp|_{\mathfrak{u}}$ is diffeo. for $|t| \ll 1$. Let $\gamma(t_0) = \exp(t_0 X)$.

$\gamma(\frac{t_0}{k}) = \exp(\frac{t_0 X}{k})$ since $\exp|_{\mathfrak{u}}$ is diffeo. $\Rightarrow \gamma(\frac{j t_0}{k}) = \exp(\frac{j t_0 X}{k}) \Rightarrow \gamma(t) = \exp(tX)$.

2. Def. $\gamma: \mathfrak{g} \rightarrow \mathfrak{h}$ by $\Phi(\exp tX) = \exp t\varphi(X)$ by step 1. φ cont.



$$\exp(\varphi(X+Y)) = \Phi(\exp(X+Y))$$

$$= \lim_{n \rightarrow \infty} \left(\Phi\left(\exp \frac{X}{n}\right) \Phi\left(\exp \frac{Y}{n}\right) \right)^n$$

$$= \lim_{n \rightarrow \infty} \left(\exp \varphi\left(\frac{X}{n}\right) \exp \varphi\left(\frac{Y}{n}\right) \right)^n$$

$$= \exp(\varphi(X) + \varphi(Y)).$$

$$\Rightarrow \varphi(X+Y) = \varphi(X) + \varphi(Y) \text{ for } X, Y \text{ small}$$

$$\Rightarrow \text{by multip. by } k.$$

$$\Phi \circ \exp = \exp \circ \varphi \text{ sm. } \Rightarrow \Phi \text{ sm. around 1}$$

loc. diffeo around 1

$$\underbrace{L_g \circ \Phi \circ L_{g^{-1}}}_{\text{sm. around 1}} = \tilde{\Phi} \text{ sm. around } g.$$

φ is lin.:

$$\varphi\left(\frac{j}{k}X\right) = \frac{j}{k}\varphi(X)$$

$$\Rightarrow \varphi(aX) = a\varphi(X).$$