Wednesday, December 30, 2015 3:37 PM

- Left and right actions on a set (g.h: g or h acts first), and one-one correspondence between them (g.x = x.g^(-1))
- Action on manifold: G times M -> M smooth. Smooth map G -> Diffeo(M)
- Example: SL(2,Z) < SL(2,C) act on upper-half-plane < P^1
- Right G action => Lie(G) -> vf(M) Lie alg homo
- Left action: anti-homo
- Group action <-> {orbit maps theta_p: G -> M} satisfying associativity $(p\cdot g)\cdot h = p\cdot(g\cdot h) \Leftrightarrow \theta_{\theta_p(g)}(h) = \theta_p(g\cdot h)$.
- Push forward of left-invariant vf of G by orbit map G -> M (G-equivariant map)
- Thm: Complete g action: g -> complete vf(M) => G action (need G simply connected)
- Proof by Frobenius theorem (trouble: don't know exp is surjective, well-def, and asso.)
- Complete g action determines G action
- Ex. Give a counterexample when G not simply connected.

$\mathfrak{g} \longrightarrow vf(M)$ Lie alg. homo.: (tang. map of $G \longrightarrow Diff(M)$)

$$[X_\#, Y_\#] f_{(p)} = X_\# \frac{d}{dt}\Big|_{t=0} f(p\cdot \exp tY) - Y_\# X_\# f(p)$$

$$= \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \left(f(p\cdot(\exp sX \cdot \exp tY)\right) - Y_\# X_\# \cdot f(p)$$

$$= \frac{d}{ds}\frac{d}{dt}\Big|_{s=t=0} f\left(p\cdot \exp(tY)\left(\exp(-tY)\exp(sX)\exp(tY)\right)\right) - Y_\# X_\# f(p)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\left(\exp(tY) X \exp(tY)\right)_\# f\right)\left(p\cdot \exp(tY)\right) - Y_\# X_\# f(p)$$

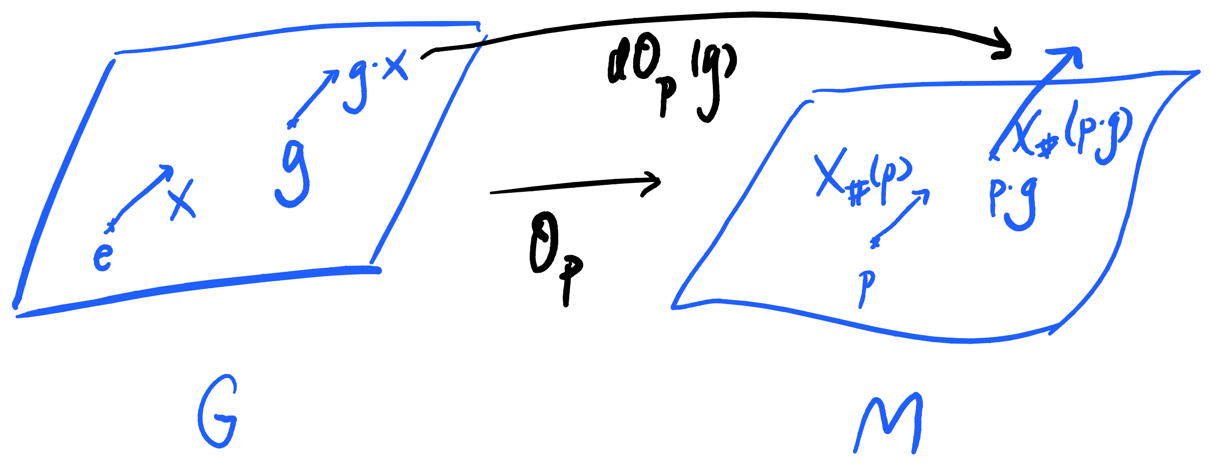
$$= (-\mathcal{L}_Y X)_\# f(p) + \cancel{\frac{d}{dt}\Big|_{t=0} (X_\# f)(p \exp tY) - Y_\# X_\# f(p)}$$

Orbit map:

$G \longrightarrow M \ni p$ fixed, $\circlearrowleft G$ G-equiv.

