- Lie correspondence: simply connected Lie group $\leftrightarrow$ Lie algebra
- Eg $\mathbb{R}^2$ and $T^2$. $\text{SU}(2)$ simply connected; $\text{SO}(3)$ is not; $\text{SU}(2)$ is double cover of $\text{SO}(3)$, hence same Lie algebra
- Simple Lie algebra are classified by Dynkin diagram. \{upper triangular matrices\} is not semi-simple!

- $\mathfrak{g} < \mathfrak{gl}_N \xrightarrow{\text{leaf}} G < \mathbb{C} \mathcal{L}_N \ni \text{Lie}(G) = \mathfrak{g}$. Take $\tilde{G} : \text{uni. cover} \to \text{homotopy class of paths from } I$

  $\Phi : G \rightarrow H \ni \Phi_x = \varphi$.

- $G$ simply connected, $\text{Lie}(G) \cong \mathfrak{g}$, $\text{Lie}(H) \cong \mathfrak{h}$.

  $\Phi : G \rightarrow H$,

  $\Phi : (g) = \exp \varphi (g)$.

  $\Phi$ is homo:

  If $x = \exp X$,

  $\Phi(g_1 g_2) = (\varphi_x \circ \varphi) \cdot g_2 = \text{walk along } \varphi(x) \text{ from } \Phi(g_1)$

  $\Phi(g_1) \cdot (\text{walk along } \varphi(x) \text{ from } \varphi(1)) = \Phi(g_1) \cdot \Phi(g_2)$.

  Any $g = \exp X_1 \cdots \exp X_k$.

  $\Phi$ is unique: $\Phi(\exp X) = \exp \varphi(X)$. Any $g = (\exp X_1) \cdots (\exp X_k)$.
20-20. Let $G$ be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Prove that the kernel of $\text{Ad}: G \to \text{GL}(\mathfrak{g})$ is the center of $G$, that is, the set of elements of $G$ that commute with every element of $G$.

20-21. Show that the adjoint representation of $\text{GL}(n, \mathbb{R})$ is given by $\text{Ad}(A)Y = AYA^{-1}$ for $A \in \text{GL}(n, \mathbb{R})$ and $Y \in \mathfrak{gl}(n, \mathbb{R})$. Show that it is not faithful.

20-22. If $\mathfrak{g}$ is a Lie algebra, the center of $\mathfrak{g}$ is the set of all $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. Suppose $G$ is a connected Lie group. Show that the center of $\text{Lie}(G)$ is the Lie algebra of the center of $G$.

\[
\text{Ad}(g)X = X \cdot g^{-1} \quad \text{(as left inv. v.f.)},
\]

\[
d(\text{Ad})_x(Y) = [Y, X].
\]
\[ \text{Ad}(g) = \text{Id} \iff X \cdot g^{-1} = X \quad \forall X \iff R_g^{-1} \text{ maps } \exp tX \text{ to } g \cdot \exp tX \quad \forall X \]

\[ (\exp tX) \cdot g^{-1} = g \cdot \exp tX \quad \forall X \]

\[ g \text{ commutes with anything (any } g = \exp Y_1 \ldots \exp Y_k \text{.)} \]

e.g. \( c : I \in \text{center}\left( \text{GL}(n, \mathbb{R}) \right) \) acts trivially on \( \text{gl}(n, \mathbb{R}) \).

\[ \text{center} \left( \text{Lie}(G) \right) = \text{Lie} \left( \text{center}(G) \right) \]

\[ \Rightarrow \exp tX \in \text{center}(G) \Rightarrow \text{Ad}(\exp tX) = \text{Id} \Rightarrow \frac{d}{dt} \text{Ad}_t(X) = 0 \]

\[ \Rightarrow \frac{d}{dt} Y \cdot \exp (-tX) = Y \cdot \exp (-tX) \cdot [X, Y \cdot \exp tX] = 0 \quad \forall t, Y \]

\[ \Rightarrow Y \cdot \exp (tX) = Y \]

\[ \text{trajectory of } Y \text{ are preserved} \]

\[ \Rightarrow \exp s Y \cdot \exp (-tX) = \exp (-tX) \cdot \exp (sY) \]

\[ \Rightarrow \exp (-tX) \in \text{center}(G) \quad (\text{any } g = \exp Y_1 \ldots \exp Y_k) \Rightarrow X \in \text{Lie} \left( \text{center}(G) \right) \]