Lie correspondence

Wednesday, December 30, 2015 3:53 PM

- Lie correspondence: simply connected Lie group <-> Lie algebra
- Eg R^2 and T^2. SU(2) simply connected; SO(3) is not; SU(2) is double cover of SO(3), hence same Lie algebra
- Simple Lie algebra are classified by Dynkin diagram. {upper triangular matrices} is not semi-simple!

$\{\text{s.c. } G\} \underset{\text{int.}}{\overset{\mathrm{Lie}(\cdot)}{\rightleftarrows}} \{\mathfrak{g}\}$: $\circlearrowright = \mathrm{Id}$ by def.

$\circlearrowleft = \mathrm{Id}$: uniqueness

- $\mathfrak{g} < \mathfrak{gl}_N \xrightarrow{\text{leaf}_1} G < GL_N$ w/ $\mathrm{Lie}(G) = \mathfrak{g}$. Take $\tilde{G}$: univ. cover $= \{$homotopy class of paths from $1\}$ (still has group str.)
- Uniqueness $\mathrm{Lie}(G) \simeq \mathrm{Lie}(H)$ $(G, H \text{ s.c})$ $\Rightarrow G \simeq H$:

$$\mathrm{Lie}(G) \xrightarrow[\varphi]{} \mathrm{Lie}(H) \quad (G \text{ s.c.})$$

$$\Rightarrow \exists!\ \Phi : G \longrightarrow H \text{ w/ } \Phi_* = \varphi.$$

Pf:

$$\begin{tikzcd} & \mathrm{Lie}\,H \arrow[dl] & & & \\ H & \mathrm{Lie}\,G \arrow[l] \arrow[u, "\varphi"] & G \text{ s.c.} \arrow[r] & H & G \arrow[l] \end{tikzcd}$$

$\Phi(g) \doteq 1_H \cdot g$.

Φ is homo:

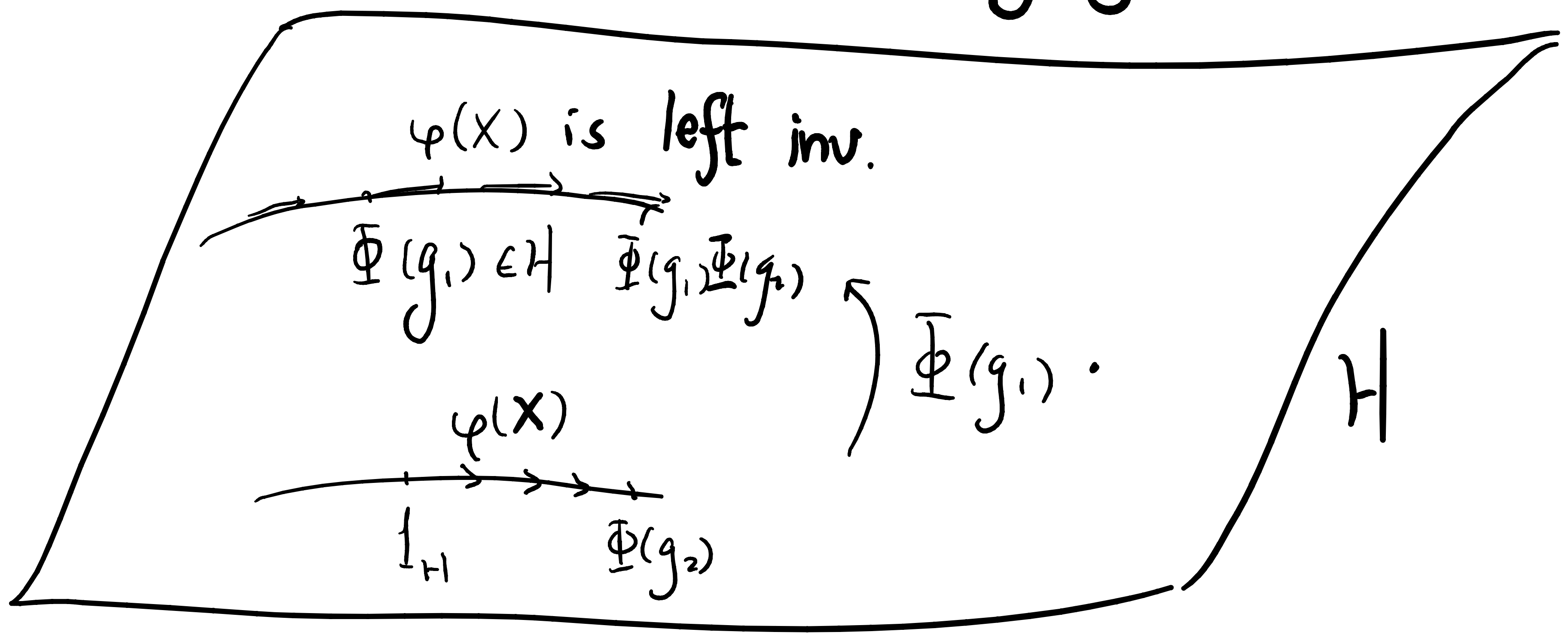
$$\Phi(g_1 g_2) = \underbrace{(1_H \cdot g_1)}_{\Phi(g_1)} \cdot \overset{\text{if } \exp X}{g_2} = \text{walk along } \varphi(X) \text{ from } \Phi(g_1)$$

$$= \Phi(g_1) \cdot (\text{walk along } \varphi(X) \text{ from } 1_H)$$

$$= \Phi(g_1) \cdot \Phi(g_2).$$

Any $g_2 = \exp X_1 \ldots \exp X_k$.