$\theta_p(g) = p\cdot g$

$$d\theta_p(g)(X|_g) = \frac{d}{dt}\Big|_{t=0} p\cdot\left(g\cdot(\exp tX)\right) = (p\cdot g)\cdot X = X_\#(p\cdot g).$$



Pf that complete $\mathfrak{g}$-action & G simply conn. $\Longrightarrow$ G-action. (compare w/ $\mathfrak{h} < \mathfrak{g} \longleftrightarrow H \underset{\text{conn}}{<} G$)

(can't just use $p\cdot\mathfrak{g} \subset T_pM$: only get orbits of G but pts are not labelled by $g\in G$.)

Distribution $D_{g,p} = \{\underbrace{(\underbrace{gX}_{X|_g}, \underbrace{p\cdot X}_{X_\#(p)})}_{\tilde X} : X \in \mathfrak{g}\}$ $\subset G\times M$

$[\underset{(X,X_\#)}{\tilde X}, \underset{(Y,Y_\#)}{\tilde Y}] = \underset{([X,Y],[X,Y]_\#)}{\widetilde{[X,Y]}}$ $\therefore$ involutive

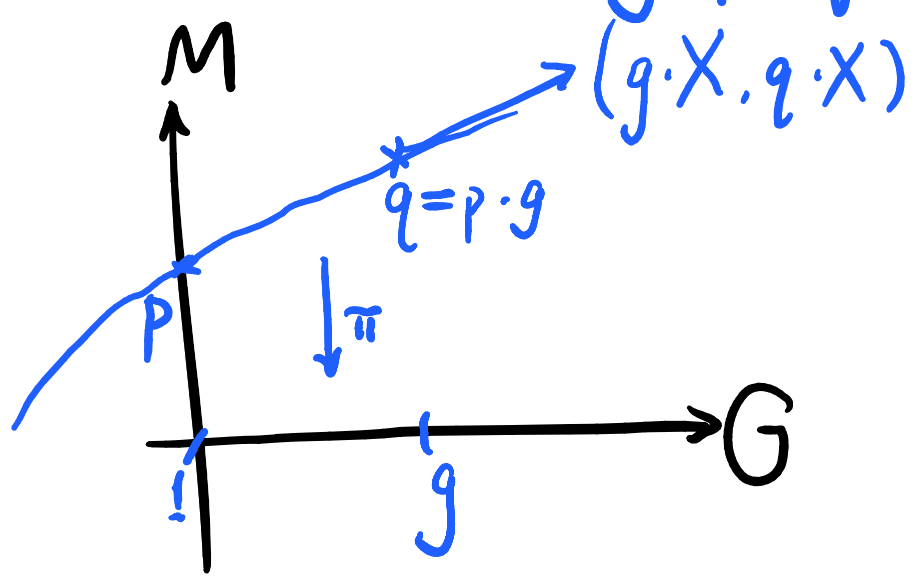
$S_{(g,p)} \triangleq$ leaf thru (g,p).

Want: Define $p\cdot g = q$ if $S_{(e,p)} = S_{(g,q)}$.

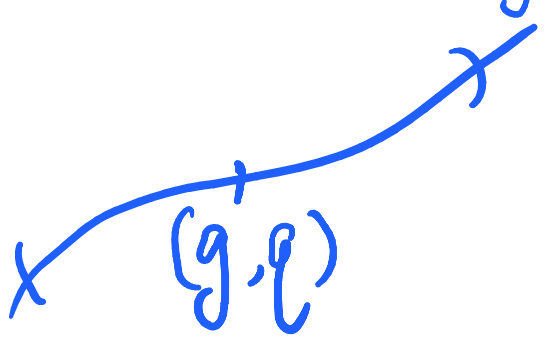
$S_{(e,p)}$ is a graph, i.e. $S_{(e,p)} \xrightarrow{\pi} G$ diffeo: π is a cover & G is simply conn.

(get around the diff^{ty} exp is not surj. just need $\exp U$ gen. but not surj.)

$S_{(1,p)}$: graph of orbit map O_p



$\pi^{-1}(g\cdot\exp(U))$

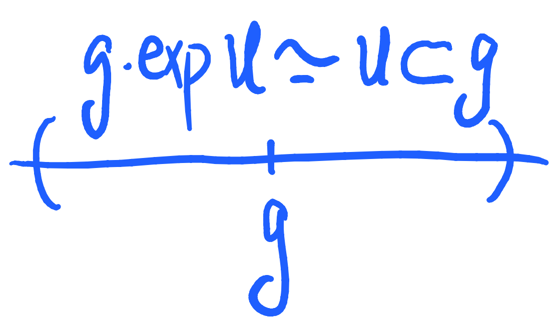


$q\cdot\exp X \triangleq$ walk along int. curve of $\hat X$ (complete) for time 1.

