# Closed subgroup theorem

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- Subgroup + closed => submanifold. (a priori only weakly embedded)
- Eg. SL, SO and so on are all defined by equations => They are embedded Lie groups. Irrational slope in torus is not!
- $(\exp tX)(\exp tY) = \exp\left(t(X+Y) + t^2 Z(t)\right)$

  for some $Z : (-\varepsilon, \varepsilon) \to \mathfrak{g}$.

$\therefore \lim_{n\to\infty} \left((\exp \frac{t}{n} X)(\exp \frac{t}{n} Y)\right)^n = \exp t(X+Y)$.

$\Downarrow$ closed

Pf of closed subgp thm:

$\mathfrak{h} \triangleq \{ X : \exp tX \in H \ \ \forall t \}$ is a v.s.

To get chart of $H$:

$$\begin{tikzcd}
 & & (\exp U)\cap H \arrow[d, phantom, "\cap"] \\
\mathfrak{h}\cap U \arrow[r, phantom, "\subset"] \arrow[urr, bend left, "\exp", "\sim"'] & U \arrow[r, "\exp", "\sim"'] \arrow[d, phantom, "\cap\ \text{open}"] & \exp U \\
 & \mathfrak{g} &
\end{tikzcd}$$

chart at Id.

chart at $h$: $h \cdot \exp(\cdot)$.

- Every Lie subgroup is a dense subset of an embedded Lie subgroup (Prob. 20.10)

- Any continuous homo G -> H is smooth (Prob. 20.11)

1. any cont. homo. $\mathbb{R} \xrightarrow[\gamma]{} H$ is sm.: need to show $\gamma(t) = \exp tX$ for some $X$. Same as pf of closed gp thm.

$\gamma(t_0) \in \exp(U)$ where $\exp|_U$ is diffeo. for $|t_0| \ll 1$. Let $\gamma(t_0) = \exp(t_0 X)$.

$\gamma\left(\frac{t_0}{k}\right) = \exp\left(\frac{t_0 X}{k}\right)$ since $\exp|_U$ is diffeo. $\Rightarrow \gamma\left(\frac{j t_0}{k}\right) = \exp\left(\frac{j t_0 X}{k}\right) \Rightarrow \gamma(t) = \exp(tX)$.

2. Def. $\underset{X \in}{\mathfrak{g}} \xrightarrow{\varphi} \mathfrak{h}$ by $\Phi(\exp tX) = \exp t\varphi(X)$ by step 1.

$\varphi$ cont.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & \mathfrak{h} \\ \downarrow & \circlearrowright & \downarrow \exp \\ G & \xrightarrow{\Phi} & H \end{array}$$

$\Phi \circ \underbrace{\exp}_{\text{loc. diffeo around } 1} = \exp \circ \varphi$ sm. $\Rightarrow \Phi$ sm. around $1$

$L_g \circ \underbrace{\Phi}_{\text{sm. around } 1} \circ L_{g^{-1}} = \Phi$ sm. around $g$.

$\varphi$ is lin.:

$\varphi\left(\frac{j}{k} X\right) = \frac{j}{k}\varphi(X)$

$\Rightarrow \varphi(aX) = a\varphi(X)$.

$$\begin{aligned} \exp(\varphi(X+Y)) &= \Phi(\exp(X+Y)) \\ &= \lim_{n\to\infty}\left(\left(\Phi\left(\exp \frac{X}{n}\right)\right)\left(\Phi\left(\exp\frac{Y}{n}\right)\right)\right)^n \\ &= \lim_{n\to\infty}\left(\left(\exp \varphi\left(\frac{X}{n}\right)\right)\left(\exp\varphi\left(\frac{Y}{n}\right)\right)\right)^n \\ &= \exp(\varphi(X)+\varphi(Y)). \end{aligned}$$

$\Rightarrow \varphi(X+Y) = \varphi(X)+\varphi(Y)$ for $X, Y$ small

$\Rightarrow$ " by multip. by $k$.