• Review: simply connected Lie group <-> Lie algebra
• Connected subgroup (may not be closed) <-> Lie sub-algebra
• **Connected normal subgroup** <-> ideal of Lie algebra (NEED G CONNECTED)
• Most subgroups are not normal (since quotient is no longer a group). Any subgroup for Abelian group is normal. GL(1) and SL(n) in GL(n) are normal.

\[ \text{def: } \frac{\text{GL}(n)}{\text{SL}(n)} \rightarrow \mathbb{R}^*. \]

\[ \text{(left) } \quad \text{Adjoint action: } G \rightarrow G^\text{ad} = \mathfrak{g}. \]

\[ \text{Ad}(g) \cdot X = g \cdot X \cdot g^{-1} = X \cdot g^t \]

\[ \text{as } x \in T_xG \text{ as left inv v.s. } \]

\[ \frac{d}{dt} \text{Ad}(\exp tY) \cdot X = \mathcal{L}_Y X = [X,Y] \text{ by def. } \]

\[ \text{(linear: } G \rightarrow \mathfrak{gl}(g) ) \quad \text{Ad} \leftrightarrow \text{ad} \quad \text{(linear: } g \rightarrow gl(g) ) \]

\[ \text{normal } \Rightarrow \text{ideal } \]

\[ \exp tX \exp \gamma \exp (-tX) \in H \quad \Rightarrow \quad \text{Ad}(\exp tX) : Y \in h \]

\[ \frac{d}{dt} \bigg|_{t=0} \]
\begin{align*}
\iff \text{Check } H \text{ is normal: } X \in \mathfrak{g}, \ Y \in h, \quad & GL(g) \\
(\exp X) \cdot Y \cdot \exp(-X) = \Ad_{\exp X} Y = \exp (\text{Ad}_X)=Y = \sum_{k=0}^{\infty} \frac{1}{k!} ([X,Y])^k \in h, \\
\therefore (\exp X) \cdot (\exp Y) \cdot \exp(-X) = \exp (\text{Ad}_X Y) \in H. \\
\therefore (\exp X)(\exp Y_1)(\exp Y_2)(\exp Y_3) \cdots (\exp X)(\exp Y_1)(\exp Y_2) \in H. \\
(\exp X_1)(\exp X_2) \cdot \text{Ad}_{\exp X_1} \cdots \text{Ad}_{\exp X_2} \in H. \\
\text{any } g \text{ is of this form} \quad & \text{if } G \text{ is connected} \\
\text{Ad}_g (X) = X \Rightarrow \text{Conj}_g (\exp X) = \exp (\text{Ad}_g X) = \exp X. \\
\text{any } h = (\exp X_1) \cdots (\exp X_n) \Rightarrow \text{Conj}_g h = h \quad \forall h \text{ } \Rightarrow \text{ } g \in \text{center}. \\
\text{c:Id acts trivially.} \\
\text{center (Lie } G) = \text{Lie (center (G))}. \\
\text{ad}_X = 0 \quad \forall Y \Leftrightarrow \text{Ad}_{\exp X} = \exp (\text{ad}_X) = \text{Id} \\
\Leftrightarrow \exp X \in \text{center (G)} \Leftrightarrow X \in \text{Lie (center (G))}.
\end{align*}