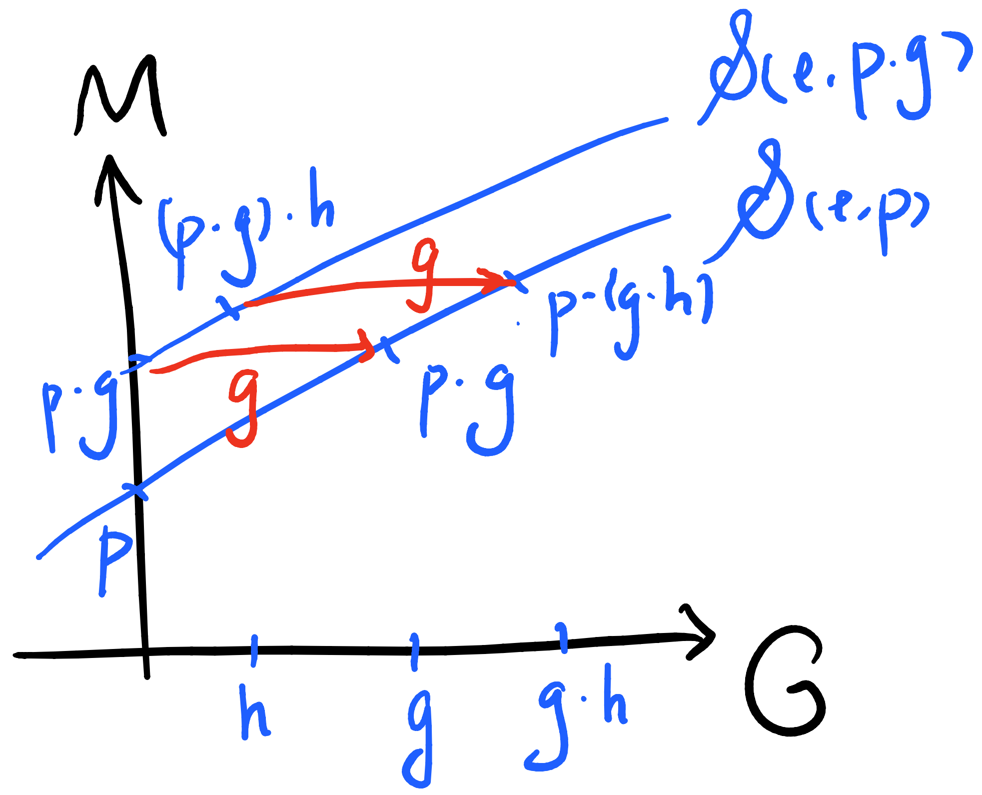
$\pi^{-1}(g\cdot\exp(U)) = \bigsqcup_{q\in\pi^{-1}\{g\}} (g\exp U, q\exp U)$: $\supset$ is obvious.

$\subset$: $(g\exp X, \alpha) \in$ left. Take $q = \alpha\cdot\exp(-X)$. $(g\exp X, \alpha) = (g\exp X, q\exp X) \in$ right.

disjoint : $(g,q)\exp X_1 = (g,\alpha)\exp X_2 \underset{\exp|_U \text{ is diffeo.}}{\Rightarrow} X_1 = X_2 \Rightarrow q = \alpha$.



asso. :



$$(h, (p\cdot g)\cdot h) \underset{\text{leaf}}{\overset{\text{same}}{\sim}} (e, p\cdot g).$$

left g on 1st factor

$$(gh, (p\cdot g)\cdot h) \sim (g, p\cdot g) \sim (e, p) \sim (gh, p\cdot(gh)).$$

graph $\Rightarrow (p\cdot g)\cdot h = p\cdot(g\cdot h)$.

send leaf to leaf:

$$D_{h,p} \ni (h\cdot X, p\cdot X) \xrightarrow{d(L_g)} (g\cdot h\cdot X, p\cdot X) \in D_{(gh, p)}.$$

unique: If $G \overset{\exists \text{ another}}{\curvearrowright_{*}} M$ w/ same $\mathfrak{g} \to$ v.f.(M):

$$\{(g, p*g) : g \in G\} \overset{\text{graph}}{\subset} G\times M.$$

$$T_{(g,q)} = \{(g\cdot X, \underset{q\cdot X}{\underset{\parallel}{q*X}})\} = D_{(g,q)}$$

∴ same graphs, hence same action.