- Review: simply connected Lie group <-> Lie algebra
- Connected subgroup (may not be closed) <-> Lie sub-algebra
- **Connected normal subgroup <-> ideal of Lie algebra (NEED G CONNECTED)**
- Most subgroups are not normal (since quotient is no longer a group). Any subgroup for Abelian group is normal. GL(1) and SL(n) in GL(n) are normal.

normal subgroup $H: GHG^{-1} \in H$.

ideal $\mathfrak{h}: [\mathfrak{g}, \mathfrak{h}] \in \mathfrak{h}$.

$\det: GL(n)/SL(n) \xrightarrow{\sim} \mathbb{R}^\times$.

(left)
Adjoint action: $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}$, $ad = \mathcal{L}$.

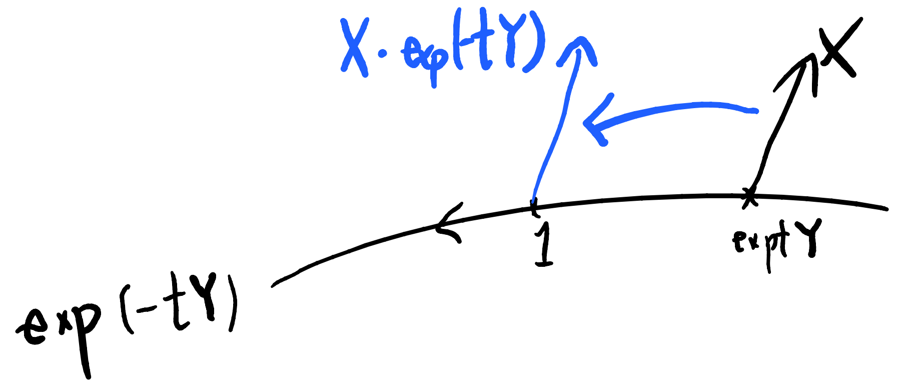
$G \overset{Ad}{\curvearrowright} \mathfrak{g}$.

$G \overset{conj.}{\curvearrowright} G$.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Ad_g"] \arrow[d, "\exp"] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "conj.(g)"'] & G \end{tikzcd}$$

$$Ad(g) \cdot X = \underset{\text{as vector} \in T_1 G}{g \cdot X \cdot g^{-1}} = \underset{\text{as left inv. v.f.}}{X \cdot g^{-1}}$$

$$\left.\frac{d}{dt}\right|_{t=0} Ad(\exp tY) \cdot X = \mathcal{L}_Y X = [Y, X] \text{ by def.}$$




Hence
$$\{G \curvearrowright \mathfrak{g}\} \xleftrightarrow{1-1} \{\mathfrak{g} \overset{\text{complete}}{\curvearrowright} \mathfrak{g}\}$$
$$(\text{linear}: G \to GL(\mathfrak{g})) \quad Ad \longleftrightarrow ad \quad (\text{linear}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}))$$

$$\boxed{Ad(\exp X) = \exp(ad_X).} \quad (\text{walk along v.f. } ad_X \in \mathfrak{gl}(\mathfrak{g}).) \quad \exp(ad_X) = \sum_{k=0}^{\infty} \frac{(ad_X)^k}{k!}.$$

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normal $\Rightarrow$ ideal)

$$(\exp t\underset{\mathfrak{g}}{\underset{\cap}{X}})(\exp s\underset{\mathfrak{h}}{\underset{\cap}{Y}})(\exp(-tX)) \in H \underset{\left.\frac{d}{ds}\right|_{s=0}}{\Rightarrow} Ad(\exp tX) \cdot Y \in \mathfrak{h}$$

$$\underset{\left.\frac{d}{dt}\right|_{t=0}}{\Rightarrow} [X, Y] \in \mathfrak{h}.$$

$\Leftarrow$) Check $H$ is normal: $X \in \mathfrak{g}$. $Y \in \mathfrak{h}$.

$$(\exp X)\cdot Y \cdot \exp(-X) = \mathrm{Ad}_{\exp X} Y = \overbrace{\exp(\underbrace{\mathrm{ad}_X}_{\in \mathfrak{gl}(\mathfrak{g})})}^{GL(\mathfrak{g})}\cdot Y = \sum_{k=0}^{\infty} \frac{1}{k!}\left([X,Y]\right)^k \underset{\text{ideal}}{\in} \mathfrak{h}.$$

$$\therefore (\exp X)\cdot\underbrace{(\exp Y)}_{\in H}\exp(-X) = \exp(\underbrace{\mathrm{Ad}_{\exp X} Y}_{\in \mathfrak{h}}) \in H.$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}_g"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{conj.}(g)"'] & G \end{tikzcd}$$

$$(\exp X)\underbrace{(\exp Y_1)\dots(\exp Y_k)}_{\text{any } h \in H \;\uparrow \text{conn.}}(\exp -X) = \underbrace{(\exp X)(\exp Y_1)(\exp -X)}_{\in H} \dots \underbrace{(\exp X)(\exp Y_k)(\exp -X)}_{\in H} \in H.$$

$$\underbrace{(\exp X_1)\dots(\exp X_\ell)}_{\text{any } g \text{ is of this form}} \cdot h\, (\exp X_1 \dots \exp X_\ell)^{-1} = \mathrm{Ad}_{\exp X_1} \cdots \mathrm{Ad}_{\exp X_\ell} \cdot h \in H.$$

$\boxed{\text{if } G \text{ is connected}}$

20-20. Let $G$ be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Prove that the kernel of $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is the ***center of*** $G$, that is, the set of elements of $G$ that commute with every element of $G$.

$\mathrm{Ad}_g(X) = X \Rightarrow \mathrm{Conj}_g(\exp X) = \exp(\mathrm{Ad}_g X) = \exp X.$

any $h = (\exp X_1)\dots(\exp X_k) \Rightarrow \mathrm{Conj}_g\, h = h \;\; \forall h \Leftrightarrow g \in \text{center}.$

20-21. Show that the adjoint representation of $\mathrm{GL}(n,\mathbb{R})$ is given by $\mathrm{Ad}(A)Y = AYA^{-1}$ for $A \in \mathrm{GL}(n,\mathbb{R})$ and $Y \in \mathfrak{gl}(n,\mathbb{R})$. Show that it is not faithful.

$c \cdot \mathrm{Id}$ acts trivially.

20-22. If $\mathfrak{g}$ is a Lie algebra, the ***center of*** $\mathfrak{g}$ is the set of all $X \in \mathfrak{g}$ such that $[X,Y] = 0$ for all $Y \in \mathfrak{g}$. Suppose $G$ is a connected Lie group. Show that the center of $\mathrm{Lie}(G)$ is the Lie algebra of the center of $G$.

center $(\mathrm{Lie}\, G) = \mathrm{Lie}(\text{center}(G))$:

$$\mathrm{ad}_X = 0 \;\; \forall Y \Leftrightarrow \mathrm{Ad}_{\exp tX} = \exp(\mathrm{ad}_{tX}) = \mathrm{Id}$$
$$\Leftrightarrow \exp tX \in \text{center}(G) \Leftrightarrow X \in \mathrm{Lie}(\text{center}(G)).$$