• Subgroup + closed => submanifold. (a priori only weakly embedded)
• Eg. SL, SO and so on are all defined by equations => They are embedded Lie groups. Irrational slope in torus is not!
  \((\exp tX)(\exp tY) = \exp (t(X+Y) + t^2 \mathcal{Z}(t))\)
  for some \(\mathcal{Z}: (-\epsilon, \epsilon) \rightarrow g\).
\[\lim_{n \rightarrow \infty} (\exp \frac{t}{n}X)(\exp \frac{t}{n}Y)^n = \exp t(X+Y).\]

**Proof of closed subgroup theorem:**

\[h = \{ X : \exp tX \in H \ \forall t \} \text{ is a v.s.} \]

To get chart of \(H:\)

\[\exp (\exp U) \cap H \]

\[h \cap U \subset U \xrightarrow{\exp} \text{exp} U \text{ chart at } \text{Id}. \]

\[\bigcap_{n=1}^{\infty} \text{exp} U \text{ chart at } h : h \cdot \exp(\cdot).\]

• Every Lie subgroup is a dense subset of an embedded Lie subgroup (Prob. 20.10)
Any continuous homo $G \to H$ is smooth (Prob. 20.11)

1. Any cont. homo $R \to H$ is sm.: need to show $\gamma(x) = \exp_t x$ for some $x$. Same as pf of direct gp thm.
   \[ \gamma(t) = \exp(tu) \text{ where } \exp|_u \text{ is diff. for } |t| \ll 1. \text{ Let } \gamma(t) = \exp(tx). \]
   \[ \gamma\left(\frac{t}{k}\right) = \exp\left(\frac{tx}{k}\right) \text{ since } \exp|_u \text{ is diff. } \Rightarrow \gamma\left(\frac{tx}{k}\right) = \exp\left(\frac{tx}{k}\right) \Rightarrow \gamma(t) = \exp(tx). \]

2. Def: $\Phi : R^+ \to H$ by $\Phi(\exp_t X) = \exp_t(\Phi(X))$ by step 1.
   \[ \Phi \text{ cont.} \]
   \[ \Phi \circ \exp = \exp \Phi \text{ sm. } \Rightarrow \Phi \text{ sm. around } 1 \]
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\[ L_g \Phi \circ L_g = \Phi \text{ sm. around } g, \]

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\[ \exp (\Phi(x + y)) = \Phi(\exp(x + y)) = \lim_{n \to \infty} \left( \frac{\Phi(\exp(X))}{\Phi(\exp(X + Y))} \right)^n \]

\[ \Phi \text{ is lin.: } \]

\[ \phi\left(\frac{x}{k}\right) = \frac{1}{k} \phi(x) \]

\[ \Rightarrow \phi(ax) = a \phi(x). \]

\[ \Rightarrow \phi(x + y) = \phi(x) + \phi(y) \text{ for } x, y \text{ small. } \]

\[ \Rightarrow \text{ by multip. by } k. \]