Φ is unique: $\Phi(\exp X) = \exp \varphi(X)$. Any $g = (\exp X_1) \ldots (\exp X_k)$.



20-20. Let G be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Prove that the kernel of $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is the ***center of*** G, that is, the set of elements of G that commute with every element of G.

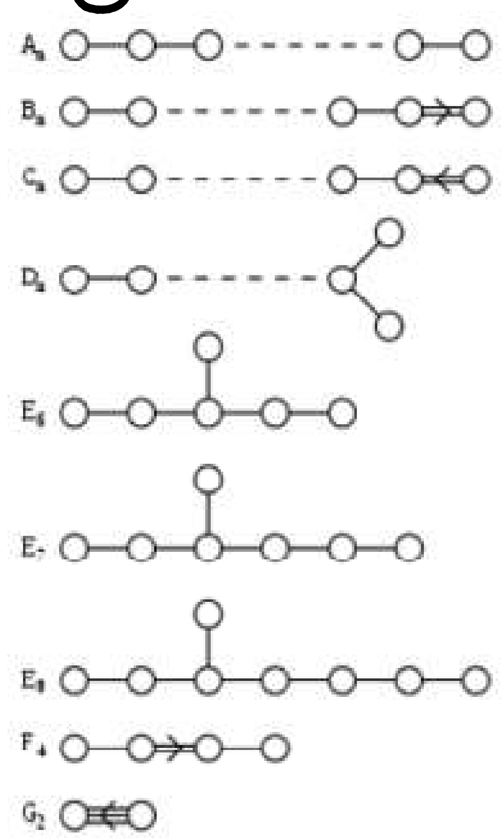
20-21. Show that the adjoint representation of $\mathrm{GL}(n,\mathbb{R})$ is given by $\mathrm{Ad}(A)Y = AYA^{-1}$ for $A \in \mathrm{GL}(n,\mathbb{R})$ and $Y \in \mathfrak{gl}(n,\mathbb{R})$. Show that it is not faithful.

20-22. If $\mathfrak{g}$ is a Lie algebra, the ***center of*** $\mathfrak{g}$ is the set of all $X \in \mathfrak{g}$ such that $[X,Y]=0$ for all $Y \in \mathfrak{g}$. Suppose G is a connected Lie group. Show that the center of $\mathrm{Lie}(G)$ is the Lie algebra of the center of G.

$\mathrm{Ad}(g)\cdot X = X\cdot g^{-1}$ (as left-inv. v.f.).

$d(\mathrm{Ad})|_1(Y) = [Y,\cdot\,]$.

Dynkin diagram:



- $\mathrm{Ad}(g) = \mathrm{Id} \Leftrightarrow X \cdot g^{-1} = X \ \forall X \Leftrightarrow R_{g^{-1}}$ maps $\underbrace{\exp tX}_{\text{int. curve thru } 1}$ to $\underbrace{g^{-1} \cdot \exp tX}_{\text{int. curve thru } g^{-1}} \ \forall X$

$$\Leftrightarrow (\exp tX) \cdot g^{-1} = g^{-1} \cdot \exp tX \quad \forall X$$

$\Leftrightarrow g$ commutes w/ anything (any $g = \exp X_1 \dots \exp X_k$.)

- e.g. $c \cdot I \in \mathrm{center}(GL(n, \mathbb{R}))$ acts trivially on $\mathfrak{gl}(n, \mathbb{R})$.

- $\mathrm{center}(\mathrm{Lie}(G)) = \mathrm{Lie}(\underbrace{\mathrm{center}(G)}_{\text{subgp}})$:

$\supset$: $\exp tX \in \mathrm{center}(G) \Rightarrow \mathrm{Ad}(\exp tX) = \mathrm{Id} \xRightarrow{\left.\frac{d}{dt}\right|_{t=0}} d\mathrm{Ad}_1(X) = 0$

$\subset$: $\frac{d}{dt} Y \cdot \exp(-tX) = Y \cdot \exp(-tX) \cdot R_{-X} = [X, Y \cdot (\exp tX)] = 0 \quad \forall t, Y$

$\Rightarrow Y \cdot \exp(-tX) = Y$

$\Rightarrow \exp sY \cdot \exp(-tX) = \exp(-tX) \cdot \exp(sY)$ ⟵ trajectories of Y are preserved

$\Rightarrow \exp(-tX) \in \mathrm{center}(G)$ (any $g = \exp Y_1 \dots \exp Y_k$) $\Rightarrow X \in \mathrm{Lie}(\mathrm{center}(G))$.